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APPLIED MECHANICS.

BY

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PREFACE.

THIS book is the result of the experience of the writer in teaching the subject of Applied Mechanics for the last twelve years at the Massachusetts Institute of Technology.

The immediate object of publishing it is, to enable him to dispense with giving to the students a large amount of notes. As, however, it is believed that it may be found useful by others, the following remarks in regard to its general plan are submitted.

The work is essentially a treatise on strength and stability; but, inasmuch as it contains some other matter, it was thought best to call it "Applied Mechanics," notwithstanding the fact that a number of subjects usually included in treatises on applied mechanics are omitted.

It is primarily a text-book; and hence the writer has endeavored to present the different subjects in such a way as seemed to him best for the progress of the class, even though it be at some sacrifice of a logical order of topics. While no attempt has been made at originality, it is believed that some features of the work are quite different from all pre-

vious efforts; and a few of these cases will be referred to, with the reasons for so treating them.

In the discussion upon the definition of "force," the object is, to make plain to the student the modern objections to the usual ways of treating the subject, so that he may have a clear conception of the modern aspect of the question, rather than to support the author's definition, as he is fully aware that this, as well as all others that have been given, is open to objection.

In connection with the treatment of statical couples, it was thought best to present to the student the actual effect of the action of forces on a rigid body, and not to delay this subject until dynamics of rigid bodies is treated, as is usually done.

In the common theory of beams, the author has tried to make plain the assumptions on which it is based. A little more prominence than usual has also been given to the longitudinal shearing of beams.

In that part of the book that relates to the experimental results on strength and elasticity, the writer has endeavored to give the most reliable results, and to emphasize the fact, that, to obtain constants suitable for use in practice, we must deduce them from tests on full-size pieces. This principle of being careful not to apply experimental results to cases very different from those experimented upon, has long been recognized in physics, and therefore needs no justification.

The government reports of tests made at the Watertown Arsenal have been extensively quoted from, as it is believed

that they furnish some of our most reliable information on these subjects.

The treatment of the strength of timber will be found to be quite different from what is usually given; but it speaks for itself, and will not be commented upon here.

In the chapter on the "Theory of Elasticity," a combination is made of the methods of Rankine and of Grashof.

In preparing the work, the author has naturally consulted the greater part of the usual literature on these subjects; and, whenever he has drawn from other books, he has endeavored to acknowledge it. He wishes here to acknowledge the assistance furnished him by Professor C. H. Peabody of the Massachusetts Institute of Technology, who has read all the proofs, and has aided him materially in other ways in getting out the work.

GAETANO LANZA.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
April, 1885.

TABLE OF CONTENTS.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

APPLIED MECHANICS.

CHAPTER I.

COMPOSITION AND RESOLUTION OF FORCES.

§ 1. **Fundamental Conceptions.** — The fundamental conceptions of Mechanics are Force, Matter, Space, Time, and Motion.

§ 2. **Relativity of Motion.** — The limitations of our natures are such that all our quantitative conceptions are relative. The truth of this statement may be illustrated, in the case of motion, by the fact, that, if we assume the shore as fixed in position, a ship sailing on the ocean is in motion, and a ship moored in the dock is at rest; whereas, if we assume the sun as our fixed point, both ships are really in motion, as both partake of the motion of the earth. We have, moreover, no means of determining whether any given point is absolutely fixed in position, nor whether any given direction is an absolutely fixed direction. Our only way of determining direction is by means of two points assumed as fixed; and the straight line joining them, we are accustomed to assume as fixed in direction. Thus, it is very customary to assume the straight line joining the sun with any fixed star as a line fixed in direction; but if the whole visible universe were in motion, so as to change the absolute direction of this line, we should have no means of recognizing it.

§ 3. **Rest and Motion.** — In order to define rest and motion, we have the following; viz., —

When a single point is spoken of as having motion or rest, some other point is always expressed or understood, which is for the time being considered as a fixed point, and some direction is assumed as a fixed direction: and we then say that the first-named point is at rest relatively to the fixed point, when the straight line joining it with the fixed point changes neither in length, nor in direction; whereas it is said to be in motion relatively to the fixed point, when this straight line changes in length, in direction, or in both.

If, on the other hand, we had considered the first-named point as our fixed point, the same conditions would determine whether the second was at rest, or in motion, relatively to the first.

A body is said to be at rest relatively to a given point and to a given direction, when all its points are at rest relatively to this point and this direction.

§ 4. **Velocity.** — When the motion of one point relatively to another, or of one body relatively to another, is such that it describes equal distances in equal times, however small be the parts into which the time is divided, the motion is said to be uniform and the velocity constant.

The velocity, in this case, is the space passed over in a unit of time, and is to be found by dividing the space passed over in any given time by the time; thus, if s represent the space passed over in time t, and v represent the velocity, we shall have

$$v = \frac{s}{t}.$$

When the motion is not uniform, if we divide the time into small parts, and then divide the space passed over in one of these intervals by the time, and then pass to the limit as these intervals of time become shorter, we shall obtain the velocity.

Thus, if Δs represent the space passed over in the interval of time Δt, then we shall have

$$v = \text{limit of } \frac{\Delta s}{\Delta t} \text{ as } \Delta t \text{ diminishes,}$$

or

$$v = \frac{ds}{dt}.$$

§ 5. **Force.** — We shall next attempt to obtain a correct definition of force, or at least of what is called force in mechanics.

It may seem strange that it should be necessary to do this; as it would appear that clear and correct definitions must have been necessary in order to make correct deductions, and therefore that there ought to be no dispute whatever over the meaning of the word *force*. Nevertheless, it is a fact in mechanics, as well as in all those sciences which attempt to deal with the facts and laws of nature, that correct definitions are only gradually developed, and that, starting with very imperfect and often erroneous views of natural laws and phenomena, it is only after these errors have been ascertained and corrected by a long range of observation and experiment, and an increased range of knowledge has been acquired, that exactness and perspicuity can be obtained in the definitions.

Now, this is precisely what has happened in the case of force.

In ancient times *rest* was supposed to be the natural state of bodies; and it was assumed that, in order to make them move, force was necessary, and that even after they had been set in motion their own innate inertia or sluggishness would cause them to come to rest unless they were constantly urged on by the application of some force, the bodies coming to rest whenever the force ceased acting.

It was under the influence of these vague notions that such terms arose as *Force of Inertia*, *Moment of Inertia*, *Vis Viva* or *Living Force*, etc.

A number of these terms are still used in mechanics; but in all such cases they have been re-defined, such new meanings having been attached to them as will bring them into accord with the more advanced ideas of the present time. Such definitions will be given in the course of this work, as the necessity may arise for the use of the terms.

Moreover, it is to be regretted that there still prevails, to some extent, in the more popular class of scientific literature, a loose usage of such terms, which is very liable to impart erroneous ideas to those whose minds are not clear as to their true meanings.

NEWTON'S FIRST LAW OF MOTION.

Ideas becoming more precise, in course of time there was framed Newton's first law of motion; and this law is as follows: —

A body at rest will remain at rest, and a body in motion will continue to move uniformly and in a straight line, unless and until some external force acts upon it.

The assumed truth of this law was based upon the observed facts of nature; viz., —

When bodies were seen to be at rest, and from rest passed into a state of motion, it was always possible to assign some cause; i.e., they had been brought into some new relationship, either with the earth, or with some other body: and to this cause could be assigned the change of state from rest to motion. On the other hand, in the case of bodies in motion, it was seen, that, if a body altered its motion from a uniform rectilinear motion, there was always some such cause that could be assigned. Thus, in the case of a ball thrown from the hand, the attraction of the earth and the resistance of the air soon caused it to come to rest. In the case of a ball rolled along the ground, friction (i.e., the continual contact and collision with the ground) gradually destroyed its motion, and brought it to

rest; whereas, when such resistances were diminished by rolling it on glass or on the ice, the motion always continued longer: hence it was inferred, that, were these resistances entirely removed, the motion would continue forever.

In accordance with these views, the definition of force usually given was substantially as follows: —

Force is that which causes, or tends to cause, a body to change its state from rest to motion, from motion to rest, or to change its motion as to direction or speed.

Under these views, uniform rectilinear motion was recognized as being just as much a condition of equilibrium, or of the action of no force or of balanced forces, as rest; and the recognition of this one fact upset many false notions, destroyed many incorrect conclusions, and first rendered possible a science of mechanics. Along with the above-stated definition of force is ordinarily given the following proposition; viz., —

Forces are proportional to the velocities that they impart, in a unit of time, to the same body. The reasoning given in support of this proposition is as follows: —

Suppose a body to be moving uniformly and in a straight line, and suppose a force to act upon it for a certain length of time t in the direction of the body's motion: the effect of the force is to alter the velocity of the body; and it is only by this alteration of velocity that we recognize the action of the force. Hence, as long as the alteration continues at the same rate, we recognize the same force as acting.

If, therefore, f represent the amount of velocity which the force would impart in one unit of time, the total increase in the velocity of the body will be ft; and, if the force now stop acting, the body will again move uniformly and in the same direction, but with a velocity greater by ft.

Hence, if we are to measure forces by their effects, it will follow that —

The velocity which a force will impart to a given (or standard)

body in a unit of time is a proper measure of the force. And we shall have, that two forces, each of which will impart the same velocity to the same body in a unit of time, are equal to each other; and a force which will impart to a given body twice the velocity per unit of time that another force will impart to the same body, is itself twice as great, or, in other words, —

Forces are proportional to the velocities that they impart, in a unit of time, to the same body.

MODERN CRITICISM OF THE ABOVE.

The scientists and the metaphysicians of the present time are recognizing two other facts not hitherto recognized, and the result is a criticism adverse to the above-stated definition of force. Other definitions have, in consequence, been proposed; but none are free from objection on logical grounds, and at the same time capable of use in mechanics in a quantitative way.

The two facts referred to are the following; viz., —

1°. That all our ideas of space, time, rest, motion, and even of direction, are relative.

2°. That, because two effects are identical, it does not follow that the causes producing those effects are identical.

Hence, in the light of these two facts, it is plain, that, inasmuch as we can only recognize motion as relative, we can only recognize force as acting when at least two bodies are concerned in the transaction; and also that if the forces are simply the causes of the motion in the ordinary popular sense of the word *cause*, we cannot assume, that, when the effects are equal, the causes are in every way identical, although we have, of course, a perfect right to say that they are identical so far as the production of motion is concerned.

I shall now proceed, in the light of the above, to deduce a definition of force, which, although not free from objection, seems as free as any that has been framed.

It is one of the facts of nature, that, when bodies are by any

means brought under certain relations to each other, certain tendencies are developed, which, if not interfered with, will exhibit themselves in the occurrence of certain definite phenomena. What these phenomena are, depends upon the nature of the bodies concerned, and on the relationships into which they are brought.

As an illustration, we know that if an apple is placed at a certain height above the surface of the earth, there is developed between the two bodies a tendency to approach each other; and if there is no interference with this tendency, it exhihits itself in the fall of the apple. If, on the other hand, the apple were hung on the hook of a spring balance in the same position as before, the spring would stretch, and there would be developed a tendency of the spring to make the apple move upwards. This tendency to make the apple move upwards would be just equal to the tendency of the earth and apple to approach each other. This would be expressed by saying that the pull of the spring is just equal and opposite to the weight of the apple.

As other illustrations of these tendencies developed in bodies when placed in certain relations to each other, we have the following cases: —

(*a*) When two bodies collide.

(*b*) When two substances, coming together, form a chemical union, as sodium and water.

(*c*) When the chemical union is entered into only by raising the temperature to some special point.

Any of these tendencies that are developed by bringing about any of these special relationships between bodies might properly be called a force; and the term might properly be, and is, used in the same sense in the mental and moral world, as well as in the physical. In mechanics, however, we have to deal only with the relative motion of bodies; and hence we give the name *force* only to tendencies to change the relative

motion of the bodies concerned; and this, whether these tendencies are unresisted, and exhibit themselves in the actual occurrence of a change of motion, or whether they are resisted by equal and opposite tendencies, and exhibit themselves in the production of a tensile, compressive, or other stress in the bodies concerned, instead of motion.

DEFINITION OF FORCE.

Hence our definition of force, as far as mechanics has to deal with it or is capable of dealing with it, is as follows; viz., —

Force is a tendency to change the relative motion of the two bodies between which that tendency exists.

Indeed, when, as in the illustration given a short time ago, the apple is hung on the hook of a spring balance, there still exists a tendency of the apple and the earth to approach each other; i.e., they are in the act of trying to approach each other; and it is this tendency, or *act of trying*, that we call the force of gravitation. In the case cited, this tendency is balanced by an opposite tendency on the part of the spring; but, were the spring not there, the force of gravitation would cause the apple to fall.

Professor Rankine calls force "an action between two bodies, either causing or tending to cause change in their relative rest or motion;" and if the *act of trying* can be called an *action*, my definition is equivalent to his.

For the benefit of any one who wishes to follow out the discussions that have lately taken place, I will enumerate the following articles that have been written on the subject: —

(*a*) "Recent Advances in Physical Science," by P. G. Tait, Lecture XIV.

(*b*) Herbert Spencer, "First Principles of Philosophy" (certain portions of the book).

(*c*) Discussion by Messrs. Spencer and Tait, "Nature," Jan. 2, 9, 16, 1879.

(*d*) Force and Energy, "Nature," Nov. 25, Dec. 2, 9, 16, 1880.

§ 6. **External Force.** — We thus see, that, in order that a force may be developed, there must be two bodies concerned in the transaction; and we should speak of the force as that developed or existing between the two bodies.

But we may confine our attention wholly to the motion or condition of one of these two bodies; and we may refer its motion either to the other body as a fixed point, or to some body different from either; and then, in speaking of the force, we should speak of it as the force acting on the body under consideration, and call it an external force. It is the tendency of the other body to change the motion of the body under consideration relatively to the point considered as fixed.

§ 7. **Relativity of Force.** — In adopting the above-stated definition of force, we acknowledge our incapacity to deal with it as an absolute quantity; for we have defined it as a tendency to change the relative motion of a pair of bodies. Hence it is only through relative motion that we recognize force; and hence force is relative, as well as motion.

§ 8. **Newton's First Law of Motion.** — In the light of the above discussion, we might express Newton's first law of motion as follows: —

A body at rest, or in uniform rectilinear motion relatively to a given point assumed as fixed, will continue at rest, or in uniform motion in the same direction, unless and until some external force acts either on the body in question, or on the fixed point, or on the body which furnishes us our fixed direction. This law is really superfluous, as it has all been embodied in the definition.

§ 9. **Measure of Force.** — We next need some means of comparing forces with each other in magnitude; and, subse-

quently, we need to select one force as our unit force, by means of which to estimate the magnitude of other forces.

Let us suppose a body moving uniformly and in a straight line, relatively to some fixed point; as long as this motion continues, we recognize no unbalanced force acting on it; but, if the motion changes, there must be a tendency to change that motion, or, in other words, an unbalanced force is acting on the body from the instant when it begins to change its motion.

Suppose a body to be moving uniformly, and a force to be applied to it, and to act for a length of time t, and to be so applied as not to change the direction of motion of the body, but to increase its velocity; the result will be, that the velocity will be increased by equal amounts in equal times, and if f represent the amount of velocity the force would impart in one unit of time, the total increase in velocity will be ft. This results merely from the definition of a force; for if the velocity produced in one (a standard) body by a given force is twice as great as that produced by another given force, then is the tendency to produce velocity twice as great in the first case as in the second, or, in other words, the first force is twice as great as the second. Hence —

Forces are proportional to the velocities which they will impart to a given (or standard) body in a unit of time.

We may thus, by using one standard body, determine a set of equal forces, and also the proportion between different forces.

§ 10. **Measure of Mass.** — After having determined, as shown, a set of equal (unit) forces, if we apply two of them to different bodies, and let them act for the same length of time on each, and find that the resulting velocities are unequal, these bodies are said to have unequal *masses;* whereas, if the resulting velocities are equal, they are said to have equal *masses.*

Hence we have the following definitions: —

1°. *Equal forces are those which, by acting for equal times on the same or standard body, impart to it equal velocities.*

2°. *Equal masses are those masses to which equal forces will impart equal velocities in equal times.*

§ 11. Suppose two bodies of equal mass moving side by side with the same velocity, and uniformly, let us apply to one of them a force F in the direction of the body's motion: the effect of this force is to increase the velocity with which the body moves; and if we wish, at the same time, to increase the velocity of the other, so that they will continue to move side by side, it will be necessary to apply an equal force to that also.

We are thus employing a force $2F$ to impart to the two bodies the required increment of velocity.

If we unite them into one, it still requires a force $2F$ to impart to the one body resulting from their union the required increment of velocity: hence, if we double the *mass* to which we wish to impart a certain velocity, we must double the force, or, in other words, employ a force which would impart to the first mass alone a velocity double that required. Hence —

Forces are proportional to the masses to which they will impart the same velocity in the same time.

§ 12. **Momentum.** — The product obtained by multiplying the number of units of mass in a body by its velocity is called the momentum of the body.

§ 13. **Relation between Force and Momentum.** — The number of units of momentum imparted to a body in a unit of time by a given force, is evidently identical with the number of units of velocity that would be imparted by the same force, in the same time, to a unit mass. Hence —

Forces are proportional to the momenta (or velocities per unit of mass) which they will generate in a unit of time.

Hence, if F represent a force which generates, in a unit of time, a velocity f in a body whose mass is m, we shall have

$$F \propto mf;$$

and, inasmuch as the choice of our units is still under our control, we so choose them that

$$F = mf;$$

i.e., the force F contains as many units of force as mf contains units of momentum ; in other words, —

The momentum generated in a body in a unit of time by a force acting in the direction of the body's motion, is taken as a measure of the force.

§ 14. **Statical Measure of Force.** — When the forces are prevented from producing motion by being resisted by equal and opposite forces, as is the case in that part of mechanics known as *Statics*, they must be measured by a direct comparison with other forces. An illustration of this has already been given in the case of an apple hung on the hook of a spring balance. In that case the pull of the spring is equal in magnitude to the weight of the apple : indeed, it is very customary to adopt for forces what is known as the *gravity measure*, in which case we take as our unit the gravitation, or *tendency to fall*, of a given piece of metal, at a given place on the surface of the earth ; in other words, its weight at a given place.

The gravity unit may thus be the kilogram, the pound, or the ounce, etc.

It is evident, moreover, from our definition of force, and the subsequent discussion, that whatever we take as our unit of mass, the *statical measure* of a force is proportional to its *dynamical measure;* i.e., the numbers representing the magnitudes of any two forces, in pounds, are proportional to the momenta they will impart to any body in a unit of time.

§ 15. **Gravity Measure of Mass.** — If we assume one pound as our unit of force, one foot as our unit of length, and

one second as our unit of time, the ratio between the number of pounds in any given force and the momentum it will impart to a body on which it acts unresisted for a unit of time, will depend on our unit of mass; and, as we are still at liberty to fix this as we please, it will be most convenient so to choose it that the above-stated ratio shall be unity, so that there shall be no difference in the measure of a force, whether it is measured statically or dynamically. Now, it is known that a body falling freely under the action of its own weight acquires, every second, a velocity of about thirty-two feet per second: this number is denoted by g, and varies for different distances from the centre of the earth, as does also the weight of the body.

Now, if W represent the weight of the body in pounds, and m the number of units of mass in its mass, we must have, in order that the statical and dynamical measures may be equal,

$$W = mg.$$

Hence

$$m = \frac{W}{g};$$

i.e., the number of units of mass in a body is obtained by dividing the weight in pounds, by the value of g at the place where the weight is determined.

The values of W and of g vary for different positions, but the value of m remains always the same for the same body.

UNIT OF MASS.

If $m = 1$, then $W = g$; or, in words, —

The weight in pounds of the unit of mass (when the gravity measure is used) is equal to the value of g *in feet per second for the same place.*

§ 16. **Newton's Second Law of Motion.** — Newton's second law of motion is as follows: —

"*Change of momentum is proportional to the impressed moving force, and occurs along the straight line in which the force is impressed*."

Newton states further in his "Principia:" —

"If any force generate any momentum, a double force will generate a double, a triple force will generate a triple, momentum, whether simultaneously and suddenly, or gradually and successively impressed. And if the body was moving before, this momentum, if in the same direction as the motion, is added; if opposite, is subtracted; or if in an oblique direction, is annexed obliquely, and compounded with it, according to the direction and magnitude of the two."

Part of this law has reference to the proportionality between the force and the momentum imparted to the body; and this has been already embodied in our definition of force, and illustrated in the discussion on the measure of forces.

The other part is properly a law of motion, and may be expressed as follows: —

If a body have two or more velocities imparted to it simultaneously, it will move so as to preserve them all.

The proof of this law depends merely upon a proper conception of motion. To illustrate this law when two velocities are imparted simultaneously to a body, let us suppose a man walking on the deck of a moving ship: he then has two motions in relation to the shore, his own and that of the ship.

Suppose him to walk in the direction of motion of the ship at the rate of 10 feet per second, while the ship moves at 25 feet per second relatively to the shore: then his motion in relation to the shore will be $25 + 10 = 35$ feet per second. If, on the other hand, he is walking in the opposite direction at the same rate, his motion relatively to the shore will be $25 - 10 = 15$ feet per second.

Suppose a body situated at A (Fig. 1) to have two motions imparted to it simultaneously, one of which would carry it to B

in one second, and the other to C in one second; and that it is required to find where it will be at the end of one second, and what path it will have pursued.

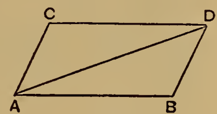


FIG. 1.

Imagine the body to move in obedience to the first alone, during one second: it would thus arrive at B; then suppose the second motion to be imparted to the body, instead of the first, it will arrive at the end of the next second at D, where BD is equal and parallel to AC. When the two motions are imparted simultaneously, instead of successively, the same point D will be reached in one second, instead of two; and by dividing AB and AC into the same (any) number of equal parts, we can prove that the body will always be situated at some point of the diagonal AD of the parallelogram, hence that it moves along AD. Hence follows the proposition known as the parallelogram of motions.

PARALLELOGRAM OF MOTIONS.

If there be simultaneously impressed on a body two velocities, which would separately be represented by the lines AB *and* AC, *the actual velocity will be represented by the line* AD, *which is the diagonal of the parallelogram of which* AB *and* AC *are the adjacent sides.*

§ 17. **Polygon of Motions.** — In all the above cases, the point reached by the body at the end of a second when the two motions take place simultaneously is the same as that which would be reached at the end of two seconds if the motions took place successively; and the path described is the straight line joining the initial position of the body, with its position at the end of one second when the motions are simultaneous.

The same principle applies whatever be the number of velocities that may be imparted to a body simultaneously. Thus, if we suppose the several velocities imparted to be (Fig. 2) AB, AC, AD, AE, and AF, and it be required to

determine the resultant velocity, we first let the body move with the velocity AB for one second; at the end of that second it is found at B; then let it move with the velocity AC only, and at the end of another second it will be found at c; then with AD only, and at the end of the third second it will be found at d; at the end of the fourth at e; at the end of the fifth at f. Hence the resultant velocity, when all are imparted simultaneously, is Af, or the closing side of the polygon.

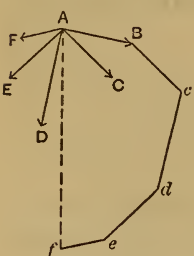


FIG. 2.

This proposition is known as the polygon of motions.

POLYGON OF MOTIONS.

If there be simultaneously impressed on a body any number of velocities, the resulting velocity will be represented by the closing side of a polygon of which the lines representing the separate velocities form the other sides.

§ 18. **Characteristics of a Force.** — A force has three characteristics, which, when known, determine it; viz., *Point of Application*, *Direction*, and *Magnitude*. These can be represented by a straight line, whose length is made proportional to the magnitude of the force, whose direction is that of the motion which the force imparts, or tends to impart, and one end of which is the point of application of the force; an arrow-head being usually employed to indicate the direction in which the force acts.

§ 19. **Parallelogram of Forces.**

PROPOSITION. — *If two forces acting simultaneously at the same point be represented, in point of application, direction, and magnitude, by two adjacent sides of a parallelogram, their resultant will be represented by the diagonal of the parallelogram, drawn from the point of application of the two forces.*

PROOF. — In the last part of § 16 was proved the propo-

sition known as the *Parallelogram of Motions*, for the statement of which the reader is referred to the close of that section.

We have also seen that forces are proportional to the velocities which they impart, or tend to impart, in a unit of time, to the same body.

Hence the lines representing the two impressed forces are coincident in direction with, and proportional to, the lines representing the velocities they would impart in a unit of time to the same body; and moreover, since the resultant velocity is represented by the diagonal of the parallelogram drawn with the component velocities as sides, the resultant force must coincide in direction with the resultant velocity, and the length of the line representing the resultant force will bear to the resultant velocity the same ratio that one of the component forces bears to the corresponding velocity. Hence it follows, that the resultant force will be represented by the diagonal of the parallelogram having for sides the two component forces.

§ 20. **Parallelogram of Forces: Algebraic Solution.**

PROBLEM. — *Given two forces* F *and* F_1 *acting at the same point* A (*Fig.* 3), *and inclined to each other at an angle* θ; *required the magnitude and direction of the resultant force.*

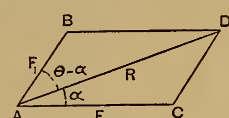


FIG. 3

Let AC represent F, AB represent F_1, and let angle $BAC = \theta$; then will $R = AD$ represent in magnitude and direction the resultant force. Also let angle $DAC = \alpha$; then from the triangle DAC we have

$$AD^2 = AC^2 + CD^2 - 2AC \, . \, CD \cos ACD.$$

But

$$ACD = 180^\circ - \theta \qquad \therefore \quad \cos ACD = -\cos\theta$$

$$\therefore \quad R^2 = F^2 + F_1^2 + 2FF_1 \cos\theta$$

$$\therefore \quad R = \sqrt{F^2 + F_1^2 + 2FF_1 \cos\theta}.$$

This determines the magnitude of R. To determine its direction, let angle $CAD = \alpha \quad \therefore$ angle $BAD = \theta - \alpha$, and we shall have from the triangle DAC

$$CD : AD = \sin CAD : \sin ACD,$$

or

$$F_1 : R = \sin\alpha : \sin\theta$$

$$\therefore \quad \sin\alpha = \frac{F_1}{R}\sin\theta,$$

and similarly

$$\sin(\theta - \alpha) = \frac{F}{R}\sin\theta.$$

EXAMPLES.

1°. Given $F = 47.34$, $F_1 = 75.46$, $\theta = 73°\ 14'\ 21''$; find R and α.
2°. Given $F = 5.36$, $F_1 = 4.27$, $\theta = 32°\ 10'$; find R and α.
3°. Given $F = 42.00$, $F_1 = 31.00$, $\theta = 150°$; find R and α.
4°. Given $F = 47.00$, $F_1 = 75.00$, $\theta = 253°$; find R and α.

§ 21. **Parallelogram of Forces when $\theta = 90°$.** — When the two given forces are at right angles to each other, the formulæ become very much simplified, since the parallelogram becomes a rectangle.

From Fig. 4 we at once deduce

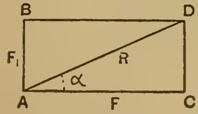


FIG. 4.

$$R = \sqrt{F^2 + F_1^2},$$

$$\sin\alpha = \frac{F_1}{R},$$

$$\cos\alpha = \frac{F}{R}.$$

EXAMPLES.

1°. Given $F = 3.0$, $F_1 = 5.0$; find R and α.
2°. Given $F = 3.0$, $F_1 = -5.0$; find R and α.
3°. Given $F = 5.0$, $F_1 = 12.0$; find R and α.
4°. Given $F = 23.2$, $F_1 = 21.3$; find R and α.

§ 22. **Triangle of Forces.** — *If three forces be represented, in magnitude and direction, by the three sides of a triangle taken in order, then, if these forces be simultaneously applied at one point, they will balance each other.*

Conversely, three forces which, when simultaneously applied at one point, balance each other, can be correctly represented in magnitude and direction by the three sides of a triangle taken in order.

These propositions, which find a very extensive application, especially in the determination of the stresses in roof and bridge trusses, are proved as follows: —

If we have two forces, AC and AB (see Fig. 3), acting at the point A, their resultant is, as we have already seen, AD; and hence a force equal in magnitude and opposite in direction to AD will balance the two forces AC and AB. Now, the sides of the triangle $ACDA$, if taken in order, represent in magnitude and direction the force AC, the force CD or AB, and a force equal and opposite to AD; and these three forces, if applied at the same point, would balance each other. Hence follows the proposition.

Moreover, we have

$$AC : CD : DA = \sin ADC : \sin CAD : \sin ACD,$$

or

$$F : F_1 : R = \sin(\theta - \alpha) : \sin\alpha : \sin\theta;$$

or each force is, in this case, proportional to the sine of the angle between the other two.

§ 23. **Decomposition of Forces in one Plane.** — It is often convenient to resolve a force into two components, in two given directions in a plane containing the force. Thus, suppose we have the force $R = AD$ (Fig. 3), and we wish to resolve it into two components acting respectively in the directions AC and AB; i.e., we wish to find two forces acting respectively in these directions, of which AD shall be the resultant: we

determine these components graphically by drawing a parallelogram, of which AD shall be the diagonal, and whose sides shall have the directions AC and AB respectively. The algebraic values of the magnitudes of the components can be determined by solving the triangle ADC. In the case when the directions of the components are at right angles to each other, let the force R (Fig. 5), applied at O, make an angle α with OX. We may, by drawing the rectangle shown in the figure, decompose R into two components, F and F_1, along OX and OY respectively; and we shall readily obtain from the figure,

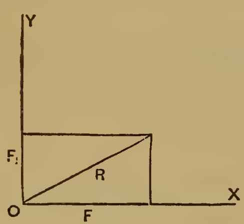


FIG. 5.

$$F = R\cos\alpha, \quad F_1 = R\sin\alpha.$$

EXAMPLES.

1°. The force exerted by the steam upon the piston of a steam-engine at the moment when it is in the position shown in the figure is $AB =$ 1000 lbs. The resistance of the guides upon the cross-head DE is vertical. Determine the force acting along the connecting-rod AC and the pressure on the guides; also resolve the force acting along the connecting-rod into two components, one along, and the other at right angles to, the crank OC.

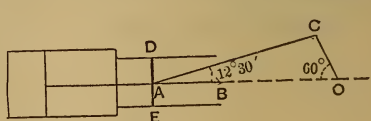


FIG. 6.

2°. A load of 500 lbs. is placed at the apex C of the frame ACB: find the stresses in AC and CB respectively.

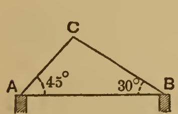


FIG. 7.

3°. A load of 4000 lbs. is hung at C, on the crane ABC: find the pressure in the boom BC, and the pull on the tie AC, where BC makes an angle of 60° with the horizontal, and AC an angle of 15°.

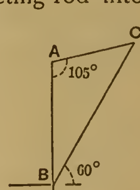


FIG. 8.

4°. A force whose magnitude is 7 is resolved into two forces whose magnitudes are 5 and 3: find the angles they make with the given force.

§ 24. Composition of any Number of Forces in One Plane, all applied at the Same Point.

(*a*) GRAPHICAL SOLUTION. — Let the forces be represented (Fig. 2) by AB, AC, AD, AE, and AF respectively. Draw Bc ∥ and $=$ AC, cd ∥ and $=$ AD, de ∥ and $=$ AE, and ef ∥ and $=$ AF; then will Af represent the resultant of the five forces. This solution is to be deduced from § 17 in the same way as § 19 is deduced from § 16.

(*b*) ALGEBRAIC SOLUTION. — Let the given forces (Fig. 9), of which three are represented in the figure, be F, F_1, F_2, F_3, F_4, etc.; and let the angles made by these forces with the axis OX be α, α_1, α_2, α_3, α_4, etc., respectively.

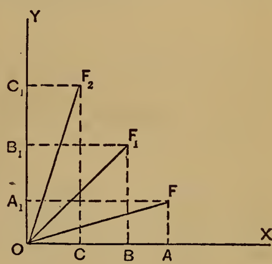


FIG. 9.

Resolve each of these forces into two components, in the directions OX and OY respectively. We shall obtain for the components along OX

$$OA = F\cos\alpha, \quad OB = F_1\cos\alpha_1, \quad OC = F_2\cos\alpha_2, \quad \text{etc.};$$

and for those along OY

$$OA_1 = F\sin\alpha, \quad OB_1 = F_1\sin\alpha_1, \quad OC_1 = F_2\sin\alpha_2, \quad \text{etc.}$$

These forces are equivalent to the following two; viz., a force $F\cos\alpha + F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 +$ etc. along OX, and a force $F\sin\alpha + F_1\sin\alpha_1 + F_2\sin\alpha_2 + F_3\sin\alpha_3 +$ etc. along OY. The first may be represented by $\Sigma F\cos\alpha$, and the second by $\Sigma F\sin\alpha$, where Σ stands for *algebraic sum*. There remains only to find the resultant of these two, the magnitude of which is given by the equation

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\sin\alpha)^2};$$

and, if we denote by α_r the angle made by the resultant with OX, we shall have

$$\cos\alpha_r = \frac{\Sigma F\cos\alpha}{R}, \quad \sin\alpha_r = \frac{\Sigma F\sin\alpha}{R}.$$

EXAMPLES.

1°. Given $\left\{\begin{array}{l} F = 47 \\ F_1 = 73 \\ F_2 = 43 \\ F_3 = 23 \end{array}\right. \quad \left.\begin{array}{l} \alpha = 21^\circ \\ \alpha_1 = 48^\circ \\ \alpha_2 = 82^\circ \\ \alpha_3 = 112^\circ \end{array}\right\}$ Find the resultant force and its direction.

Solution.

F.	α.	$\cos\alpha$.	$\sin\alpha$.	$F\cos\alpha$.	$F\sin\alpha$.
47	21°	0.93358	0.35837	43.87826	16.84339
73	48°	0.66913	0.74315	48.84649	54.24995
43	82°	0.13917	0.99027	5.98431	42.58161
23	112°	−0.37461	0.92718	−8.61603	21.32414
				90.09303	134.99909

$$\therefore\ \Sigma F\cos\alpha = 90.09303, \quad \Sigma F\sin\alpha = 134.99909,$$

$$\therefore\ R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\sin\alpha)^2} = 162.2976.$$

$$\begin{aligned} \log\Sigma F\cos\alpha &= 1.954691 \\ \log R &= 2.210331 \\ \hline \log\cos\alpha_r &= 9.744360 \\ \alpha_r &= 56^\circ\ 17'. \end{aligned}$$

OBSERVATION. — It would be perfectly correct to use the minus sign in extracting the square root, or to call $R = -162.2976$; but then we should have

$$\cos\alpha_r = \frac{90.09303}{-162.2976}, \quad \text{and} \quad \sin\alpha_r = \frac{134.99909}{-162.2976},$$

or

$$\alpha_r = 180^\circ + 56^\circ\ 17' \qquad = 236^\circ\ 17';$$

a result which, if plotted, would give the same force as when we call

$$R = 162.2976 \quad \text{and} \quad \alpha_r = 56^\circ\ 17'.$$

Hence, since it is immaterial whether we use the plus or the minus sign in extracting the square root provided the rest of the computation be consistent with it, we shall, for convenience, use always plus.

2°.

$$\begin{aligned} F &= 4, & \alpha &= 77^\circ, \\ F_1 &= 3, & \alpha_1 &= 82^\circ, \\ F_2 &= 10, & \alpha_2 &= 163^\circ, \\ F_3 &= 5, & \alpha_3 &= 275^\circ. \end{aligned}$$

3°.

$$\begin{aligned} F &= 5, & \alpha &= \cos^{-1}\tfrac{4}{5}, \\ F_1 &= 4, & \alpha_1 &= 0, \\ F_2 &= 3, & \alpha_2 &= 90^\circ. \end{aligned}$$

§ 25. **Polygon of Forces.** — *If any number of forces be represented in magnitude and direction by the sides of a polygon taken in order, then, if these forces be simultaneously applied at one point, they will balance each other.*

Conversely, any number of forces which, when simultaneously applied at one point, balance each other, can be correctly represented in magnitude and direction by the sides of a polygon taken in order.

These propositions are to be deduced from § 24 (*a*) in the same way as the *triangle of forces* is deduced from the parallelogram of forces.

§ 26. **Composition of Forces all applied at the Same Point, and not confined to One Plane.** — This problem can be solved by the polygon of forces, since there is nothing in the demonstration of that proposition that limits us to a plane rather than to a *gauche* polygon.

The following method, however, enables us to determine algebraic values for the magnitude of the resultant and for its direction.

We first assume a system of three rectangular axes, OX, OY, and OZ (Fig. 10), whose origin is at the common point of the given forces. Now, let $OE = F$ be one of the given forces. First resolve it into two forces, OC and OD, the first of which lies in the z axis, and the second perpendicular to OZ, or, as it is usually called, in the z plane; the plane perpendicular to OX being the x plane, and that perpendicular to OY the y plane. Then resolve OD into two components, OA along OX, and OB along OY. We thus obtain three forces, OA, OB, and OC respectively, which are equivalent to the single force OE. These three components are the edges of a rectangular parallelopiped, of which $OE = F$ is the diagonal.

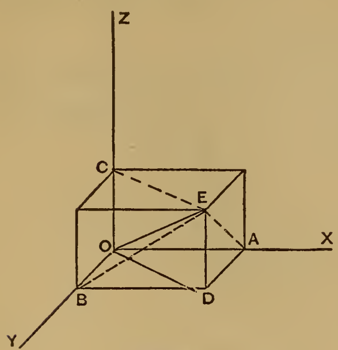


FIG. 10.

Let, now,

$$\text{angle } EOX = \alpha, \quad EOY = \beta, \quad \text{and} \quad EOZ = \gamma;$$

and we have, from the right-angled triangles EOA, EOB, and EOC respectively,

$$OA = F\cos\alpha, \quad OB = F\cos\beta, \quad OC = F\cos\gamma.$$

Moreover,

$$OA^2 + OB^2 = OD^2 \text{ and } OD^2 + OC^2 = OE^2$$
$$\therefore \quad OA^2 + OB^2 + OC^2 = OE^2,$$

and by substituting the values of OA, OB, and OC, given above, we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

a purely geometrical relation existing between the three angles that any line makes with three rectangular co-ordinate axes.

When two of the angles α, β, and γ are given, the third can be determined from the above equation.

Resolve, in the same way, each of the given forces into three components, along OX, OY, and OZ respectively, and we shall thus reduce our entire system of forces to the following three forces: —

1°. A single force $\Sigma F \cos \alpha$ along OX.
2°. A single force $\Sigma F \cos \beta$ along OY.
3°. A single force $\Sigma F \cos \gamma$ along OZ.

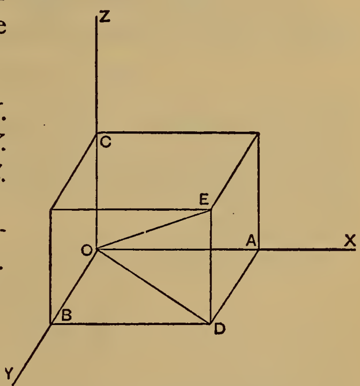


FIG. 11.

We next proceed to find a single resultant for these three forces.

Let (Fig. 11)

$$OA = \Sigma F \cos \alpha$$
$$OB = \Sigma F \cos \beta,$$
$$OC = \Sigma F \cos \gamma.$$

Compounding OA and OB, we find OD to be their resultant; and this, compounded with OC, gives OE as the resultant of the entire system. Moreover,

$$OE^2 = OD^2 + OC^2 = OA^2 + OB^2 + OC^2,$$

or

$$R^2 = (\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2$$

$$\therefore \quad R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2};$$

and if we let $EOX = \alpha_r$, $EOY = \beta_r$, and $EOZ = \gamma_r$, we shall have

$$\cos \alpha_r = \frac{OA}{OE} = \frac{\Sigma F \cos \alpha}{R}, \quad \cos \beta_r = \frac{\Sigma F \cos \beta}{R}, \quad \text{and } \cos \gamma_r = \frac{\Sigma F \cos \gamma}{R}.$$

This gives us the magnitude and direction of the resultant.

The same observation applies to the sign of the radical for R as in the case of forces confined to one plane.

DETERMINATION OF THE THIRD ANGLE FOR ANY ONE FORCE.

When two of the angles α, β, and γ are given, the cosine of the third may be determined from the equation, —

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

but, as we may use either the plus or the minus sign in extracting the square root, we have no means of knowing which of the two supplementary angles whose cosine has been deduced is to be used.

Thus, suppose $\alpha = 45°$, $\beta = 60°$, then

$$\cos\gamma = \pm\sqrt{1 - \tfrac{1}{2} - \tfrac{1}{4}} = \pm\tfrac{1}{2}$$
$$\therefore \quad \gamma = 60°, \text{ or } 120°;$$

but which of the two to use we have no means of deciding.

This indetermination will be more clearly seen from the following geometrical considerations: —

The angle α (Fig. 12), being given as 45°, locates the line representing the force on a right circular cone, whose axis is OX, and whose semi-vertical angle is $AOX = BOX = 45°$. On the other hand, the statement that $\beta = 60°$ locates the force on another right circular cone, having OY for axis, and a semi-vertical angle of 60°; both cones, of course, having their vertices at O. Hence, when α and β are given, we know that the line representing the force is an element of both cones; and this is all that is given.

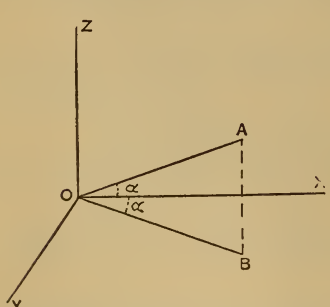


Fig. 12.

(*a*) Now, if the sum of the two given angles is less than 90°, the cones will not intersect, and the data are consequently inconsistent.

(*b*) If, on the other hand, one of the given angles being greater than 90°, their difference is greater than 90°, the cones will not intersect, and the data are again inconsistent.

(*c*) If $\alpha + \beta = 90°$, the cones are tangent to each other, and $\gamma = 90°$.

(*d*) If $\alpha + \beta > 90°$, and $\alpha - \beta$ or $\beta - \alpha < 90°$, the cones intersect, and have two elements in common; and we have no means of determining, without more data, which intersection is intended, this being the indetermination that arises in the algebraic solution.

EXAMPLES.

1. Given $\left\{\begin{array}{lll} F = 63 & \alpha = 53° & \beta = 42° \\ F_1 = 49 & \alpha = 87° & \gamma = 72° \\ F_2 = 2 & \beta = 70° & \gamma = 45° \end{array}\right\}$ Find the magnitude and direction of the resultant.

Solution.

F	α.	β.	γ.	$\cos\alpha$.	$\cos\beta$.	$\cos\gamma$.	$F\cos\alpha$.	$F\cos\beta$.	$F\cos\gamma$.
63	53°	42°		0.60182	0.74314	0.29250	37.91466	46.81782	18.42750
49	87°		72°	0.05234	0.94961	0.30902	2.56466	46.53089	15.14198
2		70°	45°	0.61888	0.34202	0.70711	1.23776	0.68404	1.41422
							41.71708 $\Sigma F\cos\alpha$	94.03275 $\Sigma F\cos\beta$	34.98370 $\Sigma F\cos\gamma$

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\cos\beta)^2 + (\Sigma F\cos\gamma)^2} = 108.6569.$$

$\log\Sigma F\cos\alpha$	$= 1.620314$	$\log\Sigma F\cos\beta$	$= 1.973279$	$\log\Sigma F\cos\gamma$	$= 1.543866$
$\log R$	$= 2.036057$	$\log R$	$= 2.036057$	$\log R$	$= 2.036057$
$\log\cos\alpha_r$	$= 9.584257$	$\log\cos\beta_r$	$= 9.937222$	$\log\cos\gamma_r$	$= 9.507809$
α_r	$= 67° 25' 20''$	β_r	$= 30° 4' 14''$	γ_r	$= 71° 13' 5''$

	F.	α.	β.		F.	α.	β.	γ.
2.	4.3	47° 2′	65° 7′	3.	5	90°	90°	
	87.5	88° 3′	10° 5′		7	0°		
	6.4	68° 4′	83° 2′		4		0°	
					75	73°		45°

§ 27. **Conditions of Equilibrium for Forces applied at a Single Point.**

1°. When the forces are not confined to one plane, we have already found, for the square of the resultant,

$$R^2 = (\Sigma F \cos\alpha)^2 + (\Sigma F \cos\beta)^2 + (\Sigma F \cos\gamma)^2.$$

But this expression can reduce to zero only when we have

$$\Sigma F \cos\alpha = 0, \quad \Sigma F \cos\beta = 0, \quad \text{and} \quad \Sigma F \cos\gamma = 0;$$

for the three terms, being squares, are all positive quantities, and hence their sum can reduce to zero only when they are separately equal to zero.

Hence: *If a set of balanced forces applied at a single point be resolved into components along three directions at right angles to each other, the algebraic sum of the components of the forces along each of the three directions must be equal to zero, and conversely.*

2°. When the forces are all confined to one plane, let that plane be the z plane; then $\gamma = 90°$ in each case, and

$$\begin{aligned} \therefore\ \beta &= 90° - \alpha \\ \therefore\ \cos\beta &= \sin\alpha \\ \therefore\ R^2 &= (\Sigma F \cos\alpha)^2 + (\Sigma F \sin\alpha)^2. \end{aligned}$$

Hence, for equilibrium we must have

$$(\Sigma F \cos\alpha)^2 + (\Sigma F \sin\alpha)^2 = 0;$$

and, since this is the sum of two squares,

$$\Sigma F \cos \alpha = 0, \text{ and } \Sigma F \sin \alpha = 0.$$

Hence: *If a set of balanced forces, all situated in one plane, and acting at one point, be resolved into components along two directions at right angles to each other, and in their own plane, the algebraic sum of the components along each of the two given directions must be equal to zero respectively; and conversely.*

§ 28. **Statics of Rigid Bodies.** — A rigid body is one that does not undergo any alteration of shape when subjected to the action of external forces. Strictly speaking, no body is absolutely rigid; but different bodies possess a greater or less degree of rigidity according to the material of which they are composed, and to other circumstances. When a force is applied to a rigid body, we may have as the result, not merely a rectilinear motion in the direction of the force, but, as will be shown later, this may be combined with a rotary motion; in short, the criterion by which we determine the ensuing motion is, that the effect of the force will distribute itself through the body in such a way as not to interfere with its rigidity.

What this mode of distribution is, we shall discuss hereafter; but we shall first proceed to some propositions which can be proved independently of this consideration.

§ 29. **Principle of Rectilinear Transferrence of Force in Rigid Bodies.** — If a force be applied to a rigid body at the point A (Fig. 13) in the direction AB, whatever be the motion that this force would produce, it will be prevented from taking place if an equal and opposite force be applied at A, B, C, or D, or at any point along the line of action of the force: hence we have the principle that —

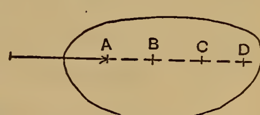


FIG. 13.

The point of application of a force acting on a rigid body, may be transferred to any other point which lies in the line of

action of the force, and also in the body, without altering the resulting motion of the body, although it does alter its state of stress.

§ 30. **Composition of two Forces in a Plane acting at Different Points of a Rigid Body, and not Parallel to Each Other.** — Suppose the force F (Fig. 14) to be applied at A, and F_1 at B, both in the plane of the paper, and acting on the rigid body *abcdef*. Produce the lines of direction of the forces till they meet at O, and suppose both F and F_1 to act at O. Construct the parallelogram $ODHE$, where $OD = F$ and $OE = F_1$; then will $OH = R$ represent the resultant force in magnitude and in direction. Its point of application may be conceived at any point along the line OH, as at C, or any other point; and a force equal and opposite to OH, applied at any point of the line OH, will balance F at A, and F_1 at B.

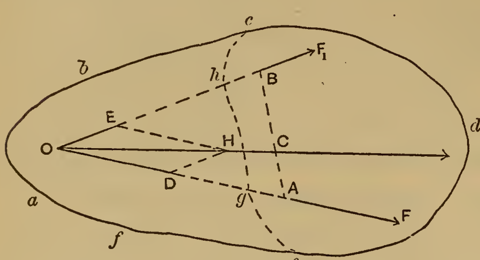


FIG. 14.

The above reasoning has assumed the points A, B, C and O, all within the body: but, since we have shown, that when this is the case, a force equal and opposite to R at C will balance F at A, and F_1 at B, it follows, that were these three forces applied, equilibrium would still subsist if we were to remove the part *bafegho* of the rigid body; or, in other words, —

The same construction holds even when the point O *falls outside the rigid body.*

§ 31. **Moment of a Force with Respect to an Axis Perpendicular to the Force.**

DEFINITION. — The moment of a force with respect to an axis perpendicular to the force, and not intersecting it, is the

product of the force by the common perpendicular to (shortest distance between) the force and the axis.

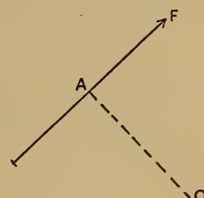


FIG. 15.

Thus, in Fig. 15 the moment of F about an axis through O and perpendicular to the plane of the paper is $F(OA)$. The sign of the moment will depend on the sign attached to the force and that attached to the perpendicular. These will be assumed in this book in such a manner as to render the following true; viz., —

The moment of a force with respect to an axis is called positive when, if the axis were supposed fixed, the force would cause the body on which it acts to rotate around the axis in the direction of the hands of a watch as seen by the observer looking at the face. It will be called negative when the rotation would take place in the opposite direction.

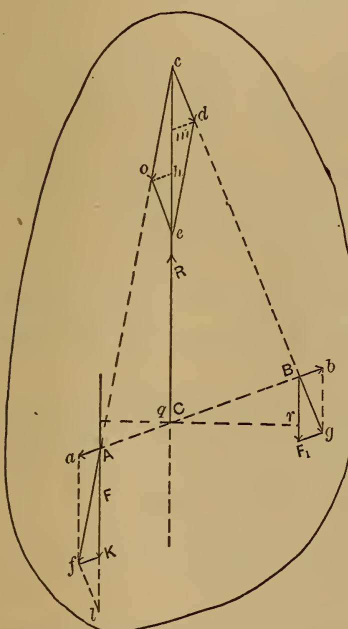


FIG. 16.

§ 32. **Equilibrium of Three Parallel Forces applied at Different Points of a Rigid Body.** — Let it be required to find a force (Fig. 16) that will balance the two forces F at A, and F_1 at B. Apply at A and B respectively, and in the line AB, the equal and opposite forces Aa and Bb. Their introduction will produce no alteration in the body's motion.

The resultant of F and Aa is Af, that of F_1 and Bb is Bg. Compound these by the method of § 30, and we obtain as resultant ce. A force equal in magnitude and opposite in direction

to *ce*, applied at any point of the line *cC*, will be the force required to balance F at A and F_1 at B; and, as is evident from the construction, this line is in the plane of the two forces. Moreover, by drawing triangle *fKl* equal to *Bbg*, we can readily prove that triangles *oce* and *Afl* are equal: hence the angle *oce* equals the angle *fAl*, and R is parallel to F and F_1. Also

$$R = ce = ch + he = AK + Kl = F + F_1,$$

$$\frac{Cc}{AC} = \frac{AK}{fK} = \frac{F}{Aa},$$

and

$$\frac{Cc}{BC} = \frac{bg}{Bb} = \frac{F_1}{Bb};$$

$\therefore$ since $Aa = Bb$,

$$\frac{AC}{BC} = \frac{F_1}{F} \therefore \frac{F}{BC} = \frac{F_1}{AC} = \frac{F + F_1}{AB} \therefore \frac{F}{Cr} = \frac{F_1}{Cq} = \frac{F + F_1}{qr},$$

where *qr* is any line passing through *C*.

Hence we have the following propositions; viz., —

If three parallel forces balance each other, —

1°. *They must lie in one plane.*

2°. *The middle one must be equal in magnitude and opposite in direction to the sum of the other two.*

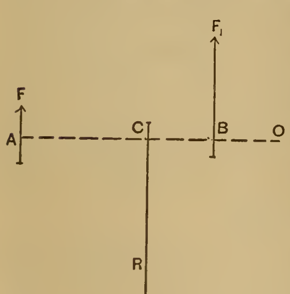


Fig. 17.

3°. *Each force is proportional to the distance between the lines of direction of the other two as measured on any line intersecting all of them.*

The third of the above-stated conditions may be otherwise expressed, thus: —

The algebraic sum of the moments of the three forces about any axis perpendicular to the forces must be zero.

PROOF. — Let F, F_1, and R (Fig. 17) be the forces; and let the axis referred to pass through O. Draw OA perpendicular to the forces. Then we have

$$\begin{aligned} F(OA) + F_1(OB) &= F(OC + CA) + F_1(OC - BC) \\ &= (F + F_1)OC + F(AC) - F_1(BC). \end{aligned}$$

But, from what we have already seen,

$$F + F_1 = -R$$

and

$$\frac{F}{BC} = \frac{F_1}{AC}$$

$$\therefore \quad F(AC) = F_1(BC)$$

$$\therefore \quad F(OA) + F_1(OB) = -R(OC) + 0$$

$$\therefore \quad F(OA) + F_1(OB) + R(OC) = 0,$$

or the algebraic sum of the moments of the forces about the axis through O is equal to zero.

§ 33. **Resultant of a Pair of Parallel Forces.** — In the preceding case, the resultant of any two of the three forces F, F_1, and R, in Fig. 16 or Fig. 17, is equal and opposite to the third force. Hence follow the two propositions: —

I. If two parallel forces act in the same direction, their resultant lies in the plane of the forces, is equal to their sum, acts in the same direction, and cuts the line joining their points of application, or any common perpendicular to the two forces, at a point which divides it internally into two segments inversely as the forces.

II. If two unequal parallel forces act in opposite directions, their resultant lies in the plane of the forces, is equal to their difference, acts in the direction of the larger force, and cuts the line joining their points of application, or any common perpendicular to them, at a point which (lying nearer the larger force)

divides it externally into two segments which are inversely as the forces.

Another mode of stating the above is as follows:—

1°. The resultant of a pair of parallel forces lies in the plane of the forces.

2°. It is equal in magnitude to their algebraic sum, and coincides in direction with the larger force.

3°. The moment of the resultant about an axis perpendicular to the plane of the forces is equal to the algebraic sum of the moments about the same axis.

EXAMPLES.

1. Find the length of each arm of a balance such that 1 ounce at the end of the long arm shall balance 1 pound at the end of the short arm, the length of beam being 2 feet, and the balance being so proportioned as to hang horizontally when unloaded.
2. Given beam = 28 inches, 3 ounces to balance 15.
3. Given beam = 36 inches, 5 ounces to balance 25 ounces.

MODE OF DETERMINING THE RESULTANT OF A PAIR OF PARALLEL FORCES REFERRED TO A SYSTEM OF THREE RECTANGULAR AXES.

Let both forces (Fig. 18) be parallel to OZ; then we have, from what has preceded,

$$\frac{F}{bc} = \frac{F_1}{ab} = \frac{F + F_1}{ac} = \frac{R}{ac}, \text{ where } R = F + F_1.$$

But from the figure

$$\frac{bc}{x_3 - x_2} = \frac{ab}{x_2 - x_1} \quad \therefore \quad \frac{F}{x_3 - x_2} = \frac{F_1}{x_2 - x_1}$$

$$\therefore \quad Fx_2 - Fx_1 = F_1x_3 - F_1x_2$$

$$\therefore \quad (F + F_1)x_2 = Fx_1 + F_1x_3,$$

or

$$Rx_2 = Fx_1 + F_1x_3;$$

and similarly we may prove that

$$Ry_2 = Fy_1 + F_1 y_3,$$

or

1°. The resultant of two parallel forces is parallel to the forces and equal to their algebraic sum.

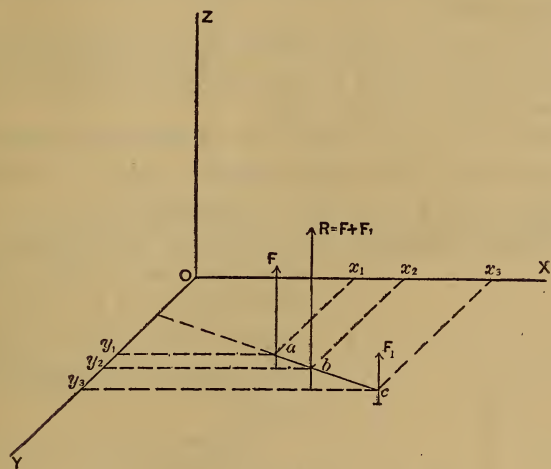


FIG. 18.

2°. The moment of the resultant with respect to OX is equal to the algebraic sum of their moments with respect to OX; and likewise when the moments are taken with respect to OY.

§ 34. **Resultant of any Number of Parallel Forces.**— Let it be required to find the resultant of any number of parallel forces.

In any such case, we might begin by compounding two of them, and then compounding the resultant of these two with a third, this new resultant with a fourth, and so on. Hence, for the magnitude of any one of these resultants, we simply add to the preceding resultant another one of the forces; and for the moment about any axis perpendicular to the forces, we add

to the moment of the preceding resultant the moment of the new force.

Hence we have the following facts in regard to the resultant of the entire system : —

1°. *The resultant will be parallel to the forces and equal to their algebraic sum.*

2°. *The moment of the resultant about any axis perpendicular to the forces will be equal to the algebraic sum of the moments of the forces about the same axis.*

The above principles enable us to determine the resultant in all cases, except when the algebraic sum of the forces is equal to zero. This case will be considered later.

§ 35. **Composition of any System of Parallel Forces when all are in One Plane.** — Refer the forces to a pair of rectangular axes, OX, OY (Fig. 19), and assume OY parallel to the forces.

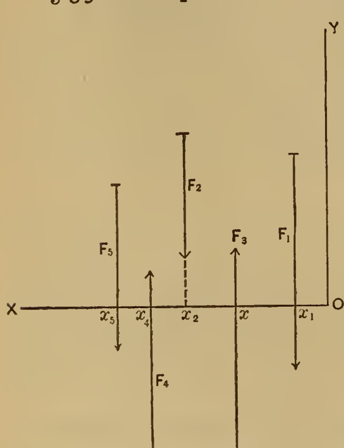


FIG. 19.

The forces and the co-ordinates of their lines of direction being as indicated in the figure, if we denote by R the resultant, and by x_0 the co-ordinate of its line of direction, we shall have, from the preceding,

$$R = \Sigma F; \qquad (1)$$

and if moments be taken about an axis through O, and perpendicular to the plane of the forces, we shall also have

$$Rx_0 = \Sigma Fx. \qquad (2)$$

Hence

$$R = \Sigma F \quad \text{and} \quad x_0 = \frac{\Sigma Fx}{\Sigma F}$$

determine the resultant in magnitude and in line of action, except when $\Sigma F = 0$, which case will be considered later.

§ 36. **Composition of any System of Parallel Forces not confined to One Plane.** — Refer the forces to a set of rectangular axes so chosen that OZ is parallel to their direction. If we denote the forces by F_1, F_2, F_3, F_4, etc., and the co-ordinates of their lines of direction by (x_1, y_1), (x_2, y_2), etc., and if we denote their resultant by R, and the co-ordinates of its line of direction by (x_0, y_0), we shall have, in accordance with what has been proved in § 34, —

1°. *The magnitude of the resultant about* OY *is equal to the algebraic sum of the forces, or*

$$R = \Sigma F.$$

2°. *The moment of the resultant about* OY *is equal to the sum of the moments of the forces about* OY, *or*

$$x_0 \Sigma F = \Sigma Fx.$$

3°. *The moment of the resultant about* OX *is equal to the sum of the moments about* OX, *or*

$$y_0 \Sigma F = \Sigma Fy.$$

Hence

$$R = \Sigma F, \quad x_0 = \frac{\Sigma Fx}{\Sigma F}, \quad y_0 = \frac{\Sigma Fy}{\Sigma F},$$

determine the resultant in all cases, except when $\Sigma F = 0$.

§ 37. **Conditions of Equilibrium of any Set of Parallel Forces.** — If the axes be assumed as before, so that OZ is parallel to the forces, we must have

$$\Sigma F = 0, \quad \Sigma Fx = 0, \quad \text{and} \quad \Sigma Fy = 0.$$

To prove this, compound all but one of the forces. Then equilibrium will subsist only when the resultant thus obtained is equal and directly opposed to the remaining force; i.e, it must be equal, and act along the same line and in the opposite direction. Hence, calling R_a the resultant above referred to, and (x_a, y_a) the co-ordinates of its line of direction, and calling F_n the

remaining force, and (x_n, y_n) the co-ordinates of its line of direction, we must have

$$
\begin{array}{llll}
 & R_a = -F_n, & x_a = x_n, & y_a = y_n, \\
\therefore & R_a + F_n = 0, & R_a x_a + F_n x_n = 0, & R_a y_a + F_n y_n = 0, \\
\therefore & \Sigma F = 0, & \Sigma Fx = 0, & \Sigma Fy = 0.
\end{array}
$$

When the forces are all in one plane, the conditions become

$$\Sigma F = 0, \quad \Sigma Fx = 0.$$

§ 38. **Centre of a System of Parallel Forces.** — The resultant of the two parallel forces F and F_1 (Fig. 20), applied at A and B respectively, is a force $R = F + F_1$, whose line of action cuts the line AB at a point C, which divides it into two segments inversely as the forces. If the forces F and F_1 are turned through the same angle, and assume the positions AO and BO_1 respectively, the line of action of the resultant will still pass through C, which is called the centre of the two parallel forces F and F_1. Inasmuch as a similar reasoning will apply in the case of any number of parallel forces, we may give the following definition: —

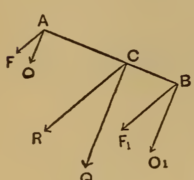


Fig. 20.

The centre of a system of parallel forces is the point through which the line of action of the resultant always passes, no matter how the forces are turned, provided only —

1°. *Their points of application remain the same.*

2°. *Their relative magnitudes are unchanged.*

3°. *They remain parallel to each other.*

Hence, in finding the centre of a set of parallel forces, we may suppose the forces turned through any angle whatever, and the centre of the set is the point through which the line of action of the resultant always passes.

§ 39. **Co-ordinates of the Centre of a Set of Parallel Forces.** — Let F_1 (Fig. 21) be one of the forces, and (x_1, y_1, z_1) the co-ordinates of its point of application. Let F_2 be another, and (x_2, y_2, z_2) co-ordinates of its point of application. Turn all the forces around till they are parallel to OZ, and find the line of direction of the resultant force when they are in this position. The co-ordinates of this line are

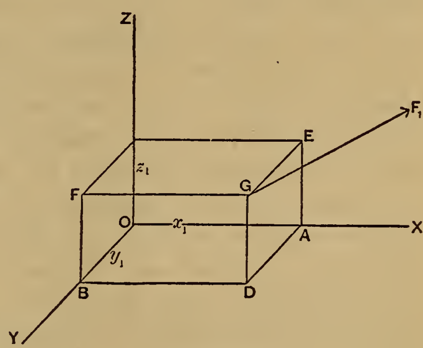


Fig. 21.

$$x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F};$$

and, since the centre of the system is a point on this line, the above are two of the co-ordinates of the centre. Then turn the forces parallel to OX, and determine the line of action of the resultant. We shall have for its co-ordinates

$$y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F}.$$

Hence, for the co-ordinates of the centre of the system, we have

$$x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F}.$$

When $\Sigma F = 0$ the co-ordinates would be ∞, therefore such a system has no centre.

§ 40. **Distributed Forces.** — While we have thus far assumed our forces as acting at single points, no force really acts at a single point, but all are distributed over a certain surface

or through a certain volume; nevertheless, the propositions already proved are all applicable to the resultants of these distributed forces. We shall proceed to discuss distributed forces only when all the elements of the distributed force are parallel to each other. As a very important example of such a distributed force, we may mention the force of gravity which is distributed through the mass of the body on which it acts. Thus, the weight of a body is the resultant of the weights of the separate parts or particles of which it is composed. As another example we have the following: if a straight rod be subjected to a direct pull in the direction of its length, and if it be conceived to be divided into two parts by a plane cross-section, the stress acting at this section is distributed over the surface of the section.

§ 41. **Intensity of a Distributed Force.** — Whenever we have a force uniformly distributed over a certain area, we obtain its *intensity* by dividing its total amount by the area over which it acts, thus obtaining the amount per unit of area.

If the force be not uniformly distributed, or if the intensity vary at different points, we must adopt the following means for finding its intensity. Assume a small area containing the point under consideration, and divide the total amount of force that acts on this small area by the area, thus obtaining the *mean intensity* over this small area: this will be an approximation to the intensity at the given point; and the intensity is the limit of the ratio obtained by making the division, as the area used becomes smaller and smaller.

Thus, also, the *intensity*, at a given point, of a force which is distributed through a certain volume, is the limit of the ratio of the force acting on a small volume containing the given point, to the volume, as the latter becomes smaller and smaller.

§ 42. **Resultant of a Distributed Force.** — 1°. Let the force be distributed over the straight line AB (Fig. 22), and

let its intensity at the point E where $AE = x$, be represented by $EF = p = \phi(x)$, a function of x; then will the force acting on the portion $Ee = \Delta x$ of the line be $p\Delta x$: and if we denote by R the magnitude of the resultant of the force acting on the entire line AB, and by x_0 the distance of its point of application from A, we shall have

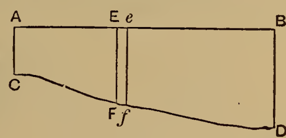


FIG. 22.

$$R = \Sigma p\Delta x \text{ approximately,}$$

or

$$R = \int pdx \text{ exactly;}$$

and, by taking moments about an axis through A perpendicular to the plane of the force, we shall have

$$x_0 R = \Sigma x(p\Delta x) \text{ approximately,}$$

or

$$x_0 R = \int pxdx \text{ exactly;}$$

whence we have the equations

$$R = \int pdx, \qquad x_0 = \frac{\int pxdx}{\int pdx}.$$

2°. Let the force be distributed over a plane area $EFGH$ (Fig. 23), let this area be referred to a pair of rectangular axes OX and OY, in its own plane, and let the intensity of the force per unit of area at the point P, whose co-ordinates are x and y, be $p = \phi(x, y)$; then will $p\Delta x\Delta y$ be approximately the force acting on the small rectangular area $\Delta x\Delta y$. Then, if we represent by R the magnitude of the resultant of the distributed force, and by x_0, y_0, the co-ordi-

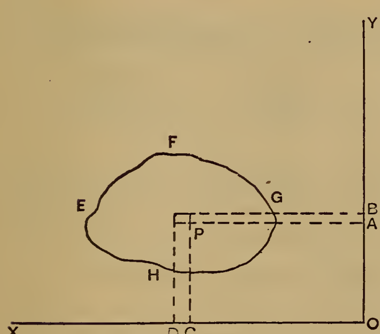


FIG. 23.

nates of the point at which the line of action of the resultant cuts the plane of $EFGH$, we shall have

$$R = \Sigma p\Delta x\Delta y \text{ approximately,}$$
$$x_0 R = \Sigma x(p\Delta x\Delta y) \quad \text{"}$$
$$y_0 R = \Sigma y(p\Delta x\Delta y) \quad \text{"}$$

or, as exact equations, we shall have

$$R = \int\int pdxdy,$$

$$x_0 = \frac{\int\int pxdxdy}{\int\int pdxdy}, \qquad y_0 = \frac{\int\int pydxdy}{\int\int pdxdy}.$$

3°. Let the force be distributed through a volume, let this volume be referred to a system of rectangular axes, OX, OY, and OZ, let ΔV represent the elementary volume, whose co-ordinates are x, y, z, and let $p = \phi(x, y, z)$ be the intensity of the force per unit of volume at the point (x, y, z); then, if we represent by R the magnitude of the resultant, and by x_0, y_0, z_0, the co-ordinates of the centre of the distributed force, we shall have, from the principles explained in § 38 and § 39, the approximate equations

$$R = \Sigma p\Delta V, \; x_0 R = \Sigma x(p\Delta V), \; y_0 R = \Sigma y(p\Delta V), \; z_0 R = \Sigma z(p\Delta V);$$

and these give, on passing to the limit, the exact equations

$$R = \int pdV, \quad x_0 = \frac{\int pxdV}{\int pdV}, \quad y_0 = \frac{\int pydV}{\int pdV}, \quad z_0 = \frac{\int pzdV}{\int pdV}.$$

§ 43. **Centre of Gravity.** — The weight of a body, or system of bodies, is the resultant of the weight of the separate parts or particles into which it may be conceived to be divided; and the *centre of gravity* of the body, or system of bodies, is the centre of the above-stated system of parallel forces, i.e., the point through which the resultant always passes, no matter how the forces are turned. The weight of any one particle is the force which gravity exerts on that particle: hence, if we repre-

sent the weight per unit of volume of a body, whether it be the same for all parts or not, by w, we shall have, as an approximation,

$$W = \Sigma w \Delta V, \quad x_0 = \frac{\Sigma wx\Delta V}{\Sigma w\Delta V}, \quad y_0 = \frac{\Sigma wy\Delta V}{\Sigma w\Delta V}, \quad z_0 = \frac{\Sigma wz\Delta V}{\Sigma w\Delta V};$$

and as exact equations,

$$W = \int wdV, \quad x_0 = \frac{\int wxdV}{\int wdV}, \quad y_0 = \frac{\int wydV}{\int wdV}, \quad z_0 = \frac{\int wzdV}{\int wdV}, \qquad (1)$$

where W denotes the entire weight of the body, and x_0, y_0, z_0, the co-ordinates of its centre of gravity.

If, on the other hand, we let $M =$ entire mass of the body, $dM =$ mass of volume dV, and $m =$ mass of unit of volume, we shall have

$$W = Mg, \quad w = mg, \quad wdV = mgdV = gdM.$$

Hence the above equations reduce to

$$M = \int dM, \quad x_0 = \frac{\int xdM}{\int dM}, \quad y_0 = \frac{\int ydM}{\int dM}, \quad z_0 = \frac{\int zdM}{\int dM}. \qquad (2)$$

Equations (1) and (2) are both suitable for determining the centre of gravity; one of the sets being sometimes most convenient, and sometimes the other.

§ 44. **Centre of Gravity of Homogeneous Bodies.** — If the body whose centre of gravity we are seeking is homogeneous, or of the same weight per unit of volume throughout, we shall have, that $w =$ a constant in equations (1); and hence these reduce to

$$W = w\int dV, \quad x_0 = \frac{\int xdV}{\int dV}, \quad y_0 = \frac{\int ydV}{\int dV}, \quad z_0 = \frac{\int zdV}{\int dV}.$$

§ 45. **Effect of a Single Force applied at the Centre of a Straight Rod of Uniform Section and Material.** — If a straight rod of uniform section and material have imparted to it

a motion, such that the velocity imparted in a unit of time to each particle of the rod is the same, and if we represent this velocity by f, then if at each point of the rod, we lay off a line xy (Fig. 24) in the direction of the motion, and representing the velocity imparted to that point, the line bounding the other ends of the lines xy will be straight, and parallel to the rod. If we conceive the rod to be divided into any number of small equal parts, and denote the mass of one of these parts by ΔM, then will $f\Delta M$ contain as many units of momentum as there are units of force in the force required to impart to this particle the velocity f in a unit of time; and hence $f\Delta M$ is the measure of this force.

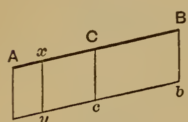


FIG. 24.

Hence the resultant of the forces which impart the velocity f to every particle of the rod will have for its measure

$$fM,$$

where M is the entire mass of the rod; and its point of application will evidently be at the middle of the rod.

It therefore follows that —

The effect of a single force applied at the middle of a straight rod of uniform section and material is to impart to the rod a motion of translation in the direction of the force, all points of the rod acquiring equal velocities in equal times.

§ 46. **Translation and Rotation combined.** — Suppose that we have a straight rod AB (Fig. 25), and suppose that such a force or such forces are applied to it as will impart to the point A in a unit of time the velocity Aa, and to the point B the (different) velocity Bb in a unit of time, both being perpendicular to the length of the rod. It is required to determine the motion of any other point of the rod and that of the entire rod.

Lay off Aa and Bb (Fig. 25), and draw the line ab, and produce it till it meets AB produced in O: then, when these velocities Aa and Bb are imparted to the points A and B, the rod is in the act of rotating around an axis through O perpendicular to the plane of the paper; for when a body is rotating around an axis, the linear velocity of any point of the body is perpendicular to the line joining the point in question with the axis (i.e., the perpendicular dropped from the point in question upon the axis), and proportional to the distance of the point from the axis.

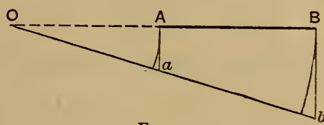


FIG. 25.

Hence: *If the velocities of two of the points in the rod are given, and if these are perpendicular to the rod, the motion of the rod is fixed, and consists of a rotation about some axis at right angles to the rod.*

Another way of considering this motion is as follows: Suppose, as before, the velocities of the points A and B to be represented by Aa and Bb respectively, and hence the velocity of any other point, as x (Fig. 26), to be represented by xy, or the length of the line drawn perpendicular to AB, and limited by AB and ab.

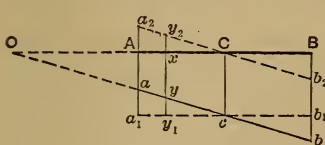


FIG. 26.

Then, if we lay off $Aa_1 = Bb_1 = \frac{1}{2}(Aa + Bb) = Cc$, and draw a_1b_1, and if we also lay off $Aa_2 = a_1a$, and $Bb_2 = b_1b$, we shall have the following relations; viz., —

$$\begin{aligned} Aa &= Aa_1 - Aa_2, \\ Bb &= Bb_1 + Bb_2, \\ xy &= xy_1 - xy_2, \quad \text{etc.,} \end{aligned}$$

or we may say that the actual motion imparted to the rod in a unit of time may be considered to consist of the following two parts: —

1°. A velocity of translation represented by Aa_1, the mean velocity of the rod; all points moving with this velocity.

2°. A varying velocity, different for every different point, and such that its amount is proportional to its distance from C, the centre of the rod, as graphically shown in the triangles Aa_2CBb_2. In other words, the rod has imparted to it two motions:—

1°. A translation with the mean velocity of the rod.

2°. A rotation of the rod about its centre.

§ 47. **Effect of a Force applied to a Straight Rod of Uniform Section and Material, not at its Centre.**—If the force be not at right angles to the rod, resolve it into two components, one acting along the rod, and the other at right angles to it. The first component evidently produces merely a translation of the rod in the direction of its length: hence the second component is the only one whose effect we need to study.

To do this we shall proceed to show, that, when such a rod has imparted to it the motion described in § 46, the single resultant force which is required to impart this motion in a unit of time is a force acting at right angles to the rod, at a point different from its centre; and we shall determine the relation between the force and the motion imparted, so that one may be deduced from the other.

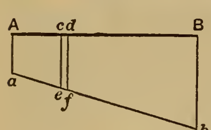


FIG. 27.

Let A be the origin (Fig. 27), and let

$Ac \;= x$, $cd = dx$.

$AB = l =$ length of the rod.

$ce \;= f =$ velocity imparted per unit of time at distance x from A.

$Aa \;= f_1$, $Bb = f_2$.

$w \;=$ weight per unit of length.

$m \;=$ mass per unit of length $= \frac{w}{g}$.

W = entire weight of rod.

M = entire mass of rod $= \dfrac{W}{g}$.

R = single resultant force acting for a unit of time to produce the motion.

x_0 = distance from A to point of application of R.

Then we shall have,

$$f = f_1 + \frac{f_2 - f_1}{l}x.$$

Hence, from § 42,

$$R = m\int_0^l fdx = m(\text{area } AabB) = \frac{m}{2}(f_1 + f_2)l = \frac{M}{2}(f_1 + f_2). \quad (1)$$

$$x_0R = m\int_0^l fxdx = \frac{m}{6}(f_1 + 2f_2)l^2 = \frac{M}{6}(f_1 + 2f_2)l. \quad (2)$$

$$\therefore \quad x_0 = \frac{1}{3}\frac{f_1 + 2f_2}{f_1 + f_2}l. \quad (3)$$

We thus have a force R, perpendicular to AB, whose magnitude is given by equation (1), and whose point of application is given by equation (3); the respective velocities imparted by the force being shown graphically in Fig. 27.

EXAMPLES.

1. Given Weight of rod = W = 100 lbs.,
   Length of rod = 3 feet,
   Assume g = 32 feet per second,
   Force applied = R = 5 lbs.,
   Point of application to be 2.5 feet from one end;

determine the motion imparted to the rod by the action of the force for one second.

Solution.

Equation (1) gives us,

$$5 = \left(\frac{100}{32}\right)\frac{f_1 + f_2}{2}, \text{ or } f_1 + f_2 = 3.2.$$

Equation (2) gives,

$$(2.5)(5) = \left(\frac{100}{32}\right)\left(\frac{1}{6}\right)(3)(f_1 + 2f_2), \text{ or } f_1 + 2f_2 = 8$$

$$\therefore \; f_2 = 4.8, \qquad f_1 = -1.6.$$

Hence the rod at the end nearest the force acquires a velocity of 4.8 feet per second, and at the other end a velocity of -1.6 feet per second. The mean velocity is, therefore, 1.6 feet per second; and we may consider the rod as having a motion of translation in the direction of the force with a velocity of 1.6 feet per second, and a rotation about its centre with such a speed that the extreme end (i.e., a point $\frac{3}{2}$ feet from the centre) moves at a velocity $4.8 - 1.6 = 3.2$ feet per second. Hence angular velocity $= \frac{3.2}{1.5} = 2.14$ per second $= 122°.6$ per second.

2. Given $W = 50$ lbs., $l = 5$ feet. It is desired to impart to it, in one second, a velocity of translation at right angles to its length, of 5 feet per second, together with a rotation of 4 turns per second: find the force required, and its point of application.

3. Assume in example 2 that the velocity of translation is in a direction inclined 45° to the length of the rod, instead of 90°. Solve the problem.

4. Given a force of 3 lbs. acting for one-half a second at a distance of 4 feet from one end of the rod, and inclined at 30° to the rod: determine its motion.

5. Given the same conditions as in example 4, and also a force of 4 lbs., parallel and opposite in direction to the 3-lb. force, and acting also for one-half a second, and applied at 3 feet from the other end: determine the resulting motion.

6. Given two equal and opposite parallel forces, each acting at right angles to the length of the rod, and each equal to 4 lbs., one being applied at 1 foot from one end, and the other at the middle of the rod; find the motion imparted to the rod through the joint action of these forces for one-third of a second.

§ 48. **Moment of the Forces causing Rotation.** — Referring again to Fig. 26, and considering the motion of the rod as a combination of translation and rotation, if we take moments about the centre C, and compare the total moment of the forces causing the rotation alone, whose accelerations are represented by the triangles aa_1cb_1b, with the total moment of the actual forces acting, whose accelerations are represented by the trapezoid $AabB$, we shall find these moments equal to each other; for, as far as the forces represented by the rectangle are concerned, every elementary force $m(xy_1)dx$ on one side of the centre C has its moment $(Cx)\{m(xy_1)dx\}$ equal and opposite to that of the elementary force at the same distance on the other side of C: hence the total moment of the forces represented graphically by the rectangle Aa_1b_1B is zero, and hence —

The moment about C *of those represented by the trapezoid equals the moment of those represented by the triangles.*

Hence, from the preceding, and from what has been previously proved, we may draw the following conclusions: —

1°. If a force be applied at the centre of the rod, it will impart the same velocity to each particle.

2°. If a force be applied at a point different from the centre, and act at right angles to its length, it will cause a translation of the rod, together with a rotation about the centre of the rod.

3°. In this latter case, the moment of the forces imparting the rotation alone is equal to the moment of the single resultant force about the centre of the rod, and the velocity of translation imparted in a unit of time is equal to the number of units of force in the force, divided by the entire mass of the rod.

§ 49. **Effect of a Pair of Equal and Opposite Parallel Forces applied to a Straight Rod of Uniform Section and Material.** — Suppose the rod to be AB (Fig. 28), and let the two equal and opposite parallel forces be Dd and Ee, each equal to F, applied at D and E respectively.

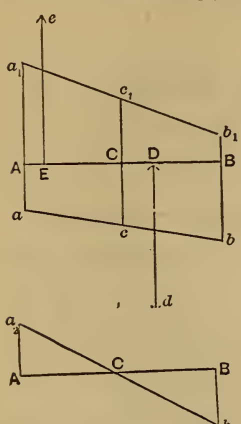


Fig. 28.

The mean velocity imparted in a unit of time by either force will be $\frac{F}{M}$; and, from what we have already seen, the trapezoid $AabB$ will furnish us the means of representing the actual velocity imparted to any point of the rod by the force Dd. The relative magnitudes of Aa and Bb, the accelerations at the ends, will depend, of course, on the position of D; but we shall always have $Cc = \frac{1}{2}(Aa + Bb) = \frac{F}{M}$, a quantity depending only on the *magnitude* of the force. So, likewise, the trapezoid Aa_1b_1B will represent the velocities imparted by the force Ee; and while the relative magnitude of Aa_1 and Bb_1 will depend upon the position of E, we shall always have $Cc_1 = \frac{1}{2}(Aa_1 + Bb_1) = \frac{F}{M}$. Hence, since $Cc = Cc_1$, the centre C of the rod has no motion imparted to it by the given pair of forces, hence the motion of the rod is one of rotation about its centre C.

The resulting velocity of any point of the rod will be the difference between the velocities imparted by the two forces; and if these be laid off to scale, we shall have the second figure. Hence —

A pair of equal and opposite parallel forces, applied to a straight rod of uniform section and material, produce a rotation of the rod about its centre. Also, —

Such a rotation about the centre of the rod cannot be pro-

duced by a single force, but requires a pair of equal and opposite parallel forces.

§ 50. **Statical Couple.** — A pair of equal and opposite parallel forces is called a statical couple.

§ 51. **Effect of a Single Force applied at the Centre of Gravity of a Straight Rod of Non-Uniform Section and Material.** — In the case of a straight rod of non-uniform section and material, we may consider the rod as composed of a set of particles of unequal mass: and if we imagine each particle to have imparted to it the same velocity in a unit of time, then, using the same method of graphical representation as before (Fig. 24), the line ab, bounding the other ends of the lines representing velocities, will be parallel to AB; but if we were to represent by the lines xy, not the velocities imparted, but the forces per unit of length, the line bounding the other ends of these forces would not, in this case, be parallel to AB. Moreover, since these forces are proportional to the masses, and hence to the weights of the several particles, their resultant would act at the centre of gravity of the rod. Hence —

A force applied at the centre of gravity of a straight rod will impart the same velocity to each point of the rod; i.e., will impart to it a motion of translation only.

§ 52. **Effect of a Statical Couple on a Straight Rod of Non-Uniform Section and Material.** — Let such a rod have imparted to it only a motion of rotation about its centre of gravity, and let us adopt the same modes of graphical representation as before.

Let the origin be taken at O (Fig. 29), the centre of gravity of the rod.

Let $Aa = f_1 =$ velocity imparted to A.
$Bb = f_2 =$ velocity imparted to B.
$OA = a,\ OB = b,\ OC = x$.
$CD = f =$ velocity imparted to C.
$dM =$ elementary mass at C.

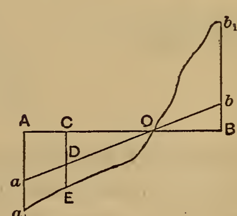


FIG. 29.

Then, from similar triangles, we have

$$f = \frac{f_1}{a}x = \frac{f_2}{b}x,$$

and hence for the force acting on dM we have

$$dF = (CE)dx = \frac{f_1}{a}xdM.$$

Hence the whole force acting on AO, and represented graphically by Aa_1O, is

$$\frac{f_1}{a}\int_{x=0}^{x=a} xdM,$$

and that acting on OB, and represented by BOb_1, is

$$\frac{f_2}{b}\int_{x=-b}^{x=0} xdM = \frac{f_1}{a}\int_{x=-b}^{x=0} xdM.$$

Hence for the resultant, or the algebraic sum, of the two, we have

$$R = \frac{f_1}{a}\int_{x=-b}^{x=a} xdM.$$

But from § 43 we have for the co-ordinate x_0 of the centre of gravity of the rod

$$x_0 = \frac{\int_{x=-b}^{x=a} xdM}{M};$$

but, since the origin is at the centre of gravity, we have

$$x_0 = 0,$$

and hence

$$\int_{x=-b}^{x=a} xdM = 0 \qquad \therefore \quad R = 0.$$

Hence the two forces represented by Aa_1O and Bb_1O are equal in magnitude and opposite in direction: hence the rotation about the centre of gravity is produced by a *Statical Couple.*

Now, a train of reasoning similar to that adopted in the case of a rod of uniform section and material will show that a single force applied at some point which is not the centre of gravity of the rod will produce a motion which consists of two parts; viz., a motion of translation, where all points of the rod have equal velocities, and a motion of rotation around the centre of gravity of the rod.

§ 53. **Moment of a Couple.** — The *moment of a statical couple* is the product of either force by the perpendicular distance between the two forces, this perpendicular distance being called the arm of the couple.

§ 54. **Measure of the Rotatory Effect.** — Before proceeding to examine the effect of a statical couple upon any rigid body whatever, we will seek a means of measuring its effect in the cases already considered.

The measure adopted is the moment of the couple; and, in order to show that it is proper to adopt this measure, it will be necessary to show —

That the moment of the couple is proportional to the angular velocity imparted to the same rod in a unit of time; and from this it will follow —

That two couples in the same plane with equal moments will balance each other if one is right-handed and the other left-handed.

If we assume the origin of co-ordinates at C (Fig. 30), the centre of gravity of the rod, and if we denote by α the angular velocity imparted in a unit of time by the forces F and $-F$, and let $CD = x_1$, $CE = x_2$, then we have for the linear velocity of a particle situated at a distance x from C the value

$$\alpha x.$$

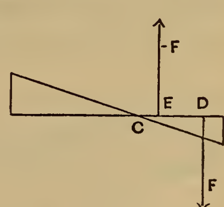


Fig. 30.

The force which will impart this velocity in a unit of time to the mass dM is

$$\alpha x dM.$$

The total resultant force is

$$\alpha \int x dM,$$

which, as we have seen, is equal to zero. The moment of the elementary force about C is

$$x(\alpha x dM) = \alpha x^2 dM,$$

and the sum of the moments for the whole rod is

$$\alpha \int x^2 dM,$$

and this, as is evident if we take moments about C, is equal to

$$Fx_1 - Fx_2 = F(x_1 - x_2) = F(DE).$$

Now, $\int x^2 dM$ is a constant for the same rod: hence any quantity proportional to $F(DE)$ is also proportional to α.

The above proves the proposition.

Moreover, we have

$$F(DE) = \alpha \int x^2 dM$$

$$\therefore \alpha = \frac{F(DE)}{\int x^2 dM},$$

whence it follows, that when the moment of the couple is given, and also the rod, we can find the angular velocity imparted in a unit of time by dividing the former by $\int x^2 dM$.

§ 55. **Effect of a Couple on a Straight Rod when the Forces are inclined to the Rod.** — We shall next show that the effect of such a couple is the same as that of a couple of equal moment whose forces are perpendicular to the rod.

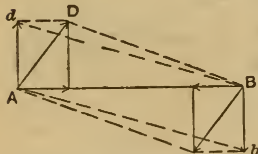


Fig. 31.

In this case let AD and BC be the forces (Fig. 31). The moment of this couple is the product of AD by the perpendicular distance between AD and BC, the graphical representation of this being the area of the parallelogram $ADBC$.

Resolve the two forces into components along and at right angles to the rod. The former have no effect upon the motion of the rod: the latter are the only ones that have any effect upon its motion. The moment of the couple which they form is the product of Ad by AB, graphically represented by parallelogram $AdBb$; and we can readily show that

$$ADBC = AdBb.$$

Hence follows the proposition.

§ 56. **Effect of a Statical Couple on any Rigid Body.** — Refer the body (Fig. 32) to three rectangular axes, OX, OY, and OZ, assuming the origin at the centre of gravity of the body, and OZ as the axis about which the body is rotating. Let the mass of the particle P be ΔM, and its co-ordinates be x, y, z.

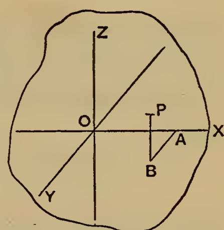


FIG. 32.

Then will the force that would impart to the mass ΔM the angular velocity α in a unit of time be

$$\alpha r \Delta M,$$

where $r =$ perpendicular from P on OZ, or

$$r = \sqrt{x^2 + y^2}.$$

This force may be resolved into two, one parallel to OY and the other to OX; the first component being $\alpha x \Delta M$, and the second $\alpha y \Delta M$.

Proceeding in the same way with each particle, and finding the resultant of each of these two sets of parallel forces, we shall obtain, finally, a single force parallel to OY and equal to

$$\alpha \Sigma x \Delta M,$$

and another parallel to OX, equal to

$$\alpha \Sigma y \Delta M.$$

But, since OZ passes through the centre of gravity of the body, we shall have

$$\Sigma x \Delta M = 0 \quad \text{and} \quad \Sigma y \Delta M = 0.$$

Hence the resultant is in each case, not a single force, but a *statical couple*; the moment of the first couple being

$$\alpha \Sigma x^2 \Delta M,$$

and that of the second

$$\alpha \Sigma y^2 \Delta M.$$

These couples produce the same effect in whatever plane perpendicular to OZ they are situated. Hence, suppose them both in the plane XOY, then representing them as in (Fig. 33), we make

$$F_1(AB) = \alpha \Sigma x^2 \Delta M \quad \text{and} \quad F_2(CD) = \alpha \Sigma y^2 \Delta M.$$

Now, compound the force at D with that at B, and the force at C with that at A, and we obtain as a result two equal and opposite parallel forces, or *one* statical couple, whose moment will be shown to be equal to

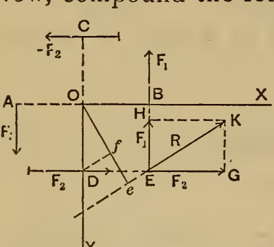


FIG. 33.

$$\alpha(\Sigma x^2 \Delta M + \Sigma y^2 \Delta M) = \alpha \Sigma (x^2 + y^2) \Delta M = \alpha \Sigma r^2 \Delta M.$$

To show this, let

$OB = p_1 =$ one-half the arm of $\alpha \Sigma x^2 \Delta M$,
$OD = p_2 =$ one-half the arm of $\alpha \Sigma y^2 \Delta M$,
$Oe = p =$ one-half the arm of the resultant couple.

Let angle $DOe = \theta = GEK$: we shall then have

$$p = Oe = Of + fe = p_2 \cos\theta + p_1 \sin\theta;$$

but
$$\cos\theta = \frac{F_2}{R}, \quad \sin\theta = \frac{F_1}{R},$$
$$\therefore \quad p \quad = \frac{p_2F_2}{R} + \frac{p_1F_1}{R}$$
$$\therefore \quad Rp \quad = F_1p_1 + F_2p_2.$$

Hence the moment of the resultant couple is equal to the sum of the moments of the separate couples, or

$$R(2p) = \alpha\Sigma r^2\Delta M = \alpha\Sigma x^2\Delta M + \alpha\Sigma y^2\Delta M.$$

Hence: *To impart to a body a rotation about an axis passing through its centre of gravity requires the action of a statical couple, and conversely a statical couple so applied will cause such a rotation as that described.*

Hence we may generalize all our propositions in regard to the effect of statical couples and we may conclude that —

In order that two couples may have the same effect, it is necessary —

1°. *That they be in the same or parallel planes.*

2°. *That they have the same moment.*

3°. *That they tend to cause rotation in the same direction* (*i.e., both right-handed or both left-handed when looked at from the same side*).

It also follows, that, for a given statical couple, we may substitute another having the magnitudes of its forces different, provided only the moment of the couple remains the same.

§ 57. **Composition of Couples in the Same or Parallel Planes.** — If the forces of the couples are not the same, reduce them to equivalent couples having the same force, transfer them to the same plane, and turn them so that their arms shall lie in the same straight line, as in Fig. 34; the first couple consisting of the force F at A and $-F$ at B, and the second of F at B and $-F$ at C.

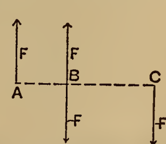


Fig. 34.

The two equal and opposite forces counterbalance each other, and we have left a couple with force F and arm

$$AC = AB + BC$$

$$\therefore \quad \text{Resultant moment} = F.AC = F(AB) + F(BC).$$

Hence: *The moment of the couple which is the resultant of two or more couples in the same or parallel planes is equal to the algebraic sum of the moments of the component couples.*

EXAMPLES.

1. Convert a couple whose force is 5 and arm 6 to an equivalent couple whose arm is 3. Find the resultant of this and another couple in the same plane and sense whose force is 7 and arm 8; also find the force of the resultant couple when the arm is taken as 5.

Solution.

Moment of first couple	$= 5 \times 6$	$= 30$
When arm is 3, force	$= \frac{30}{3}$	$= 10$
Moment of second couple	$= 7 \times 8$	$= 56$
Moment of resultant couple	$= 30 + 56$	$= 86$
When arm is 5, force	$= \frac{86}{5}$	$= 17\frac{1}{5}$

2. Given the following couples in one plane: —

Force.	Arm.		Force.	Arm.
12	17		5	
3	8			6
5	7	Convert to equivalent	8	
6	9	couples having the	6	
12	12	following: —		7
10	9		4	
14	6			20

The first and the last three are right-handed; the second, third, and fourth are left-handed. Find the moment of the resultant couple, and also its force when it has an arm 11.

§ 58. **Representation of a Couple by a Line.** — From the preceding we see that the effect of a couple remains the same as long as —

1°. *Its moment does not change.*

2°. *The direction of its axis (i.e., of the line drawn perpendicular to the plane of the couple) does not change.*

3°. *The direction in which it tends to make the body turn (right-handed or left-handed) remains the same.*

Hence a couple may be represented by drawing a line in the direction of its axis (perpendicular to its plane), and laying off on this line a distance containing as many units of length as there are units of moment in the couple, and indicating by a dot, an arrow-head, or some other means, in what direction one must look along the line in order that the rotation may appear right-handed.

This line is called the *Moment Axis* of the couple.

§ 59. **Composition of Couples situated in Planes inclined to Each Other.** — Suppose we have two couples situated neither in the same plane nor in parallel planes, and that we wish to find their resultant couple. We may proceed as follows: Substitute for them equivalent couples with equal arms, then transfer them in their own plane respectively to such positions that their arms shall coincide, and lie in the line of intersection of the two planes.

This having been done, let OO_1 (Fig. 35) be the common arm, F and $-F$ the forces of one couple, F_1 and $-F_1$ those of the other. The forces F and F_1 have for their resultant R, and $-F$ and $-F_1$ have $-R_1$. Moreover, we may readily show that R and $-R_1$ are equal and

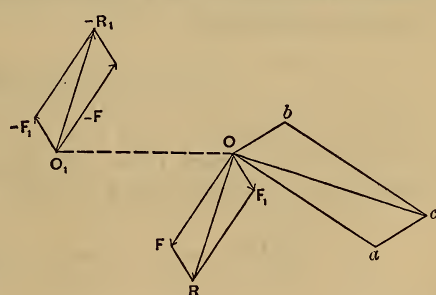


FIG. 35.

parallel, both being perpendicular to OO_1. The resultant of the two couples is, therefore, a couple whose arm is OO_1 and force R, the diagonal of the parallelogram on F and F_1, so that

$$R = \sqrt{F^2 + F_1^2 + 2FF_1 \cos\theta},$$

where θ is the angle between the planes of the couples. Now, if we draw from O the line Oa perpendicular to OO_1 and to F, and hence perpendicular to the plane of the first couple, and if we draw in the same manner Ob perpendicular to the plane of the second couple, so that there shall be in Oa as many units of length as there are units of moment in the first couple, and in Ob as many units of length as there are units of moment in the second couple, we shall have —

1°. The lines Oa and Ob are the moment axes of the two given couples respectively.

2°. The lines Oa and Ob lie in the same plane with F and F_1, this plane being perpendicular to OO_1.

3°. We have the proportion

$$Oa : Ob = F \,.\, OO_1 : F_1 \,.\, OO_1 = F : F_1.$$

4°. If on Oa and Ob as sides we construct a parallelogram, it will be similar to the parallelogram on F and F_1. We shall have the proportion

$$Oc : R = Oa : F = Ob : F_1 ;$$

and since the sides of the two parallelograms are respectively perpendicular to each other, the diagonals are perpendicular to each other; and since we have also

$$Oc = \frac{R \,.\, Oa}{F} \quad \text{and} \quad Oa = F \,.\, OO_1 \quad \therefore \quad Oc = R \,.\, OO_1,$$

it follows that Oc is perpendicular to the plane of the resultant couple, and contains as many units of length as there are units of moment in the moment of the resultant couple; in other

words, Oc will represent the *moment axis* of the resultant couple, and we shall have

$$Oc = \sqrt{Oa^2 + Ob^2 + 2\,Oa\,.\,Ob\cos aOb};$$

or, if we let

$$Oa = L, \quad Ob = M, \quad Oc = G, \quad aOb = \theta,$$

$$G = \sqrt{L^2 + M^2 + 2LM\cos\theta}.$$

This determines the moment of the resultant couple; and, for the direction of its moment axis, we have

$$\sin aOc = \frac{M}{G}\sin\theta$$

and

$$\sin bOc = \frac{L}{G}\sin\theta.$$

Hence we can compound and resolve couples just as we do forces, provided we represent the couples by their *moment axes*

EXAMPLES.

1. Given $L = 43$, $M = 15$, $\theta = 65°$; find resultant couple.
2. Given $L = 40$, $M = 30$, $\theta = 30°$; find resultant couple.
3. Given $L = 1$, $M = 5$, $\theta = 45°$; find resultant couple.

§ 60. **Resultant of a Couple and a Single Force in the Same Plane.** — Let M (Fig. 36 or 37) be the moment of the

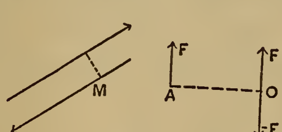


FIG. 36.

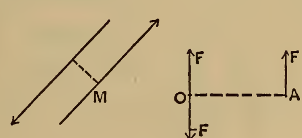


FIG. 37.

given couple, and let $OF = F$ be the single force. For the given couple substitute an equivalent couple, one of whose forces is $-F$ at O, equal and directly opposed to the single

force F, these two counterbalancing each other, and leaving only the other force of the couple, which is equal and parallel to the original single force F, and acts along a line whose distance from O is $OA = \frac{M}{F}$. Hence —

The resultant of a single force and a couple in the same plane is a force equal and parallel to the original force, having its line of direction at a perpendicular distance from the original force equal to the moment of the couple divided by the force.

Fig. 36 shows the case when the couple is right-handed, and Fig. 37 when it is left-handed.

§ 61. **Composition of Parallel Forces in General.** — In each case of composition of parallel forces (§§ 34, 35, and 36) it was stated that the method pursued was applicable to all cases except those where

$$\Sigma F = 0.$$

We were obliged, at that time, to reserve this case, because we had not studied the action of a statical couple; but now we will adopt a method for the composition of parallel forces which will apply in all cases.

(*a*) *When all the forces are in one plane.* Assume, as we did in § 35, the axis OY to be parallel to the forces; assume the forces and the co-ordinates of their lines of direction, as shown in the figure (Fig. 38). Now place at the origin O, along OY, two equal and opposite forces, each equal to F_1; then these three forces, viz., F_1 at D, OA, and OB, produce the same effect as F_1 at D alone; but F_1 at D and OB form a couple (left-handed in the figure) whose moment is $-F_1x_1$. Hence the force F_1 is equivalent to —

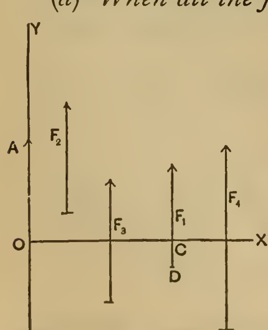


Fig. 38.

1°. An equal and parallel force at the origin, and

2°. A statical couple whose moment is $-F_1x_1$.

Likewise the force F_2 is equivalent to (1°) an equal and parallel force at the origin, and (2°) a couple whose moment is $-F_2x_2$, etc.

Hence we shall have, if we proceed in the same way with all the forces, for resultant of the entire system a single force

$$R = \Sigma F \text{ along } OY,$$

and a single resultant couple

$$M = -\Sigma Fx.$$

(Observe that downward forces and left-handed couples are to be accounted negative.)

Now, there may arise two cases.

1°. When $\Sigma F = 0$, and

2°. When $\Sigma F >< 0$.

CASE I. When $\Sigma F = 0$, the resultant force along OY vanishes, and the resultant of the entire system is a statical couple whose moment is

$$M = -\Sigma Fx.$$

CASE II. When $\Sigma F >< 0$, we can reduce the resultant to a single force.

Let (Fig. 39) OB represent the resultant force along OY, $R = \Sigma F$. With this, compound the couple whose moment is $M = -\Sigma Fx$, and we obtain as resultant (§ 60) a single force

$$R = \Sigma F,$$

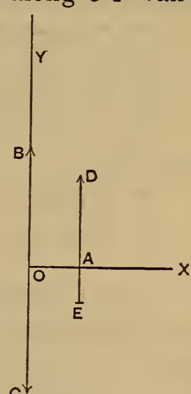


FIG. 39.

whose line of action is at a perpendicular distance from OY equal to

$$AO = x_r = \frac{\Sigma Fx}{\Sigma F}.$$

(*b*) *When the forces are not confined to one plane.* Assume, as before (Fig. 40), OZ parallel to the forces, and let F acting through A be one of the given forces, the co-ordinates of A being x and y. Place at O two equal and opposite forces, each equal to F, and also at B two equal and opposite forces, each equal to F. These five forces produce the same effect as F alone at A, and they may be considered to consist of —

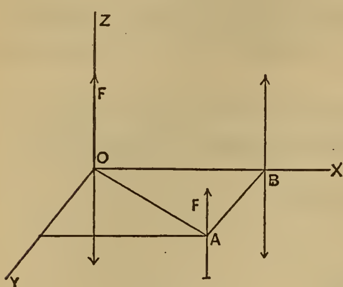


FIG. 40.

1°. A single force F at the origin.

2°. A couple whose forces are F at B and $-F$ at O, and whose moment is $-Fx$ acting in the y plane.

3°. A couple whose forces are F at A and $-F$ at B, and whose moment is Fy acting in the x plane. Treating each of the forces in the same way, we shall have, in place of the entire system of parallel forces, the following forces and couples: —

1°. A single force $R = \Sigma F$ along OZ.

2°. A couple $M_y = -\Sigma Fx$ in the y plane.

3°. A couple $M_x = +\Sigma Fy$ in the x plane.

Now, there may be two cases: —

1°. When $\Sigma F >< 0$.

2°. When $\Sigma F = 0$.

CASE I. When $\Sigma F >< 0$, we can reduce to a single resultant force having a fixed line of direction. Lay off (Fig. 41) along OZ, $OH = \Sigma F$.

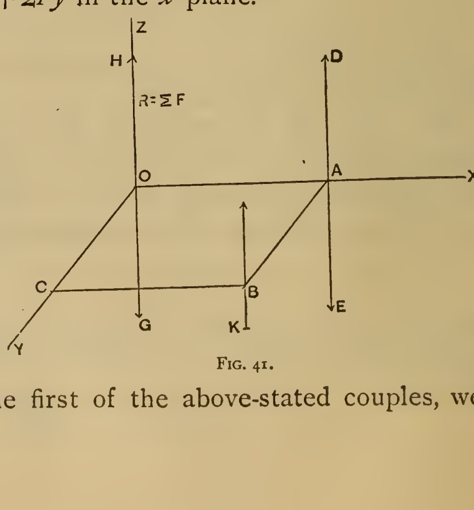


FIG. 41.

Combining this with the first of the above-stated couples, we

obtain a force $R = \Sigma F$ at A, where $OA = \dfrac{\Sigma Fx}{\Sigma F} = x_r$. Then combine with this resultant force $R = \Sigma F$ at A, the second couple, and we shall have as single resultant of the entire system a single force

$$R = \Sigma F$$

acting through B, where

$$AB = y_r = \frac{\Sigma Fy}{\Sigma F}.$$

Hence the resultant is a force whose magnitude is

$$R = \Sigma F,$$

the co-ordinates of its line of direction being

$$x_r = \frac{\Sigma Fx}{\Sigma F}, \quad y_r = \frac{\Sigma Fy}{\Sigma F}.$$

CASE II. When $\Sigma F = 0$, there is no single resultant force; but the system reduces to two couples, one in the x plane and one in the y plane, and these two can be reduced to one single resultant couple. (Observe that couples are to be accounted positive when, on being looked at by the observer from the positive part of the axis towards the origin, they are right-handed; otherwise they are negative.)

The moment axis of the couple in the x plane will be laid off on the axis OX from the origin towards the positive side if the moment is positive, or towards the negative side if it is negative, and likewise for the couple in the y plane.

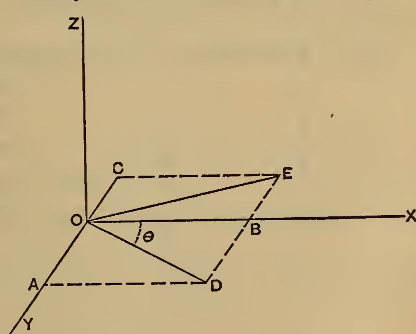


FIG. 42.

Hence lay off (Fig. 42) $OB = M_x$, $OA = M_y$, and by completing the rectangle we shall have OD as the moment axis of the resultant couple; hence the resultant couple lies in a plane perpendicular to OD, and its moment bears to OD the same ratio as M_x bears to OB.

Hence we may write

$$OD = M_r = \sqrt{M_x^2 + M_y^2},$$

$$\cos DOX = \frac{M_x}{M_r} = \cos\theta.$$

If M_y had been negative, we should have OE as the moment axis of the resultant couple.

EXAMPLES.

	F.	x.	y.		F.	x.	y.
1.	5	4	3	2.	5	−4	3
	3	2	1		−2	2	−1
	1	3	5		−3	3	5

Find the resultant in each example.

§ 62. Resultant of any System of Forces acting at Different Points of a Rigid Body, all situated in One Plane. — Let $CF = F$ (Fig. 43) be one of the given forces. Let all the forces be referred to a system of rectangular axes, as in the figure, and let α = angle made by F with OX, etc. Let the co-ordinates of the point of application of F be $AO = x$, $BO = y$.

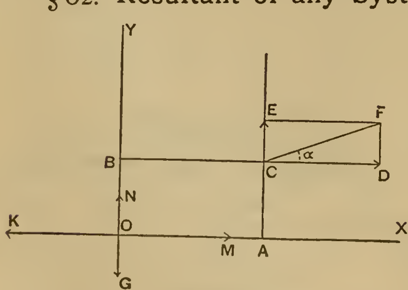


Fig. 43.

We first decompose $CF = F$ into two components, parallel respectively to OX and OY. These components are

$$CD = F\cos\alpha, \quad CE = F\sin\alpha.$$

Apply at O in the line OY two equal and opposite forces, each equal to $F\sin\alpha$, and at O in the line OX two equal and opposite forces, each equal to $F\cos\alpha$. Since these four are mutually balanced, they do not alter the effect of the single force; and hence we have, in place of F at C, the six forces CD, OM, OK, CE, ON, OG. Of these six, CE and OG form a couple whose moment is

$$-(F\sin\alpha)x = -Fx\sin\alpha,$$

CD and OK form a couple whose moment is

$$(F\cos\alpha)y = Fy\cos\alpha.$$

These two couples, being in the same plane, give as resultant moment their algebraic sum, or

$$F(y\cos\alpha - x\sin\alpha).$$

We have, therefore, instead of the single force at C, the following:—

1°. $OM = F\cos\alpha$ along OX.
2°. $ON = F\sin\alpha$ along OY.
3°. The couple $M = F(y\cos\alpha - x\sin\alpha)$ in the given plane.

Decompose in the same way each of the given forces; and we have, on uniting the components along OX, those along OY, and the statical couples respectively, the following:—

1°. A resultant force along OX, $R_x = \Sigma F\cos\alpha$.
2°. A resultant force along OY, $R_y = \Sigma F\sin\alpha$.
3°. A resultant couple in the plane, whose moment is

$$M = \Sigma F(y\cos\alpha - x\sin\alpha).$$

This entire system, on compounding the two forces at O, reduces to

1°. $R = \sqrt{R_x^2 + R_y^2} = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2}$;

making with OX an angle α_r, where

$$\cos \alpha_r = \frac{\Sigma F \cos \alpha}{R}.$$

2°. A resultant couple in the same plane, whose moment is

$$M = \Sigma F(y \cos \alpha - x \sin \alpha).$$

Now compound this resultant force and couple, and we have, for final resultant, a single force equal and parallel to R, and acting along a line whose perpendicular distance from O is equal to

$$\frac{M}{R}.$$

The equation of this line may be found as follows:

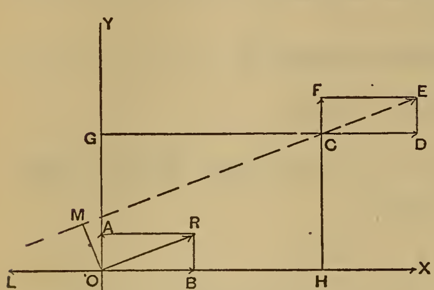


FIG. 44.

Suppose (Fig. 44) the force

$$OB = \Sigma F \cos \alpha,$$

$$OA = \Sigma F \sin \alpha,$$

$$OR = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2};$$

and let us suppose the resultant couple to be right-handed, and let

$$OM = \frac{M}{R}:$$

then will the line ME parallel to OR be the line of direction of the single resultant force.

Assuming the force R to act at any point $C(x_r, y_r)$ of this line, if we decompose it in the same way as we did the single forces previously, we obtain —

1°. The force $R \cos \alpha_r = \Sigma F \cos \alpha$ along OX.

2°. The force $R \sin \alpha_r = \Sigma F \sin \alpha$ along OY.

3°. The couple $R(y_r \cos \alpha_r - x_r \sin \alpha_r)$.

Hence we must have

$$R(y_r \cos \alpha_r - x_r \sin \alpha_r) = \Sigma F(y \cos \alpha - x \sin \alpha) = M.$$

Hence for the equation of the line of direction we have

$$y_r \cos \alpha_r - x_r \sin \alpha_r = \frac{M}{R}. \qquad (1)$$

Another form for the same equation is

$$y_r(\Sigma F \cos \alpha) - x_r(\Sigma F \sin \alpha) = M. \qquad (2)$$

§ 63. **Conditions of Equilibrium.** — If such a set of forces be in equilibrium, there must evidently be no tendency to translation and none to rotation. Hence we must have

$$R = 0 \quad \text{and} \quad M = 0.$$

Hence the conditions of equilibrium for any system of forces in a plane are three; viz., —

$$\Sigma F \cos \alpha = 0, \quad \Sigma F \sin \alpha = 0, \quad \Sigma F(y \cos \alpha - x \sin \alpha) = 0.$$

Another and a very convenient way to state the conditions of equilibrium for this case is as follows: —

If the forces be resolved into components along two directions at right angles to each other, then the algebraic sum of the components along each of these directions must be zero, and the algebraic sum of the moments of the forces about any axis perpendicular to the plane of the forces must equal zero.

EXAMPLES.

1. Given

F.	x.	y.	α.
5	3	2	31°
10	1	3	40°
−7	4	2	54°

Find the resultant, and the equation of its line of direction.

2. Given

F.	x.	y.	α.
12	27	3	15°
4	13	−5	30°
8	−5	−4	45°

Find the resultant, and the equation of its line of direction.

§ 64. **Resultant of any System of Forces not confined to One Plane.** — Suppose we have a number of forces applied at different points of a rigid body, and acting in different directions, of which we wish to find the resultant. Refer them all to a system of three rectangular axes, OX, OY, OZ (Fig. 45). Let $PR = F$ be one of the given forces. Resolve it into three components, PK, PH, and PG, parallel respectively to the three axes. Let

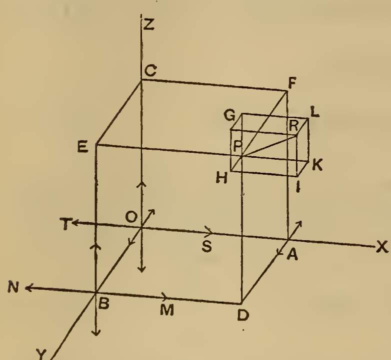


Fig. 45.

$$RPK = \alpha, \quad RPH = \beta, \quad RPG = \gamma.$$

Let $OA = x$, $OB = y$, $OC = z$, be the co-ordinates of the point of application of the force F. Now introduce at B and also at O two forces, opposite in direction, and each equal to PK. We now have, instead of the force PK, the five forces PK, BM, BN, OS, and OT. The two forces PK and BN form a couple in the y plane, whose axis is a line parallel to the axis OY, and whose moment is $(PK)(EB) = (F \cos \alpha)z = Fz \cos \alpha$. The

forces BM and OT form a couple in the z plane, whose moment is

$$(BM)(OB) = -Fy\cos\alpha.$$

Now do the same for the other forces PH and PG, and we shall finally have, instead of the force PR, three forces,

$$F\cos\alpha, \quad F\cos\beta, \quad F\cos\gamma,$$

acting at O in the directions OX, OY, and OZ respectively; together with six couples, two of which are in the x plane, two in the y plane, and two in the z plane.

They thus form three couples, whose moments are as follows:—

$$\begin{aligned}
&\text{Around } OX,\ F(y\cos\gamma - z\cos\beta);\\
&\text{Around } OY,\ F(z\cos\alpha - x\cos\gamma);\\
&\text{Around } OZ,\ F(x\cos\beta - y\cos\alpha).
\end{aligned}$$

Treat each of the given forces in the same way, and we shall have, in place of all the forces of the system, three forces,

$$\begin{aligned}
&\Sigma F\cos\alpha \text{ along } OX,\\
&\Sigma F\cos\beta \text{ along } OY,\\
&\Sigma F\cos\gamma \text{ along } OZ;
\end{aligned}$$

and three couples, whose moments are as follows:—

$$\begin{aligned}
&\text{Around } OX,\ M_x = \Sigma F(y\cos\gamma - z\cos\beta);\\
&\text{Around } OY,\ M_y = \Sigma F(z\cos\alpha - x\cos\gamma);\\
&\text{Around } OZ,\ M_z = \Sigma F(x\cos\beta - y\cos\alpha).
\end{aligned}$$

The three forces give a resultant at O equal to

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\cos\beta)^2 + (\Sigma F\cos\gamma)^2}, \qquad (1)$$

$$\cos\alpha_r = \frac{\Sigma F\cos\alpha}{R}, \quad \cos\beta_r = \frac{\Sigma F\cos\beta}{R}, \quad \cos\gamma_r = \frac{\Sigma F\cos\gamma}{R}. \qquad (2)$$

For the three couples we have as resultant

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}, \tag{3}$$

$$\cos\lambda = \frac{M_x}{M}, \quad \cos\mu = \frac{M_y}{M}, \quad \cos\nu = \frac{M_z}{M}; \tag{4}$$

λ, μ, and ν being the angles made by the moment axis of the resultant couple with OX, OY, and OZ respectively.

Thus far we have reduced the whole system to a single resultant force at the origin, and a couple. Sometimes we can reduce the system still farther, and sometimes not. The following investigation will show when we can do so. Let (Fig. 46) $OP = R$ be the resultant force, and $OC = M$ the moment axis of the resultant couple. Denote the angle between them by θ (a quantity thus far undetermined). Project $OP = R$ on OC. Its projection will be $OD = R\cos\theta$; then project, in its stead, the broken line $OABP$ on OC. By the principles of projections, the projection of this broken line will equal OD.

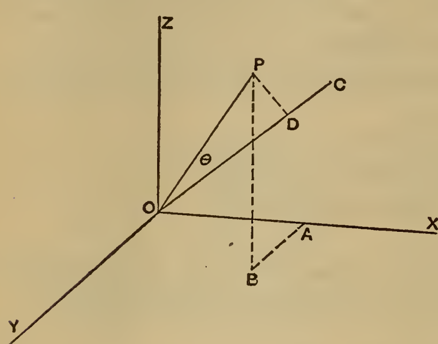


Fig. 46.

Now OA, AB, and BP are the co-ordinates of P, and make with OC the same angle as the axes OX, OY, and OZ; i.e., λ, μ, and ν respectively: hence the length of the projection is

$$OA\cos\lambda + AB\cos\mu + BP\cos\nu.$$

But

$$OA = R\cos\alpha_r, \quad AB = R\cos\beta_r, \quad BP = R\cos\gamma_r.$$

Hence

$$R\cos\theta = R\cos\alpha_r\cos\lambda + R\cos\beta_r\cos\mu + R\cos\gamma_r\cos\nu$$

$$\therefore \quad \cos\theta = \cos\alpha_r\cos\lambda + \cos\beta_r\cos\mu + \cos\gamma_r\cos\nu. \tag{5}$$

This enables us to find the angle between the resultant force and the moment axis of the resultant couple.

The following cases may arise:—

1°. When $\cos\theta = 0$, or $\theta = 90°$, the force lies in the plane of the couple, and we can reduce to a single force acting at a distance from O equal to $\frac{M}{R}$, and parallel to R at O.

2°. When $\cos\theta = 1$, or $\theta = 0$, the moment axis of the couple coincides in direction with the force: hence the plane of the couple is perpendicular to the force, and no farther reduction is possible.

3°. When θ is neither 0° nor 90°, we can resolve the couple M into two component couples, one of which, $M\cos\theta$, acts in a plane perpendicular to the direction of R, and the other, $M\sin\theta$, acts in a plane containing R. The latter, on being combined with the force R at the origin, gives an equal and parallel force whose line of action is at a distance from that of R at O, equal to

$$\frac{M\sin\theta}{R}.$$

4°. When $M = 0$, the resultant is a single force at O.

5°. When $R = 0$, the resultant is a couple.

§ 65. **Conditions of Equilibrium.**—To produce equilibrium, we must have no tendency to translation and none to rotation. Hence we must have

$$R = 0 \quad \text{and} \quad M = 0.$$

Hence we have, in general, six conditions of equilibrium; viz.,—

$$\begin{aligned} \Sigma F\cos\alpha &= 0, & \Sigma F\cos\beta &= 0, & \Sigma F\cos\gamma &= 0. \\ M_x &= 0, & M_y &= 0, & M_z &= 0. \end{aligned}$$

EXAMPLES.

1. Prove that, whenever three forces balance each other, they must lie in one plane.

2. Show how to resolve a given force into two whose sum is given.

3. A straight rod of uniform section and material is suspended by two strings attached to its ends, the strings being of given length, and attached to the same fixed point: find the position of equilibrium of the rod.

4. Two spheres are supported by strings attached to a given point, and rest against each other: find the tensions of the strings.

5. A straight rod of uniform section and material has its ends resting against two inclined planes at right angles to each other, the vertical plane which passes through the rod being at right angles to the line of intersection of the two planes: find the position of equilibrium of the rod, and the pressure on each plane, disregarding friction.

6. A certain body weighs 8 lbs. when placed in one pan of a false balance of equal arms, and 10 lbs. in the other: find the true weight of the body.

7. The points of attachment of the three legs of a three-legged table are the vertices of an isosceles right-angled triangle; a weight of 100 lbs. is supported at the middle of a line joining the vertex of one of the acute angles with the middle of the opposite side: find the pressure upon each leg.

8. A heavy body rests upon an inclined plane without friction: find the horizontal force necessary to apply, to prevent it from falling.

9. A rectangular picture is supported by a string passing over a smooth peg, the string being attached in the usual way at the sides, but one-fourth the distance from the top: find how many and what are the positions of equilibrium, assuming the absence of friction.

10. Two equal and weightless rods are jointed together, and form a right angle; they move freely about their common point: find the ratio of the weights that must be suspended from their extremities, that one of them may be inclined to the horizon at sixty degrees.

11. A weight of 100 lbs. is suspended by two flexible strings, one of which is horizontal, and the other is inclined at an angle of thirty degrees to the vertical: find the tension in each string.

CHAPTER II.

DYNAMICS.

§ 66. **Definitions.** — *Dynamics* is that part of mechanics which discusses the forces acting, when motion is the result.

Velocity, in the case of uniform motion, is the space passed over by the moving body in a unit of time; so that, if s represent the space passed over in time t, and v represent the velocity, then

$$v = \frac{s}{t}.$$

Velocity, in variable motion, is the limit of the ratio of the space (Δs) passed over in a short time (Δt), to the time, as the latter approaches zero: hence

$$v = \frac{ds}{dt}.$$

Acceleration is the limit of the ratio of the velocity (Δv) imparted to the moving body in a short time (Δt), to the time, as the time approaches zero. Hence, if a represent the acceleration,

$$a = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2}.$$

§ 67. **Uniform Motion.** — In this case the acceleration is zero, and the velocity is constant; and we have the equation

$$s = vt.$$

§ 68. **Uniformly Varying Motion.** — In this case the acceleration is constant: hence a is a constant in the equation

$$\frac{d^2s}{dt^2} = a,$$

and we obtain by one integration

$$v = \frac{ds}{dt} = at + c,$$

where c is an arbitrary constant: to determine it we observe, that, if v_0 represent the value of v when $t = 0$, we shall have

$$v_0 = 0 + c$$

$$\therefore \quad c = v_0$$

$$\therefore \quad v = \frac{ds}{dt} = at + v_0,$$

and by another integration

$$s = \tfrac{1}{2}at^2 + v_0t,$$

where s is the space passed over in time t; the arbitrary constant vanishing, because, when $t = 0$, s is also zero.

§ 69. **Measure of Force.** — It has already been seen, that, when a body is either at rest or moving uniformly in a straight line, there are either no forces acting upon it, or else the forces acting upon it are balanced. If, on the other hand, the motion of the body is rectilinear, but not uniform, the only unbalanced force acting is in the direction of the motion, and equal in magnitude to the momentum imparted in a unit of time in the direction of .the motion, or, in other words, to the limit of the ratio of the momentum imparted in a short time (Δt), to the time, as the latter approaches zero.

Thus, if F denote the force acting in the direction of the motion, m the mass, and a the acceleration, we shall have

$$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}. \tag{1}$$

From (1) we derive

$$mdv = Fdt; \tag{2}$$

and, if v_0 be the velocity of the moving body at the time when $t = t_0$, and v_1 its velocity when $t = t_1$, we shall have

$$\int_{v_0}^{v_1} mdv = \int_{t_0}^{t_1} Fdt$$

or

$$m(v_1 - v_0) = \int_{t_0}^{t_1} Fdt; \tag{3}$$

or, in words, the momentum imparted to the body during the time $t = (t_1 - t_0)$ by the force F, will be found by integrating the quantity Fdt between the limits t_1 and t_0.

§ 70. **Mechanical Work.** — Whenever a force is applied to a moving body, the force is either used in overcoming resistances (i.e., opposing forces, such as gravity or friction), and leaving the body free to continue its original motion undisturbed, or else it has its effect in altering the velocity of the body. In either case, the work done by the force is the product of the force, by the space passed through by the body in the direction of the force.

Unit of Work. — The unit of work is that work which is done when a unit of force acts through a unit of distance in the same direction as the force; thus, if one pound and one foot are our units of force and length respectively, the unit of work will be one foot-pound.

If a constant force act upon a moving body in the direction of its motion while the body moves through the space s, the work done by the force is

$$Fs;$$

and this, if the force is unresisted, is the energy, or capacity for performing work, which is imparted to the body upon which the force acts while it moves through the space s.

Thus, if a 10-pound weight fall freely through a height of 5 feet, the energy imparted to it by the force of gravity during this fall is $10 \times 5 = 50$ foot-pounds, and it would be necessary to do upon it 50 foot-pounds of work in order to destroy the velocity acquired by it during its fall. If, on the other hand, the force is a variable, the amount of work done in passing over any finite space in its own direction will be found by integrating, between the proper limits, the expression

$$\int F ds.$$

The power which a machine exerts is the work which it performs in a unit of time.

The unit of power commonly employed is the *horse-power*, which in English units is equal to 33000 foot-pounds per minute, or 550 foot-pounds per second.

§ 71. **Energy.** — The energy of a body is its capacity for performing work.

Kinetic or Actual Energy is the energy which a body possesses in virtue of its velocity; in other words, it is the work necessary to be done upon the body in order to destroy its velocity. This is equal to the work which would have to be done to bring the body from a state of rest to the velocity with which it is moving. Assume a body whose mass is m, and suppose that its velocity has been changed from v_0 to v_1. Then if F be the force acting in the direction of the motion, we shall have, from equation (2), § 69, that

$$Fvdt = mvdv; \qquad (1)$$

but

$$vdt = ds$$

$$\therefore \quad Fds = mvdv. \qquad (2)$$

Hence, by integration,

$$\int_{v_0}^{v_1} mvdv = \int Fds$$

$$\therefore \quad \tfrac{1}{2}m(v_1^2 - v_0^2) = \int Fds; \qquad (3)$$

but $\int Fds$ is the work that has been done on the body by the force, and the result of doing this work has been to increase its velocity from v_0 to v_1. It follows, that, in order to change the velocity from v_0 to v_1, the amount of work necessary to perform upon the body is

$$\tfrac{1}{2}m(v_1^2 - v_0^2) = \tfrac{1}{2}\frac{W}{g}(v_1^2 - v_0^2). \qquad (4)$$

If $v_0 = 0$, this expression becomes

$$\tfrac{1}{2}mv_1^2, \text{ or } \frac{Wv_1^2}{2g}, \qquad (5)$$

which is the expression for the kinetic energy of a body of mass m moving with a velocity v_1.

§ 72. **Atwood's Machine.** — A particular case of uniformly accelerated motion is to be found in Atwood's machine, in which a cord is passed over a pulley, and is loaded with unequal weights on the two sides. Were the weights equal, there would be no unbalanced force acting, and no motion would ensue; but when they are unequal, we obtain as a result a uniformly accelerated motion (if we disregard the action of the pulley), because we have a constant force equal to the difference of the two weights acting on a mass whose weight is the sum of the two weights. Thus, if we have a 10-pound weight on one side and a 5-pound weight on the other, the unbalanced force acting is

$$F = 10 - 5 = 5 \text{ lbs.}$$

The mass moved is $M = \frac{10 + 5}{g}$: hence the resulting acceleration is

$$a = \frac{5}{\left(\frac{15}{g}\right)} = \frac{g}{3}.$$

§ 73. **Normal and Tangential Components of the Forces acting on a Heavy Particle.** — If a body be in motion, either in a straight or in a curved line, and if at a certain instant all forces cease acting on it, the body will continue to move at a uniform rate in a straight line tangent to its path at that point where the body was situated when the forces ceased acting.

If an unresisted force be applied in the direction of the body's motion, the motion will still take place in the same straight line; but the velocity will vary as long as the force acts, and, from what we have seen, the equation

$$F = m\frac{d^2s}{dt^2} \tag{1}$$

will hold.

If an unresisted force act in a direction inclined to the body's motion, it will cause the body to change its speed, and also its course, and hence to move in a curved line. Indeed, if a force acting on a body which is in motion be resolved into two components, one of which is tangent to its path and the other normal, the tangential component will cause the body to change its speed, and the normal component will cause it to change the direction of its motion.

The measure of the tangential component is, as we have seen,

$$F = m\frac{d^2s}{dt^2};$$

and we will proceed to find an expression for the normal component otherwise known as the *Deviating Force*. For this

purpose we may substitute, for a small portion of the curve, a portion of the circle of curvature; hence we will proceed to find an expression for the centrifugal force of a body which moves uniformly with a velocity v in a circle whose radius is r.

CENTRIFUGAL FORCE.

Let AC (Fig. 47) be the space described in the time Δt. Then we have

$$AC = v\Delta t.$$

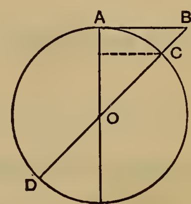


FIG. 47.

The motion AC may be approximately considered as the result of a uniform motion

$$AB = v\Delta t \text{ nearly,}$$

and a uniformly accelerated motion

$$BC = \tfrac{1}{2}a(\Delta t)^2 = s,$$

where $a =$ acceleration due to centrifugal force. But

$$(AB)^2 = BC \,.\, BD,$$

or

$$(v\Delta t)^2 = \tfrac{1}{2}a(\Delta t)^2(2r + s),$$

where

$$AO = OC = r$$

$$\therefore \quad v^2 = \tfrac{1}{2}a(2r + s) \text{ approximately}$$

$$\therefore \quad a = \frac{2v^2}{2r + s} \text{ approximately.}$$

For its true value, pass to the limit where $s = 0$.

Hence we have, for the acceleration due to the centrifugal force, the expression

$$\frac{v^2}{r}.$$

Hence the centrifugal force is equal to

$$F = \frac{mv^2}{r} = \frac{Wv^2}{gr}. \qquad (2)$$

DEVIATING FORCE.

If a body is moving in a curved path, whether circular or not, and the unbalanced force acting on it be resolved into tangential and normal components, the tangential component will be, as has already been seen,

$$m\frac{d^2s}{dt^2};$$

and the normal component will be

$$\frac{mv^2}{r} = \frac{m}{r}\left(\frac{ds}{dt}\right)^2,$$

where r is the radius of curvature of the path at the point in question.

RESULTANT FORCE.

Hence it follows that the entire unbalanced force acting on the body will be

$$F = \sqrt{\left(m\frac{d^2s}{dt^2}\right)^2 + \left(\frac{m}{r}\frac{ds^2}{dt^2}\right)^2},$$

or

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4}. \qquad (3)$$

§ 74. **Components along Three Rectangular Axes of the Velocities of, and of the Forces acting on, a Moving Body.**—If we resolve the velocity $\frac{ds}{dt}$ into three components along OX, OY, and OZ, we shall have, for these components respectively,

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt};$$

this being evident from the fact that dx, dy, and dz are respec-

tively the projections of ds on the axes OX, OY, and OZ; and, from the differential calculus, we have

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

On the other hand,

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}$$

are not only the components of the velocity $\frac{ds}{dt}$ in the directions OX, OY, and OZ, but they are also the velocities of the body in these directions respectively.

Now, the case of the accelerations is different; for, while

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \text{and} \quad \frac{d^2z}{dt^2}$$

are the accelerations in the directions OX, OY, and OZ respectively, they are not the components of the acceleration

$$\frac{d^2s}{dt^2}$$

along the three axes.

That they are the former is evident from the fact that $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ are the velocities in the directions of the axes, and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are their differential co-efficients, and hence represent the accelerations along the three axes. But if we consider the components of the force acting on the body, we shall have

for its components along OX, OY, and OZ, if α, β, and γ are the angles made by F with the axes respectively,

$$F\cos\alpha = m\frac{d^2x}{dt^2},\quad F\cos\beta = m\frac{d^2y}{dt^2},\quad F\cos\gamma = m\frac{d^2z}{dt^2},$$

$$\therefore\ F = m\sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}; \qquad (1)$$

and we found (§ 73) for F, the value

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4}. \qquad (2)$$

Hence, equating these values of F, and simplifying, we shall have the equation

$$\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2 = \left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4$$

Hence it is plain that $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$ can only be the components of the actual acceleration

$$\frac{d^2s}{dt^2}$$

when the last term $\frac{1}{r^2}\left(\frac{ds}{dt}\right)^4$ vanishes, or when $r = \infty$, i.e., when the motion is rectilinear.

Moreover, we have the two expressions (1) and (2) for the force acting upon a moving body.

The truth of the proposition just proved may also be seen from the following considerations: —

If a parallelopiped be constructed with the edges

$$\frac{dx}{dt},\quad \frac{dy}{dt},\quad \frac{dz}{dt},$$

the diagonal will be the actual velocity

$$\frac{ds}{dt},$$

and will, of course, coincide in direction with its path.

On the other hand, if a parallelopiped be constructed with the edges

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2},$$

its diagonal must coincide in direction with the force

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4},$$

and can coincide in direction with the path, and hence with the actual acceleration

$$\frac{d^2s}{dt^2},$$

only when the force is tangential to the path, and hence when the motion is rectilinear.

§ 75. **Centrifugal Force of a Solid Body.** — When a solid body revolves in a circle, the resultant centrifugal force of the entire body acts in the direction of the perpendicular let fall from the centre of gravity of the body on the axis of rotation, and its magnitude is the same as if its entire weight were concentrated at its centre of gravity.

PROOF. — Let (Fig. 48) the angular velocity $= \alpha$, and the total weight $= W$. Assume the axis of rotation perpendicular to the plane of the paper and passing through O; assume, as axis of x, the perpendicular dropped from the centre of gravity upon the axis of rotation. The co-ordinates of the centre of gravity will then be (x_0, y_0), and y_0 will be equal to zero.

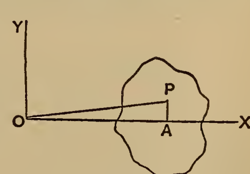


FIG. 48.

If, now, P be any particle of weight w, where $r =$ perpendicular distance from P on axis of rotation,

and $x = OA$, $y = AP$, we shall have for the centrifugal force of the particle at P

$$\frac{w}{g}\alpha^2 r;$$

but if we resolve this into two components, parallel respectively to OX and OY, we shall have for these components

$$\left(\frac{w}{g}\alpha^2 r\right)\frac{x}{r} = \frac{\alpha^2}{g}wx \quad \text{and} \quad \left(\frac{w}{g}\alpha^2 r\right)\frac{y}{r} = \frac{\alpha^2}{g}wy,$$

and, for the resultant for the entire body we shall have, parallel to OX,

$$F_x = \frac{\alpha^2}{g}\Sigma wx = \frac{\alpha^2}{g}Wx_0, \qquad (1)$$

and

$$F_y = \frac{\alpha^2}{g}\Sigma wy = \frac{\alpha^2}{g}Wy_0 = 0. \qquad (2)$$

Hence the centrifugal force of the entire body is

$$F = \frac{\alpha^2}{g}Wx_0; \qquad (3)$$

and if we let $v_0 = \alpha x_0 =$ linear velocity of the centre of gravity, we have

$$F = \frac{Wv_0^2}{gx_0},$$

which is the same as though the entire weight of the body were concentrated at its centre of gravity.

EXAMPLES.

1. A 10-pound weight is fastened by a rope 5 feet long to the centre, around which it revolves at the rate of 200 turns per minute; find the pull on the cord.

2. A locomotive weighing 50000 lbs., whose driving-wheels weigh 7000 lbs., is running at 60 miles per hour, the diameter of the drivers

being 6 feet, and the distance from the centre of the wheel to the centre of gravity of the same being 2 inches (the drivers not being properly balanced); find the pressure of the locomotive on the track (a) when the centre of gravity is directly below the centre of the wheel, and (b) when it is directly above.

3. Assume the same conditions, except that the distance between centre of the wheel and its centre of gravity is 5 inches instead of 2.

§ 76. **Uniformly Varying Rectilinear Motion.** — We have already found for this case (§ 68) the equations

$$\frac{d^2s}{dt^2} = a = \text{a constant,}$$

$$\frac{ds}{dt} = v = v_0 + at,$$

$$s = v_0t + \tfrac{1}{2}at^2;$$

and we may write for the force acting, which is, of course, coincident in direction with the motion,

$$F = m\frac{d^2s}{dt^2} = ma = \text{a constant.}$$

§ 77. **Motion of a Body acted on by the Force of Gravity only.** — A useful special case of uniformly varying motion is that of a body moving under the action of gravity only.

The downward acceleration due to gravity is represented by g feet per second, the value of g varying at different points on the surface of the earth according to the following law: —

$$g = g_1(1 - 0.00284 \cos 2\lambda)\left(1 - \frac{2h}{R}\right) \text{ feet per second,}$$

where

g_1 = 32.1695 feet,
λ = latitude of the place,
h = its elevation above mean sea-level in feet,
R = 20900000 feet.

If, now, we represent by h the height fallen through by a descending body in time t, we shall have the equations,

$$v = v_0 + gt,$$
$$h = v_0 t + \tfrac{1}{2}gt^2,$$

where v_0 is the initial downward velocity.

If, on the other hand, we represent by v_0 the initial upward velocity, and by h the height to which the body will rise in time t under the action of gravity only, we must write the equations

$$v = v_0 - gt,$$
$$h = v_0 t - \tfrac{1}{2}gt^2.$$

When $v_0 = 0$, the first set of equations gives

$$v = gt,$$
$$h = \tfrac{1}{2}gt^2,$$

which express the law of motion of a body starting from rest and subject to the action of gravity only.

Eliminate t between these equations, and we shall have

$$v^2 = 2gh \qquad \therefore \quad v = \sqrt{2gh},$$

or

$$h = \frac{v^2}{2g}:$$

h is called the *height due to the velocity* v, and represents the height through which a falling body must drop to acquire the velocity v; and

$$v = \sqrt{2gh}$$

is the velocity which a falling body will acquire in falling through the height h. Thus, if a body fall through a height of 50 feet, it will, by that fall, acquire a velocity of about

$$\sqrt{2(32\tfrac{1}{6})(50)} = \sqrt{3216.66} = 56.7 \text{ feet per second.}$$

Again: if a body has a velocity of 40 feet per second, we shall have

$$h = \frac{v^2}{2g} = \frac{1600}{64.3} = 24.8 \text{ feet};$$

and we say that the body has a velocity due to the height 24.8 feet, i. e., a velocity which it would acquire by falling through a height of 24.8 feet.

EXAMPLES.

1. A stone is dropped down a precipice, and is heard to strike the bottom in 4 seconds after it started: how high is the precipice?

2. How long will a stone, dropped down a precipice 500 feet high, take to reach the bottom?

3. What will be its velocity just before striking the ground?

4. A body is thrown vertically upwards with a velocity of 100 feet per second; to what height will it rise?

5. A body is thrown vertically upwards, and rises to a height of 50 feet. With what velocity was it thrown, and how long was it in its ascent?

6. What will be its velocity in its ascent at a point 15 feet above the point from which it started, and what at the same point in its descent?

§ 78. **Unresisted Projectile.** — In the case of an unresisted projectile, we have a body on which is impressed a uniform

motion in a certain direction (the direction of its initial motion), and which is acted on by the force of gravity only.

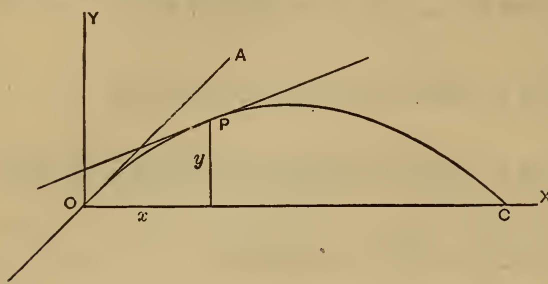


FIG. 49.

Let OPC be the path (Fig. 49), OA the initial direction, and v_0 the initial velocity, and the angle $AOX = \theta$.

Then we shall have, for the horizontal and vertical components of the unbalanced force acting, when the projectile is at P (co-ordinates x and y),

$$m\frac{d^2x}{dt^2} = 0 \text{ along } OX, \text{ and } m\frac{d^2y}{dt^2} = -mg = -W \text{ along } OY.$$

Hence

$$\frac{d^2x}{dt^2} = 0, \qquad (1) \qquad \frac{d^2y}{dt^2} = -g. \qquad (2)$$

Integrating, and observing, that, when $t = 0$, the horizontal and the vertical velocities were respectively $v_0 \cos\theta$ and $v_0 \sin\theta$, we have

$$\frac{dx}{dt} = v_0 \cos\theta, \qquad (3)$$

$$\frac{dy}{dt} = v_0 \sin\theta - gt. \qquad (4)$$

These equations could be derived directly by observing that the horizontal component of the initial velocity is $v_0 \cos\theta$, and that this remains constant, as there is no unbalanced force acting in this direction, also that $v_0 \sin\theta$ is the initial vertical velocity; and, since the body is acted on by gravity only, this velocity will in time t be decreased by gt.

Integrating equations (3) and (4), and observing that for $t = 0$, x and y are both zero, we obtain

$$x = v_0 \cos \theta . t, \tag{5}$$

$$y = v_0 \sin \theta . t - \tfrac{1}{2} g t^2. \tag{6}$$

Eliminate t, and we have

$$y = x \tan \theta - \frac{g x^2}{2 v_0^2 \cos^2 \theta} \tag{7}$$

as the equation of the path, which is consequently a parabola.

Equations (1), (2), (3), (4), (5), (6), and (7) enable us to solve any problem with reference to an unresisted projectile.

Equation (7) may be written

$$\left(y - \frac{v_0^2 \sin^2 \theta}{2g}\right) = - \frac{g}{2 v_0^2 \cos^2 \theta} \left(x - \frac{v_0^2 \sin \theta \cos \theta}{g}\right)^2 \tag{8}$$

which gives for the co-ordinates of the vertex

$$y_1 = \frac{v_0^2 \sin^2 \theta}{2g}, \qquad x_1 = \frac{v_0^2 \sin \theta \cos \theta}{g}.$$

EXAMPLES.

1. An unresisted projectile starts with a velocity of 100 feet per second at an upward angle of 30° to the horizon; what will be its velocity when it has reached a point situated at a horizontal distance of 1000 feet from its starting-point, and how long will be required for it to reach that point?

Solution.

$$v_0 = 100, \qquad \theta = 30^\circ, \qquad v_0 \cos \theta = 86.6, \qquad v_0 \sin \theta = 50,$$
$$g = 32.16.$$

Equation (5) gives us

$$1000 = 86.6\, t$$

$$\therefore \quad t = \frac{1000}{86.6} = 11.55 \text{ seconds.}$$

$$v_0 \sin\theta - gt = 50 - 371.5 = -321.5,$$

$$v = \sqrt{(86.6)^2 + (321.5)^2} = \sqrt{7500 + 103362} = 333.$$

Hence the point in question will be reached in $11\frac{1}{2}$ seconds after starting, and the velocity will then be 333 feet per second.

2. An unresisted projectile is thrown upwards from the surface of the earth at angle of 39° to the horizontal: find the time when it will reach the earth, and the velocity it will have acquired when it reaches the earth, the velocity of throwing being 30 feet per second.

3. A 10-pound weight is dropped from the window of a car when travelling over a bridge at a speed of 25 miles an hour. How long will it take to reach the ground 100 feet below the window, and what will be the kinetic energy when it reaches the ground?

4. With what horizontal velocity, and in what direction, must it be thrown, in order that it may strike the ground 50 feet forward of the point of starting?

5. Suppose the same 10-pound weight to be thrown vertically upwards from the car window with a velocity of 100 feet a minute, how long will it take to reach the ground, and at what point will it strike the ground?

§ 79. **Motion of a Body on an Inclined Plane without Friction.** — If a body move on an inclined plane along the line of steepest descent, subject to the action of gravity only, and if we resolve the force acting on it (i.e., its weight) into two components, along and perpendicular to the plane respectively, the latter component will be entirely balanced by the resistance of the plane, and the former will be the only unbalanced force acting on the body.

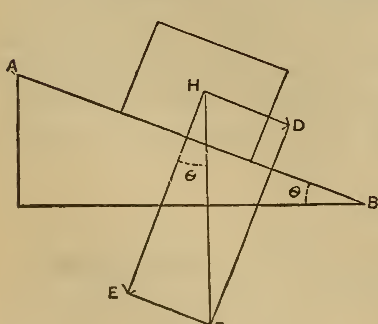


Fig. 50.

Suppose a body whose weight is represented (Fig. 50) by $HF = W$ to move along the inclined path AB under the action of gravity only. Let θ be the inclination of AB to the horizon. Resolve W into two components,

$$HD = W \sin \theta, \quad \text{and} \quad HE = W \cos \theta,$$

respectively parallel and perpendicular to the plane. The former is the only unbalanced force acting on the body, and will cause it to move down the plane with a uniformly accelerated motion; the acceleration being

$$\frac{W \sin \theta}{\left(\frac{W}{g}\right)} = g \sin \theta. \tag{1}$$

If the body is either at rest or moving downwards at the beginning, it will move downwards; whereas, if it is first moving upwards, it will gradually lose velocity, and move upwards more slowly, until ultimately its upward velocity will be destroyed, and it will begin moving downwards.

The equations for uniformly varying motion are entirely applicable to these cases. Thus, suppose that the body has an initial downward velocity v_0, this velocity will, at the end of the time t, become

$$v = \frac{ds}{dt} = v_0 + (g \sin \theta) t \tag{2}$$

$$\therefore \quad s = v_0 t + \tfrac{1}{2} g \sin \theta \, . \, t^2, \tag{3}$$

and, for the unbalanced force acting, we have

$$F = m \frac{d^2 s}{dt^2} = \frac{W}{g}(g \sin \theta) = W \sin \theta. \tag{4}$$

If, on the other hand, the body's initial velocity is upward, and we denote this upward velocity by v_0, we shall have the equations

$$v = \frac{ds}{dt} = v_0 - (g \sin\theta)t \tag{5}$$

$$s = v_0 t - \tfrac{1}{2}g \sin\theta \,.\, t^2 \tag{6}$$

$$F = -W \sin\theta. \tag{7}$$

Again, if the initial velocity is zero, equations (2) and (3) become

$$v = \frac{ds}{dt} = (g \sin\theta)t, \tag{8}$$

$$s = \tfrac{1}{2}g \sin\theta \,.\, t^2. \tag{9}$$

From these we obtain, for this case,

$$t = \sqrt{\frac{2s}{g \sin\theta}}; \tag{10}$$

and, substituting this value of t in (8), we have

$$v = \sqrt{2g(s \sin\theta)}, \tag{11}$$

or, if we let $s \sin\theta = h =$ the vertical distance through which the body has fallen, we have

$$v = \sqrt{2gh}. \tag{12}$$

Hence, *When a body, starting from rest, falls, under the action of gravity only, through a height* h, *the velocity acquired is* $\sqrt{2gh}$, *whether the path be vertical or inclined.*

EXAMPLES.

1. A body moves from the top to the bottom of a plane inclined to the horizon at 30°, under the action of gravity only: find the time required for the descent, and the velocity at the foot of the plane.

2. In the right-angled triangle shown in the figure (Fig. 51), given $AB = 10$ feet, angle $BAC = 30°$: find the time a body would require, if acted on by gravity only, to fall from rest through each of the sides respectively, AB being vertical.

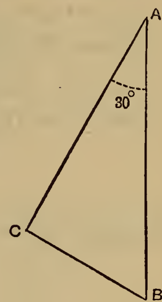


Fig. 51.

3. Given inclination of plane to the horizon $= \theta$, length of plane $= l$: compare the time of falling down the plane with the time of falling down the vertical.

4. A 100-pound weight rests, without friction, on the plane of example 3. What horizontal force is required to keep it from sliding down the plane.

5. Suppose 5 pounds horizontal force to be applied (*a*) so as to oppose the descent, (*b*) so as to aid the descent: find in each case how long it will take the weight to descend from the top to the bottom plane.

§ 80. **Motion along a Curved Line under the Action of Gravity only.** — We shall consider two questions in this regard: (*a*) the velocity at any point of the curve (*b*) the time of descent through any part of the curve.

(*a*) *Velocity at any point.* Let us suppose the body to have started from rest at A, and to have reached the point P in time t, where $AB = x$ (Fig. 52). Then, since the curved line AP may be considered as the limit of a broken line running from A to P, and as it has already been seen that the velocity acquired by falling through a certain height depends only upon the height, and not upon the inclination of the path, we shall have for a curved line also

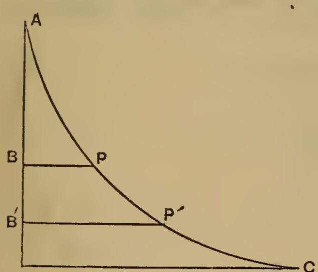


Fig. 52.

$$v = \sqrt{2gAB} = \sqrt{2gx},$$

where v is the velocity at P.

(*b*) *Time down a curve.* Referring to the same figure, let t denote the time required to go from A to P, and Δt the time to go from P to P', where $PP' = \Delta s$, and $BB' = \Delta x$; then, as we have seen that the velocity at P is $\sqrt{2gx}$, we shall have approximately for the space passed over in time Δt, the equation

$$\Delta s = \sqrt{2gx}\Delta t,$$

or, passing to the limit,

$$\frac{ds}{dt} = \sqrt{2gx}. \qquad (1)$$

This equation gives

$$dt = \frac{ds}{\sqrt{2gx}}$$

or

$$t = \int \frac{ds}{\sqrt{2gx}} = \int \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\sqrt{2gx}}, \qquad (2)$$

where, of course, the proper limits of integration must be used.

If t denote the time from A to P, we have

$$t = \int_{x=0}^{x=x} \frac{ds}{\sqrt{2gx}}.$$

EXAMPLE.

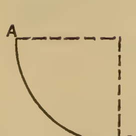


FIG. 53.

A body acted on by gravity only is constrained to move in the arc of a circle from A to C (Fig. 53), radius 10 feet. Find the time of describing the arc (quadrant) and the velocity acquired by the body when it reaches C.

§ 81. **Simple Circular Pendulum.** — To find the time occupied in a vibration of a simple circular pendulum, we take D (Fig. 54) as origin, and DC as axis of x, and the axis of y at right angles to DC. Let $AC = l$ and $BD = h$, we shall have for the time of a single oscillation from A to E

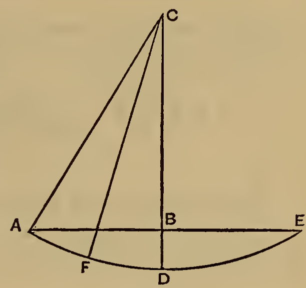


FIG. 54.

$$t = 2\int_{x=0}^{x=h} \frac{ds}{\sqrt{2g(h-x)}}.$$

Now, from the equation of the circle $AFDE$,

$$y^2 = 2lx - x^2,$$

we have

$$\frac{dy}{dx} = \frac{l-x}{y}$$

$$\therefore \quad \frac{ds}{dx} = \frac{l}{y} = \frac{l}{\sqrt{2lx - x^2}}$$

$$\therefore t = 2\int_0^h \frac{l\,dx}{\sqrt{(2lx - x^2)[2g(h-x)]}} = \frac{2l}{\sqrt{2g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}\sqrt{2l - x}}$$

or

$$t = \sqrt{\frac{l}{g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}}\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}.$$

This can only be integrated approximately.

Expanding $\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}$ we obtain

$$\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}} = 1 + \frac{x}{4l} + \frac{3}{32}\frac{x^2}{l^2} + \text{etc.},$$

$$\therefore \quad t = \sqrt{\frac{l}{g}}\int_0^h \left(1 + \frac{x}{4l} + \frac{3}{32}\frac{x^2}{l^2} + \text{etc.}\right)\frac{dx}{\sqrt{hx - x^2}}.$$

The greatest value of x is h; and if h is so small that we may omit $\frac{x}{4l}$, we shall have as our approximate result

$$t = \sqrt{\frac{l}{g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}} = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h}\right\}_0^h = \pi\sqrt{\frac{l}{g}}. \qquad (1)$$

If, however, the value of h as compared with l is too large to render it sufficiently accurate to omit $\frac{x}{4l}$, but so small that we can safely omit the higher powers of $\frac{x}{l}$, we shall have

$$t = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\int_0^h \frac{xdx}{\sqrt{hx - x^2}}\right\}_0^h$$

$$= \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\left[\frac{h}{2}\text{versin}^{-1}\frac{2x}{h} - \sqrt{hx - x^2}\right]\right\}_0^h$$

or

$$t = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right), \qquad (2)$$

a nearer approximation.

The formula

$$t = \pi\sqrt{\frac{l}{g}}$$

is the most used, and is more nearly correct, the smaller the value of h.

EXAMPLES.

1. Find the length of the simple circular pendulum which is to beat seconds at a place where $g = 32\frac{1}{6}$.

Solution.

$$t = \pi\sqrt{\frac{l}{g}} \qquad \therefore \quad l = \frac{t^2g}{\pi^2} = \frac{32\frac{1}{6}}{(3.1416)^2} = 3.259 \text{ feet.}$$

2. What is the time of vibration of a simple circular pendulum 5 feet long?

§ 82. **Simple Cycloidal Pendulum.** — The equation of the cycloid is

$$y = a \operatorname{versin}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}$$

$$\therefore \quad \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

Hence we shall have, for the time of a single oscillation,

$$t = 2\frac{\sqrt{2a}}{\sqrt{2g}} \int_0^h \frac{dx}{\sqrt{hx - x^2}}$$

or

$$t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}} \left\{ \operatorname{versin}^{-1} \frac{2x}{h} \right\}_0^h = \pi\sqrt{\frac{a}{g}}.$$

This expression is independent of h, so that the time of vibration is the same whether the arc be large or small.

A body can be made to vibrate in a cycloidal arc by suspending it by a flexible string between two cycloidal cheeks. This is shown from the fact that the evolute of the cycloid is another cycloid (Fig. 55).

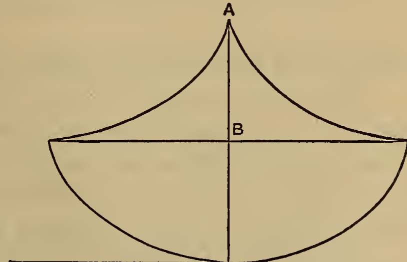


Fig. 55.

To prove this, we have, from the equation of the cycloid,

$$y = a \operatorname{versin}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}, \quad \frac{ds}{dx} = \sqrt{\frac{2a}{x}},$$

$$\frac{d^2y}{dx^2} = \frac{-a}{x^{\frac{3}{2}}\sqrt{2a - x}}.$$

Hence the radius of curvature is

$$\rho = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{-d^2y}{dx^2}} = (2)^{\frac{3}{2}}(a)^{\frac{1}{2}}\sqrt{2a - x}$$

or

$$\rho = 2(2a)^{\frac{1}{2}}\sqrt{2a - x};$$

and since we have for the evolute the relation

$$ds' = d\rho,$$

where ds' is the elementary arc of the evolute,

$$\therefore \quad s' = \int_{x=x}^{x=2a} d\rho;$$

and, observing that when $x = 2a \quad \rho = 0$, we have

$$s' = \rho,$$

$$\therefore \quad s' = 2(2a)^{\frac{1}{2}}\sqrt{2a - x};$$

or, if we transform co-ordinates to B by putting x for $2a - x$, we obtain

$$s' = 2(2ax)^{\frac{1}{2}},$$

$$\therefore \quad s'^2 = 8ax,$$

which is the equation of another cycloid just like the first.

The motion along a vertical cycloid may also be obtained by letting a body move along a groove in the form of a cycloid acted on by gravity alone; and in this case the time of descent of the body to the lowest point is precisely the same at whatever point of the curve the body is placed.

§ 83. **Effect of Grade on the Tractive Force of a Railway Train.** — As a useful particular case of motion on an inclined plane, we have the case of a railroad train moving up or down a grade. It is necessary that a certain tractive force

be exerted in order to overcome the resistances, and keep the train moving at a uniform rate along a level track. If, on the other hand, the track is not on a level, and if we resolve the weight of the train into components at right angles to and along the plane of the track, we shall have in the latter component a force which must be added to the tractive force above referred to when we wish to know the tractive force required to carry it up grade, and must be subtracted when we wish to know the tractive force required to carry it down grade. The result of this subtraction may give, if the grade is sufficiently steep and the speed sufficiently slow, a negative quantity; and in that case we must apply the brakes, instead of using steam, unless we wish the speed of the train to increase.

EXAMPLES.

1. A railroad train weighing 60000 lbs., and running at 50 miles per hour, requires a tractive force of 618 lbs. on a level; what is the tractive force necessary when it is to ascend a grade of 50 feet per mile? What when it is to descend? Also what is the amount of work per minute in each case?

Solution.

The resolution of the weight will give (Fig. 50, § 79), for the component along the plane,

$$(60000)\frac{50}{5280} = 568.2 \text{ nearly.}$$

Hence

$$\begin{aligned} \text{Tractive force for a level} &= 618.0, \\ \text{Tractive force for ascent} &= 1186.2, \\ \text{Tractive force for descent} &= 49.8. \end{aligned}$$

To ascertain the work done per minute in each case, we have —

(*a*) For a level track, $\frac{618 \times 50 \times 5280}{60} = 2719200$ foot-lbs.

(*b*) Up grade, $2719200 + \frac{60000 \times 50 \times 50}{60} = 5219200$ foot-lbs.

(*c*) Down grade, $2719200 - \frac{60000 \times 50 \times 50}{60} = 219200$ foot-lbs.

2. Suppose the tractive force required for each 2000 lbs. of weight of train to be, on a level track, for velocities of—

5.0 miles per hour,	10.0	20.0	30.0	40.0	50.0	60
6.1 lbs.,	6.6	8.3	11.2	15.3	20.6	27;

find the tractive force required to carry the train of example 1—

(*a*) Up an incline of 50 feet per mile at 30 miles per hour.

(*b*) Down an incline of 50 feet per mile at 30 miles per hour.

(*c*) Down an incline of 10 feet per mile at 20 miles per hour.

(*d*) What must be the incline down which the train must run to require no tractive force at 40 miles per hour?

3. If in the first example the tractive force remains 618 lbs. while the train is going down grade, what will be its velocity at the end of one minute, the grade being 10 feet per mile?

§ 84. **Harmonic Motion.**—If we imagine a body to be moving in a circle at a uniform rate (Fig. 56), and a second body to oscillate back and forth in the diameter *AB*, both starting from *B*, and if when the first body is at *C* the other is directly under it at *G*, etc., then is the second body said to move in harmonic motion.

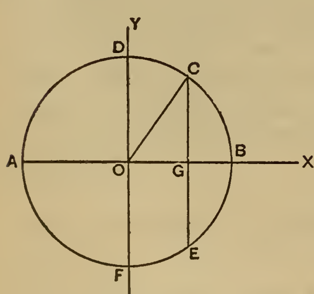


Fig. 56.

A practical case of this kind of motion is the motion of a slotted cross-head of an engine, as shown in the figure (Fig. 57); the crank moving at a uniform rate. In the case of the ordinary crank, and connecting-rod connecting the drive-wheel shaft of a stationary engine with the piston-rod,

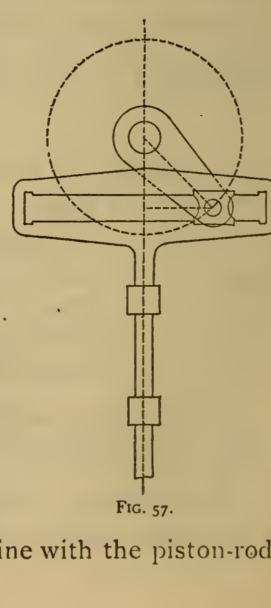


Fig. 57.

we have in the motion of the piston only an approximation to harmonic motion. We will proceed to determine the law of the force acting upon, and the velocity of, a body which is constrained to move in harmonic motion. Let the body itself and the corresponding revolving body be supposed to start from B (Fig. 56), the latter revolving in left-handed rotation with an angular velocity α, and let the time taken by the former in reaching G be t: then will the angle $BOC = \alpha t$; and we shall have, if s denote the space passed over by the body that moves with harmonic motion,

$$s = BG = OB - OC \cos \alpha t,$$

or, if

$$r = OB = OC,$$

$$s = r - r \cos \alpha t, \tag{1}$$

the velocity at the end of the time t will be

$$v = \frac{ds}{dt} = \alpha r \sin \alpha t, \tag{2}$$

and the acceleration at the end of time t will be

$$f = \frac{d^2 s}{dt^2} = \alpha^2 r \cos \alpha t. \tag{3}$$

Hence the force acting upon the body at that instant, in the direction of its motion, is

$$F = m\frac{d^2 s}{dt^2} = m\alpha^2 r \cos \alpha t = m\alpha^2 (OG). \tag{4}$$

The force, therefore, varies directly as the distance of the body from the centre of its path. It is zero when the body is at the

centre of its path, and greatest when it is at the ends of its travel, as its value is then

$$m\alpha^2 r = \frac{W}{g}\alpha^2 r;$$

this being the same in amount as the centrifugal force of the revolving body, provided this latter have the same weight as the oscillating body. On the other hand, the velocity is greatest when $\alpha t = \frac{\pi}{2}$ (i.e., at mid-stroke); and its value is then

$$v = \alpha r,$$

this being also the velocity of the crank-pin at mid-stroke.

EXAMPLE.

Given that the reciprocating parts of an engine weigh 10000 lbs., the length of crank being 1 foot, the crank making 60 revolutions per minute; find the force required to make the cross-head follow the crank, (1) when the crank stands at 30° to the line of dead points, (2) when at 60°, (3) when at the dead point.

§ 85. **Work under Oblique Force.** — If the force act in any other direction than that of the motion, we must resolve it into two components, the component in the direction of the motion being the only one that does work. Thus if the force F is variable, and θ equals the angle it makes with the direction of the motion, we shall have as our expression for the work done

$$\int F \cos\theta ds.$$

Thus if a constant force of 100 lbs. act upon a body in a direction making an angle of 30° with the line of motion, then will the work done by the force during the time in which it moves through a distance of 10 feet be

$$(100)(0.86603)(10) = 866 \text{ foot-lbs.}$$

§ 86. **Rotation of Rigid Bodies.** — Suppose a rigid body (Fig. 58) to revolve about an axis perpendicular to the plane of the paper, and passing through O; imagine a particle whose weight is w to be situated at a perpendicular distance $OA = r$ from the axis of rotation, and let the angular velocity be α: let it now be required to find the moment of the force or forces required to impart this motion in a unit of time; for we know, that, if the axis of rotation pass through the centre of gravity of the body, the motion can be imparted only by a statical couple; whereas if it do not pass through the centre of gravity, the motion can be imparted by a single force.

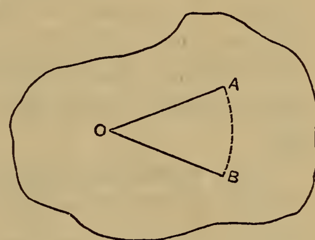


Fig. 58.

We shall have, for the particle situated at A,

Weight $= w$.
Angular velocity $= \alpha$.
Linear velocity $= \alpha r$.
Force required to impart this velocity in a unit of time to this particle $= \frac{w}{g}\alpha r$.

Moment of this force about the axis $= \frac{w}{g}\alpha r^2$.

Hence the moment of the force or forces required to impart to the entire body in a unit of time a rotation about the axis through O, with an angular velocity α, is

$$\Sigma \frac{w}{g}\alpha r^2 = \frac{\alpha}{g}\Sigma w r^2 = \frac{\alpha I}{g},$$

where I is used as a symbol to denote the limit of $\Sigma w r^2$, and is called the *Moment of Inertia of the body about the axis through* O.

§ 87. **Angular Momentum.** — This quantity, $\frac{\alpha I}{g}$, which expresses the moment of the force or forces required to impart to the body in a unit of time the angular velocity α about the axis in question is also called the *Angular Momentum of the body when rotating with the angular velocity* α *about the given axis.*

§ 88. **Actual Energy of a Rotating Body.** — If it be required to find the actual energy of the body when rotating with the angular velocity α, we have, for the actual energy of the particle at A,

$$\frac{w}{g}\frac{(\alpha r)^2}{2} = \frac{\alpha^2}{2g}wr^2,$$

and for that of the entire body

$$\frac{\alpha^2}{2g}\Sigma wr^2 = \frac{\alpha^2 I}{2g}.$$

This is the amount of mechanical work which would have to be done to bring the body from a state of rest to the velocity α, or the total amount of work which the body could do in virtue of its velocity against any resistance tending to stop its rotation.

§ 89. **Moment of Inertia.** — The term "moment of inertia" originated in a wrong conception of the properties of matter. The term has, however, been retained as a very convenient one, although the conceptions under which it originated have long ago vanished. The meaning of the term as at present used, in relation to a solid body, is as follows: —

The moment of inertia of a body about a given axis is the limit of the sum of the products of the weight of each of the elementary particles that make up the body, by the squares of their distances from the given axis.

Thus, if w_1, w_2, w_3, etc., are the weights of the particles which are situated at distances r_1, r_2, r_3, etc., respectively from

the axis, the moment of inertia of the body about the given axis is

$$I = \text{limit of } \Sigma wr^2.$$

§ 90. **Radius of Gyration.** — The radius of gyration of a body with respect to an axis is the perpendicular distance from the axis to that point at which, if the whole mass of the body were concentrated, the angular momentum, and hence the moment of inertia, of the body, would remain the same as they are in the body itself.

If ρ is the radius of gyration, the moment of inertia would be, when the mass is concentrated,

$$\rho^2 \Sigma w;$$

hence we must have

$$\rho^2 \Sigma w = \Sigma wr^2 = I,$$

whence

$$\rho^2 = \frac{\Sigma wr^2}{\Sigma w} = \frac{I}{W},$$

where $W =$ entire weight of the body.

§ 91. **Moment of Inertia of a Plane Surface.** — The term "moment of inertia," when applied to a plane figure, must, of course, be defined a little differently, as a plane surface has no weight; but, inasmuch as the quantity to which that name is given is necessary for the solution of a great many questions, and also since a knowledge of the manner of determining the moments of inertia of plane figures is very useful in simplifying the determinations of those of solid bodies, we shall now take up those of plane figures.

The moment of inertia of a plane surface about an axis, either in or not in the plane, is the limit of the sum of the products of the elementary areas into which the surface may be conceived to be divided, by the squares of their distances from the axis in question.

From this definition it will be evident, that, if the surface be referred to a pair of axes in its own plane, the moment of inertia of the surface about OY will be

$$I = \int\int x^2 dxdy, \tag{1}$$

and the moment of inertia of the surface about OX will be

$$J = \int\int y^2 dxdy. \tag{2}$$

The moment of inertia of the surface about an axis passing through the origin, and perpendicular to the plane XOY, will be

$$\int\int r^2 dxdy, \tag{3}$$

where $r =$ distance from O to the point (x, y); hence $r^2 = x^2 + y^2$, and the moment of inertia becomes

$$\int\int (x^2 + y^2) dxdy = \int\int x^2 dxdy + \int\int y^2 dxdy = I + J. \tag{4}$$

This is called the "polar moment of inertia." If polar co-ordinates be used, this last becomes

$$\int\int \rho^2 (\rho d\rho d\theta) = \int\int \rho^3 d\rho d\theta. \tag{5}$$

All these quantities are quantities that will arise in the discussion of stresses, and the letters I and J are very commonly used to denote respectively

$$\int\int x^2 dxdy \quad \text{and} \quad \int\int y^2 dxdy.$$

Another quantity that occurs also, and which will be represented by K, is

$$\int\int xydxdy; \tag{6}$$

and this is called the moment of deviation.

EXAMPLES.

The following examples will illustrate the mode of finding the moment of inertia: —

1. Find the moment of inertia of the rectangle $ABCD$ about OY (Fig. 59).

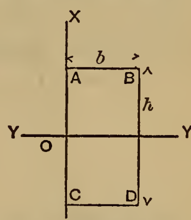


FIG. 59.

Solution.

$$I = \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} x^2 dx dy = b \int_{-\frac{h}{2}}^{\frac{h}{2}} x^2 dx = \frac{bh^3}{12}.$$

2. Find the moment of inertia of the entire circle (radius r) about the diameter OY (Fig. 60).

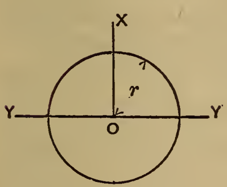


FIG. 60.

Solution.

$$I = \int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - x^2}} \int_{-r}^{r} x^2 dx dy = 2 \int_{-r}^{r} x^2 \sqrt{r^2 - x^2} dx$$

$$= 2 \left\{ -\frac{1}{4} x \left(r^2 - x^2\right)^{\frac{3}{2}} + \frac{r^2}{4} \int \sqrt{r^2 - x^2} dx \right\}_{-r}^{r}$$

$$= \frac{\pi r^4}{4} = \frac{\pi d^4}{64}.$$

3. Find the moment of inertia of the circular ring (outside radius r, inside radius r_1) about OY (Fig. 61).

Solution.

$$I = \frac{\pi r^4}{4} - \frac{\pi r_1^4}{4} = \frac{\pi (r^4 - r_1^4)}{4} = \frac{\pi (d^4 - d_1^4)}{64}.$$

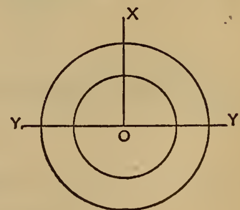


FIG. 61.

4. Find the moment of inertia of an ellipse (semi-axes a and b) about the minor axis OY.

Solution.

Equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

$$\therefore \quad I_y = \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \int_{-a}^{a} x^2 dx dy$$

$$= \frac{2b}{a}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx = \frac{2b}{a}\left(\frac{\pi a^4}{8}\right) = \frac{\pi a^3 b}{4}.$$

On the other hand, $I_x = \dfrac{\pi a b^3}{4}$.

§ 92. **Moments of Inertia of Plane Figures about Parallel Axes.**

PROPOSITION. — *The moment of inertia of a plane figure about an axis not passing through its centre of gravity is equal to its moment of inertia about a parallel axis passing through its centre of gravity increased by the product obtained by multiplying the area by the square of the distance between the two axes.*

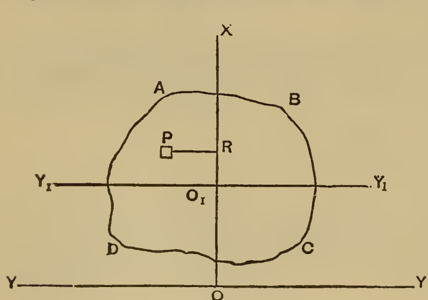


FIG. 62.

PROOF. — Let $ABCD$ (Fig. 62) be the surface; let OY be the axis not passing through the centre of gravity; let P be an elementary area $\Delta x \Delta y$, whose co-ordinates are $OP = x$ and $RP = y$; and let $OO_1 = a =$ a constant = distance between the axes.

Let $O_1R = x_1 =$ abscissa of P with reference to the axis passing through the centre of gravity,

$$\therefore \quad x = a + x_1$$

$$\therefore \quad x^2 = x_1^2 + 2ax_1 + a^2$$

$$\therefore \quad x^2\Delta x\Delta y = x_1^2\Delta x_1\Delta y + 2ax\Delta x\Delta y + a^2\Delta x\Delta y.$$

Hence, summing, and passing to the limit, we have

$$\iint x^2 dxdy = \iint x_1^2 dxdy + 2a\iint x_1 dxdy + a^2\iint dxdy; \quad (1)$$

but if we were seeking the abscissa of the centre of gravity when the surface is referred to $Y_1 O Y_{11}$, and if this abscissa be denoted by x_0, we should have

$$x_0 = \frac{\iint x_1 dxdy}{\iint dxdy}:$$

and, since $x_0 = 0$, $\therefore$ $\iint x_1 dxdy = 0$; hence, substituting this value in (1), we obtain

$$\iint x^2 dxdy = \iint x_1^2 dxdy + a^2\iint dxdy. \quad (2)$$

If, now, we call the moment of inertia about OY, I, that about $O_1 Y_{11}$, I_{11}, and let the area $= A = \iint dxdy$, we shall have

$$I = I_1 + a^2 A. \quad (3)$$

Q. E. D.

§ 93. **Polar Moment of Inertia of Plane Figures.** — *The moment of inertia of a plane figure about an axis perpendicular to the plane is equal to the sum of its moments of inertia about any pair of rectangular axes in its plane passing through the foot of the perpendicular.*

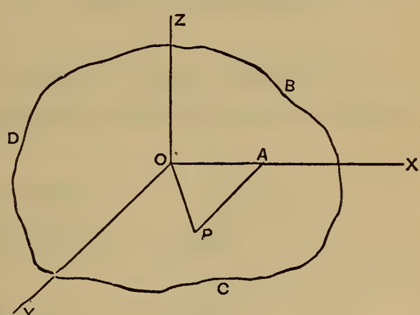


FIG. 63.

PROOF. — Let BCD (Fig. 63) be the surface, and P an elementary area, and let $OA = x$, $AP = y$, $OP = r$; then the moment of inertia of the surface about OZ will be

$$\iint r^2 dxdy = \iint (x^2 + y^2) dxdy = \iint x^2 dxdy + \iint y^2 dxdy = I + J.$$

Q. E. D.

Hence follows, also, that the sum of the moments of inertia of a plane surface relatively to a pair of rectangular axes in its own plane is isotropic; i.e., the same as for any other pair of rectangular axes meeting at the same point, and lying in its plane.

EXAMPLES.

1. To find the moment of inertia of the rectangle (Fig. 59) about an axis through its centre perpendicular to the plane of the rectangle.

Solution.

$$\text{Moment of inertia about } YY = \frac{bh^3}{12},$$

Moment of inertia about an axis through its

$$\text{centre and perpendicular to } YY = \frac{hb^3}{12};$$

hence

$$\text{Polar moment of inertia} = \frac{bh^3}{12} + \frac{hb^3}{12} = \frac{bh}{12}(h^2 + b^2).$$

2. To find the moment of inertia of a circle about an axis through its centre and perpendicular to its plane (Fig. 60).

Solution.

$$\text{Moment of inertia about } OY = \frac{\pi r^4}{4},$$

$$\text{Moment of inertia about } OX = \frac{\pi r^4}{4};$$

hence

$$\text{Polar moment of inertia} = \frac{\pi r^4}{4} + \frac{\pi r^4}{4} = \frac{\pi r^4}{2}.$$

3. To find the moment of inertia of an ellipse about an axis passing through its centre and perpendicular to its plane.

Solution.

From example 4, § 91, we have

$$I_x = \frac{\pi a b^3}{4} \qquad I_y = \frac{\pi a^3 b}{4}$$

$$\therefore \quad \text{Polar moment of inertia} = \frac{\pi a b}{4}(a^2 + b^2).$$

§ 94. **Moments of Inertia of Plane Figures about Different Axes compared.** — Given the surface KLM (Fig. 64), suppose we have already determined the quantities

$$I = \int\int x^2 dxdy, \; J = \int\int y^2 dxdy, \; K = \int\int xydxdy,$$

it is required to determine, in terms of them, the quantities

$$I_1 = \int\int x_1^2 dx_1 dy_1, \; J_1 = \int\int y_1^2 dx_1 dy_1, \; K_1 = \int\int x_1 y_1 dx_1 dy_1;$$

the angles XOY and X_1OY_1 being both right angles, and $YOY_1 = \alpha$.

We shall have, from the ordinary equations for the transformation of co-ordinates, to be found in any analytic geometry, the equations

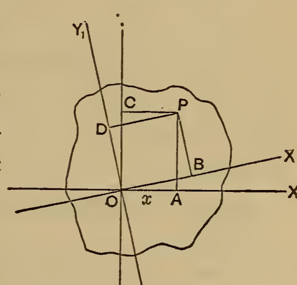


Fig. 64.

$$x_1 = x \cos \alpha + y \sin \alpha,$$

$$y_1 = y \cos \alpha - x \sin \alpha,$$

$$\therefore \quad x_1^2 = x^2 \cos^2 \alpha + y^2 \sin^2 \alpha + 2xy \cos \alpha \sin \alpha,$$

$$y_1^2 = x^2 \sin^2 \alpha + y^2 \cos^2 \alpha - 2xy \cos \alpha \sin \alpha,$$

$$x_1 y_1 = xy(\cos^2 \alpha - \sin^2 \alpha) - (x^2 - y^2) \cos \alpha \sin \alpha.$$

Hence

$$
\begin{aligned}
I_1 &= \int\int x_1^2 dx_1 dy_1 = \text{limit of } \Sigma x_1^2 \Delta A \\
&= \cos^2\alpha \text{ limit of } \Sigma x^2 \Delta A + \sin^2\alpha \text{ limit of } \Sigma y^2 \Delta A + \\
&\qquad 2\cos\alpha\sin\alpha \text{ limit of } \Sigma xy\Delta A \\
&= (\cos^2\alpha)\int\int x^2 dxdy + (\sin^2\alpha)\int\int y^2 dxdy + \\
&\qquad 2(\cos\alpha\sin\alpha)\int\int xydxdy. \\
J_1 &= \int\int y_1^2 dx_1 dy_1 = \text{limit of } \Sigma y_1^2 \Delta A \\
&= (\sin^2\alpha) \text{ limit of } \Sigma x^2\Delta A + (\cos^2\alpha) \text{ limit of } \Sigma y^2 \Delta A - \\
&\qquad 2(\cos\alpha\sin\alpha) \text{ limit of } \Sigma xy\Delta A \\
&= (\sin^2\alpha)\int\int x^2 dxdy + (\cos^2\alpha)\int\int y^2 dxdy - \\
&\qquad 2(\cos\alpha\sin\alpha)\int\int xydxdy. \\
K_1 &= \int\int x_1 y_1 dx_1 dy_1 = \text{limit of } \Sigma x_1 y_1 \Delta A \\
&= (\cos^2\alpha - \sin^2\alpha) \text{ limit of } \Sigma xy\Delta A - (\cos\alpha\sin\alpha)\{\text{limit of} \\
&\qquad \Sigma x^2\Delta A - \text{limit of } \Sigma y^2\Delta A\} \\
&= (\cos^2\alpha - \sin^2\alpha)\int\int xydxdy - (\cos\alpha\sin\alpha)\{\int\int x^2 dxdy - \\
&\qquad \int\int y^2 dxdy\}.
\end{aligned}
$$

Or, introducing the letters I, J, and K, we have

$$I_1 = I\cos^2\alpha + J\sin^2\alpha + 2K\cos\alpha\sin\alpha, \tag{1}$$

$$J_1 = I\sin^2\alpha + J\cos^2\alpha - 2K\cos\alpha\sin\alpha, \tag{2}$$

$$K_1 = -(J-I)\cos\alpha\sin\alpha + K(\cos^2\alpha - \sin^2\alpha). \tag{3}$$

The equations (1), (2), and (3) furnish the solution of the problem.

§ 95. Principal Moments of Inertia in a Plane. — *In every plane figure, a given point being assumed as origin, there is at least one pair of rectangular axes, about one of which the moment of inertia is a maximum, and a minimum about the other; these moments of inertia being called principal moments of inertia, and the axes about which they are taken being called principal axes of inertia.*

PROOF. — In order that $I_{,,}$ equation (1), § 94, may be a maximum or a minimum, we must have, as will be seen by differentiating its value, and putting the first differential co-efficient equal to zero,

$$-2I\cos\alpha\sin\alpha + 2J\cos\alpha\sin\alpha + 2K(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\therefore \quad K(\cos^2\alpha - \sin^2\alpha) - (I - J)\cos\alpha\sin\alpha = 0 \qquad (1)$$

$$\therefore \quad \frac{\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha} = -\frac{K}{I-J} \qquad \therefore \quad \tan 2\alpha = -\frac{2K}{I-J}. \qquad (2)$$

Hence, for the value of α given by (2), we have I_1 a maximum or a minimum; and as there are two values of 2α corresponding to the same value of $\tan 2\alpha$, and as these two values differ by 180°, the values of α will differ by 90°, one corresponding to a maximum and the other to a minimum.

Moreover, when the value of α is so chosen, we have

$$K_1 = 0,$$

as is proved by equation (1). Indeed, we might say that the condition for determining the principal axes of inertia is

$$K_1 = 0.$$

§ 96. **Axes of Symmetry of Plane Figures.** — An axis which divides the figure symmetrically is always a principal axis.

PROOF. — Let us assume that the y axis divides the surface symmetrically; then we shall have, with reference to this axis,

$$K = \int_{-x}^{x}\int xydydx = \left\{\int \frac{x^2}{2} ydy\right\}_{-x}^{x} = 0.$$

And, since K is zero, the axis of y is one principal axis, and of course the axis of x is the other. The same method of reasoning would show $K = 0$ if the x axis were the axis of symmetry.

Hence, whenever a plane figure has an axis of symmetry, this axis is one of the principal axes, and the other is at right angles to it. Thus, for a rectangle, when the axis is to pass through its centre of gravity, the principal axes are parallel to the sides respectively, the moment of inertia being greatest about the shortest axis, and least about the longest. Thus in an ellipse the minor axis is the axis of maximum, and the major that of minimum, moment of inertia, etc. On the other hand, in a circle, or in a square, since the maximum and minimum are equal, it follows that the moments of inertia about all axes passing through the centre are the same.

§ 97. **Conditions for Equal Values of Moment of Inertia.** — When the moments of inertia of a plane figure about three different axes passing through the same point are the same, the moments of inertia about all axes passing through this point are the same.

PROOF. — Let I be the moment of inertia about OY, I_1 about OY_1, I_2 about OY_2, and let

$$YOY_1 = \alpha, \qquad YOY_2 = \beta,$$

and let

$$I_1 = I_2 = I.$$

Then, from equation (1), § 94, we have

$$I = I\cos^2\alpha + J\sin^2\alpha + 2K\cos\alpha\sin\alpha,$$
$$I = I\cos^2\beta + J\sin^2\beta + 2K\cos\beta\sin\beta.$$

Hence

$$(I - J)\sin^2\alpha = 2K\cos\alpha\sin\alpha, \qquad (1)$$
$$(I - J)\sin^2\beta = 2K\cos\beta\sin\beta. \qquad (2)$$

Hence

$$(I - J)\tan\alpha = 2K, \qquad (3)$$
$$(I - J)\tan\beta = 2K. \qquad (4)$$

And, since $\tan\alpha$ is not equal to $\tan\beta$, we must have

$$I - J = 0 \qquad \text{and} \qquad K = 0.$$

Hence, since $K = 0$ and $I = J$, we shall have, from equa-

tion (1), § 94, for the moment of inertia I' about an axis, making any angle θ with OY,

$$I' = I\cos^2\theta + I\sin^2\theta + 0 = I. \qquad (5)$$

Hence all the moments of inertia are equal.

§ 98. **Components of Moments of Inertia of Solid Bodies.** — Refer the body to three rectangular axes, OX, OY, and OZ; and let I_x, I_y, and I_z represent its moment of inertia about each axis respectively. Then, if r denote the distance of any particle from OZ, we shall have

$$I_z = \text{limit of } \Sigma wr^2;$$

but

$$r^2 = x^2 + y^2$$

$$\therefore\ I_z = \text{limit of } \Sigma w(x^2 + y^2) = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wy^2. \quad (1)$$

In the same way we have

$$I_x = \text{limit of } \Sigma wy^2 + \text{limit of } \Sigma wz^2, \qquad (2)$$

$$I_y = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wz^2. \qquad (3)$$

§ 99. **Moments of Inertia of Solids around Parallel Axes.** — The moment of inertia of a solid body about an axis not passing through its centre of gravity is equal to its moment of inertia about a parallel axis passing through the centre of gravity, increased by the product of the entire weight of the body by the square of the distance between the two axes.

Proof. — Refer the body to a system of three rectangular axes, OX, OY, and OZ, of which OZ is the one about which the moment of inertia is taken. Let the co-ordinates of the centre of gravity of the body with reference to these axes be (x_0, y_0, z_0). Through the centre of gravity of the body draw a system of rectangular axes, parallel respectively to OX, OY, and OZ. Then we shall have for the co-ordinates of any point

$$x = x_0 + x_1,$$
$$y = y_0 + y_1,$$
$$z = z_0 + z_1.$$

Hence

$$\begin{aligned} I_z &= \text{limit of } \Sigma w(x^2 + y^2) = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wy^2 \\ &= \text{limit of } \Sigma w(x_0 + x_1)^2 + \text{limit of } \Sigma w(y_0 + y_1)^2 \\ &= x_0^2 \text{ limit of } \Sigma w + y_0^2 \text{ limit of } \Sigma w + 2x_0 \text{ limit of } \Sigma wx_1 \\ &\quad + 2y_0 \text{ limit of } \Sigma wy_1 + \text{limit of } \Sigma wx_1^2 + \text{limit of } \Sigma wy_1^2 \\ &= (x_0^2 + y_0^2) W + 2x_0 \text{ limit of } \Sigma wx_1 + 2y_0 \text{ limit of } \Sigma wy_1 \\ &\quad + \text{limit of } \Sigma wr_1^2 \\ &= r_0^2 W + I_z' + 2x_0 \text{ limit of } \Sigma wx_1 + 2y_0 \text{ limit of } \Sigma wy_1. \end{aligned}$$

But, since O_1 is the centre of gravity,

$$\therefore \quad \Sigma wx_1 = 0 \quad \text{and} \quad \Sigma wy_1 = 0.$$

Hence

$$I_z = I_z' + Wr_0^2,$$

which proves the proposition.

§ 100. **Examples of Moments of Inertia.**

1. To find the moment of inertia of a sphere whose radius is r and weight per unit of volume w, about the axis OZ drawn through its centre.

Solution.

Divide the sphere into thin slices (Fig. 65) by planes drawn perpendicular to OZ. Let the distance of the slice shown in the figure, above O be z, and its thickness dz: then will its radius be $\sqrt{r^2 - z^2}$; and we can readily see, from example 2, § 93, that its moment of inertia about OZ will be

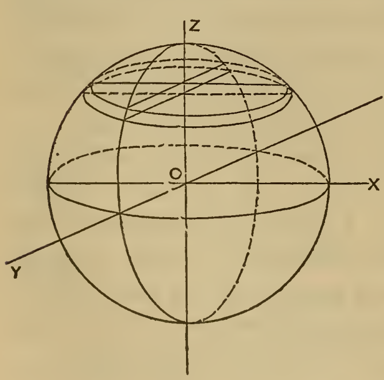


Fig. 65.

$$\frac{w\pi(r^2 - z^2)^2}{2} dz.$$

Hence the moment of inertia of the entire sphere about OZ will be

$$I_z = w\frac{\pi}{2}\int_{-r}^{r}(r^2 - z^2)^2 dz,$$

which easily reduces to

$$I_z = \frac{8}{15} w\pi r^5.$$

2. To find the moment of inertia of an ellipsoid (semi-axes a, b, c) about OZ (Fig. 66).

SOLUTION.—The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

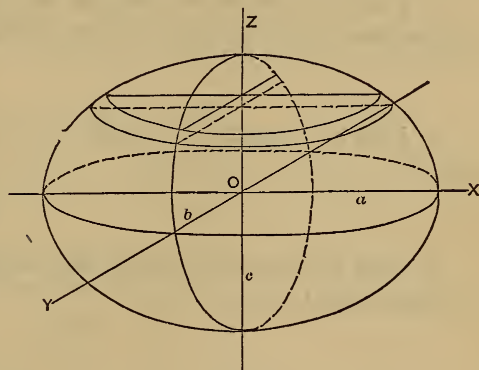


FIG. 66.

Divide it into thin slices perpendicular to OZ, and let the slice shown in the figure be at a distance z from O. Then will this slice be elliptical, and its semi-axes will be

$$\frac{a}{c}\sqrt{c^2 - z^2} \qquad \text{and} \qquad \frac{b}{c}\sqrt{c^2 - z^2};$$

and from example 3, § 93, we readily obtain, for its moment of inertia about OZ,

$$\frac{w\pi}{4}\left[\frac{ab}{c^2}(c^2 - z^2)\right]\left\{\frac{a^2}{c^2}(c^2 - z^2) + \frac{b^2}{c^2}(c^2 - z^2)\right\}dz$$
$$= \frac{w\pi ab(a^2 + b^2)}{4c^4}(c^2 - z^2)^2 dz.$$

Hence, for the moment of inertia of the ellipsoid about OZ, we have

$$I_z = \frac{w\pi ab(a^2 + b^2)}{4c^4}\int_{-c}^{c}(c^2 - z^2)^2 dz = \frac{4}{15} w\pi abc(a^2 + b^2).$$

3. Find the moment of inertia of a right circular cylinder, length a, radius r, about its axis.

Ans. $\dfrac{w\pi r^4 a}{2}$.

4. Find the moment of inertia of the same about an axis perpendicular to, and bisecting its axis.

$$Ans.\quad \frac{w\pi ar^2}{4}\left(r^2 + \frac{a^2}{3}\right).$$

5. Find the moment of inertia of an elliptic right cylinder, length $2c$, transverse semi-axes a and b, about its longitudinal axis.

$$Ans.\quad \frac{w\pi abc}{2}(a^2 + b^2).$$

6. Find the moment of inertia of the same about its transverse axis $2b$.

$$Ans.\quad 2w\pi abc\left(\frac{a^2}{4} + \frac{c^2}{3}\right).$$

7. Find the moment of inertia of a rectangular prism, sides $2a$, $2b$, $2c$, about central axis $2c$.

$$Ans.\quad \frac{8}{3}wabc(a^2 + b^2).$$

§ 101. **Centre of Percussion.** — Suppose we have a body revolving about an axis perpendicular to the plane of the paper, and passing through O, with an angular velocity α. If, with O as a centre and a radius $OA = r$, we describe an arc CB (Fig. 67), all particles situated in this arc have a linear velocity αr. The measure of the force which would impart this velocity to the particle in a unit of time is

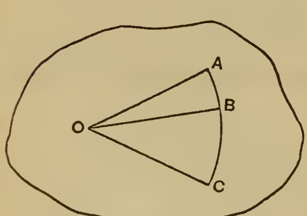


FIG. 67.

$$\frac{w}{g}\alpha r,$$

and the moment of this force about the axis is

$$\frac{w}{g}\alpha r^2;$$

hence the total angular momentum, or the total moment of the

forces which would impart to the body in a unit of time the angular velocity α, is, as has been shown already,

$$\frac{\alpha I}{g} = \frac{\alpha}{g}\Sigma wr^2.$$

The sum of the forces acting on the body is, on the other hand,

$$\frac{\alpha}{g}\Sigma wr.$$

Hence the perpendicular distance from O to the line of direction of the resultant force is

$$l = \frac{\frac{\alpha}{g}I}{\frac{\alpha}{g}\Sigma wr} = \frac{I}{\Sigma wr}; \qquad (1)$$

and if a line be drawn from O in the plane of the paper, perpendicular to the direction of the resultant, and the length l, as deduced above, laid off, the resultant force may be conceived to have its point of application at this point, and this point of application of the resultant of the forces which produce the rotation is called the *Centre of Percussion.*

If ρ = radius of gyration about the axis through O, and if r_0 = distance from O to the centre of gravity, we have

$$r_0\Sigma w = \Sigma wr.$$

Hence

$$l = \frac{\Sigma wr^2}{\Sigma wr} = \frac{I}{r_0\Sigma w} = \frac{1}{r_0}\left(\frac{I}{W}\right) = \frac{\rho^2}{r_0} \quad \therefore \quad \rho^2 = r_0 l;$$

or, in words, —

The radius of gyration is a mean proportional between the distance l, *and the distance* r_0 *between the axis of oscillation and the centre of gravity.*

The centre of percussion with respect to a given centre of oscillation O *has been defined as the point of application of the*

resultant of the forces which cause the body to rotate around the point O.

Another definition often given is, that it is *the point at which, if a force be applied, there will be no shock on the axis of oscillation;* and these two definitions are equivalent to each other.

Let the particles of the body under consideration be conceived, for the sake of simplicity, to be distributed along a single line AB, and suppose a force F applied at D (Fig. 68). Conceive two equal and opposite forces, each equal to F, applied at C, the centre of gravity of the body.

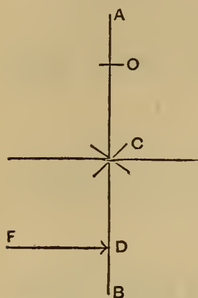


FIG. 68.

Then these three forces are equivalent to a single force F applied at the centre of gravity C, which produces translation of the whole body; and, secondly, a couple whose moment is $F(CD)$, whose effect is to produce rotation around an axis passing through the centre of gravity C. Under this condition of things it is evident that the centre of gravity C will have imparted to it in a unit of time a forward velocity equal to $\frac{F}{M}$, where M is the entire mass of the body; the point D will have imparted to it a greater forward velocity; while those points on the upper side of C will have imparted to them a less and less velocity as they recede from C, until, if the rod is sufficiently long, the particle at A will acquire a backward velocity.

Hence there must be some point which for the instant in question is at rest; i.e., where the velocity due to rotation is just equal and opposite to that due to the translation, or about which, for the instant, the body is rotating: and if this point were fixed by a pivot, there would be no stress on the pivot caused by the force applied at D.

An axis through this point is called the *Instantaneous Axis.*

§ 102. **Interchangeability of the Centre of Percussion and Centre of Oscillation.** — If we take our centre of percussion D as axis of oscillation, then will O be the new centre of percussion.

PROOF. — We have seen (§ 101) that

$$l = \frac{\rho^2}{r_0},$$

where $l = OD$, $r_0 = OC$, and $\rho =$ radius of gyration about an axis through O perpendicular to the plane of the paper.

Moreover, if ρ_0 represent the radius of gyration about an axis through C perpendicular to the plane of the paper, we shall have

$$\rho^2 = \rho_0^2 + r_0^2$$

$$\therefore \quad l = \frac{\rho_0^2}{r_0} + r_0$$

$$\therefore \quad l - r_0 = \frac{\rho_0^2}{r_0} = CD.$$

Now if D is taken as axis of oscillation, we shall have for the distance l_1 to the corresponding centre of percussion,

$$l_1 = \frac{\rho_1^2}{CD} = \frac{\rho_1^2}{l - r_0},$$

where $\rho_1 =$ radius of gyration about the axis of oscillation through D.

$$\therefore \quad l_1 = \frac{\rho_1^2}{CD} = \frac{\rho_0^2 + CD^2}{CD} = \frac{\rho_0^2}{CD} + CD = r_0 + (l - r_0) = l.$$

Hence the new centre of percussion is at O. Q. E. D.

§ 103. **Impact or Collision.** — Impact or collision is a pressure of inappreciably short duration between two bodies.

The direction of the force of impact is along the straight line drawn normal to the surfaces of the colliding bodies at their point of contact, and we may call this line the line of impact.

The action that occurs in the case of collision may be described as follows: at first the bodies undergo compression; the mutual pressure between them constantly increasing, until, when it has reached its maximum, the elasticity of the materials begins to overpower the compressive force, and restore the bodies wholly or partially to their original shape and dimensions.

Central impact occurs when the line joining the centres of gravity of the bodies coincides with the line of impact.

Eccentric impact occurs when these lines do not coincide.

Direct impact occurs when the line along which the relative motion of the bodies takes place, coincides with the line of impact.

Oblique impact occurs when these lines do not coincide.

CENTRAL IMPACT.

§ 104. **Equality of Action and Re-action.** — One fundamental principle that holds in all cases of central impact is the equality of action and re-action; in other words, we must have, that, at every instant of the time during which the impact is taking place, the pressure that one body exerts upon the other is equal and opposite to that exerted by the second upon the first.

The direct consequence of this principle is, that the algebraic sum of the momenta of the two bodies before impact remains unaltered by the impact, and hence that this sum is just the same at every instant of, and after, the impact.

If we let

m_1, m_2, be the respective masses,
c_1, c_2, their respective velocities before impact,
v_1, v_2, their respective velocities after impact,
v', v'', their respective velocities at any given instant during the time while impact is taking place,

then we must have the following two equations true; viz., —

$$m_1v_1 + m_2v_2 = m_1c_1 + m_2c_2, \tag{1}$$

$$m_1v' + m_2v'' = m_1c_1 + m_2c_2. \tag{2}$$

§ 105. **Velocity at Time of Greatest Compression.** — At the instant when the compression is greatest — i.e., at the instant when the elasticity of the bodies begins to overcome the deformation due to the impact, and to tend to restore them to their original forms — the values of v' and v'' must be equal to each other; in other words, the colliding bodies must be moving with a common velocity

$$v = v' = v''. \tag{1}$$

To determine this velocity, we have, from equation (2), § 104, combined with (1),

$$v = \frac{m_1c_1 + m_2c_2}{m_1 + m_2}. \tag{2}$$

§ 106. **Co-efficient of Restitution.** — In order to determine the values v_1, v_2, of the velocities after impact, we need two equations, and hence two conditions. One of them is furnished by equation (1), § 104. The second depends upon the nature of the material of the colliding bodies, and we may distinguish three cases: —

1°. *Inelastic Impact.* — In this case the velocity lost up to the time of greatest compression is not regained at all, and the velocity after impact is the common velocity v at the instant of greatest compression. In this case the whole of the work used up in compressing the bodies is lost, as none of it is restored by the elasticity of the material.

2°. *Elastic Impact.* — In this case the velocity regained after the greatest compression, is equal and opposite to that lost up to the time of greatest compression; therefore

$$v - v_1 = c_1 - v. \quad (1) \qquad v_2 - v = v - c_2. \quad (2)$$

We may also define this case as that in which the work lost in compressing the bodies is entirely restored by the elasticity of the material, so that

$$\frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} = \frac{m_1c_1^2}{2} + \frac{m_2c_2^2}{2}. \qquad (3)$$

Either condition will lead to the same result.

3°. *Imperfectly Elastic Impact.* — In this case a part only of the velocity lost up to the time of greatest compression is regained after that time.

If, when the two bodies are of the same material, we call e the co-efficient of restitution, then we shall so define it that

$$\frac{v - v_1}{c_1 - v} = \frac{v_2 - v}{v - c_2} = e;$$

or, in words, the co-efficient of restitution is the ratio of the velocity regained after compression to that lost previous to that time.

In this case only a part of the work done in producing the compression is regained, hence there is loss of energy. Its amount will be determined later.

Strictly speaking, all bodies belong to the third class; the value of e being always a proper fraction, and never reaching unity, the value corresponding to perfect elasticity; nor zero, the value corresponding to entire lack of elasticity.

§ 107. **Inelastic Impact.** — In this case the velocity after impact is the common velocity at the time of greatest compression; hence

$$v = v_1 = v_2 \qquad (1)$$

$$\therefore \quad v = \frac{m_1c_1 + m_2c_2}{m_1 + m_2}. \qquad (2)$$

And for the loss of energy due to impact we have

$$\frac{m_1c_1^2}{2} + \frac{m_2c_2^2}{2} - (m_1 + m_2)\frac{v^2}{2},$$

which, on substituting the value of v, reduces to

$$\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)^2. \qquad (3)$$

§ 108. **Elastic Impact.** — In this case we have, of course, the condition, equation (1), § 104,

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2,$$

and, for second equation, we may use equation (3), § 106; viz.,

$$\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2}.$$

Combining these two equations, we shall obtain

$$v_1 = c_1 - \frac{2m_2(c_1 - c_2)}{m_1 + m_2}, \qquad (1)$$

$$v_2 = c_2 + \frac{2m_1(c_1 - c_2)}{m_1 + m_2}. \qquad (2)$$

We can obtain the same result without having to solve an equation of the second degree, by using instead the equations (1) and (2) of § 106, together with (1) of § 104; i.e., —

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2;$$

$$v - v_1 = c_1 - v,$$

or

$$v_2 - v = v - c_2,$$

and (§ 105)

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}.$$

As the result of combining these equations, and eliminating v, we should obtain equations (1) and (2), as above, for the values of v_1 and v_2. In this case the energy lost by the collision is zero.

§ 109. **Special Cases of Inelastic Impact.** — (*a*) Let the mass m_2 be at rest. Then $c_2 = 0$,

$$\therefore \quad v = \frac{m_1 c_1}{m_1 + m_2} \qquad (1)$$

$$\therefore \quad \text{Loss of energy} = \frac{m_1 m_2}{m_1 + m_2} \frac{c_1^2}{2}. \qquad (2)$$

(*b*) Let m_2 be at rest, and let $m_2 = \infty$; i.e., let the mass m_1 strike against another which is at rest, and whose mass is infinite. We have

$$m_2 = \infty, \qquad c_2 = 0,$$

$$\therefore \quad v = \frac{m_1 c_1}{m_1 + m_2} = 0, \qquad (3)$$

$$\text{Loss of energy} = \frac{m_1}{\dfrac{m_1}{m_2} + 1} \frac{c_1^2}{2} = \frac{m_1 c_1^2}{2}, \qquad (4)$$

or the moving body is reduced to rest by the collision, and all its energy is expended in compression.

(*c*) Let $m_1 c_1 = -m_2 c_2$; i.e., let the two bodies move towards each other with equal momenta:

$$\therefore \quad v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2} = 0, \qquad (5)$$

$$\text{and the loss of energy} = \frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2}, \qquad (6)$$

the entire energy being lost.

§ 110. **Special Cases of Elastic Impact.** — (*a*) Let the mass m_2 be at rest. Then $c_2 = 0$,

$$v_1 = c_1 - \frac{2 m_2 c_1}{m_1 + m_2} \qquad (1)$$

$$\therefore \quad v_2 = \frac{2 m_1 c_1}{m_1 + m_2}. \qquad (2)$$

(*b*) Let m_2 be at rest, and let $m_2 = \infty$. Then we have $c_2 = 0$,

$$\therefore \quad v_1 = c_1 - \frac{2c_1}{\frac{m_1}{m_2} + 1} = c_1 - 2c_1 = -c_1, \qquad (3)$$

$$v_2 = 0. \qquad (4)$$

Hence the moving body retraces its path in the opposite direction with the same velocity.

(*c*) Let $m_1c_1 = -m_2c_2$. Then our equations of condition become

$$m_1v_1 + m_2v_2 = 0,$$

$$\frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} = \frac{m_1c_1^2}{2} + \frac{m_2c_2^2}{2}:$$

and from these we readily obtain

$$v_1 = -c_1,$$

$$v_2 = -c_2;$$

i.e., both bodies return on their path with the same velocity with which they approached each other.

§ 111. **Examples of Elastic and of Inelastic Impact.**

1. With what velocity must a body weighing 8 pounds strike one weighing 25 pounds in order to communicate to it a velocity of 2 feet per second, (*a*) when the bodies are perfectly elastic, (*b*) when wholly inelastic.

2. Suppose sixteen impacts per minute take place between two bodies whose weights are respectively 1000 and 1200 pounds, their initial velocities being 5 and 2 feet per second respectively: find the loss of energy, the bodies being inelastic.

§ 112. **Imperfect Elasticity.** — In this case we have the relations (see § 106)

$$\frac{v - v_1}{c_1 - v} = \frac{v_2 - v}{v - c_2} = e,$$

where

$$v = \frac{m_1c_1 + m_2c_2}{m_1 + m_2};$$

and we have also

$$m_1v_1 + m_2v_2 = m_1c_1 + m_2c_2.$$

Determining from them the values of v_1 and v_2, we obtain

$$v_1 = v(1 + e) - ec_1, \qquad (1)$$

$$v_2 = v(1 + e) - ec_2, \qquad (2)$$

or, by substituting for v its value,

$$v_1 = \frac{m_1c_1 + m_2c_2}{m_1 + m_2}(1 + e) - ec_1, \qquad (3)$$

$$v_2 = \frac{m_1c_1 + m_2c_2}{m_1 + m_2}(1 + e) - ec_2. \qquad (4)$$

These may otherwise be put in the form

$$v_1 = c_1 - (1 + e)\frac{m_2}{m_1 + m_2}(c_1 - c_2), \qquad (5)$$

$$v_2 = c_2 + (1 + e)\frac{m_1}{m_1 + m_2}(c_1 - c_2). \qquad (6)$$

Moreover, we have for the loss of energy due to impact

$$E = \frac{m_1}{2}(c_1^2 - v_1^2) + \frac{m_2}{2}(c_2^2 - v_2^2)$$

or

$$E = \tfrac{1}{2}\{m_1(c_1 - v_1)(c_1 + v_1) + m_2(c_2 - v_2)(c_2 + v_2)\};$$

but, from (5) and (6) respectively,

$$c_1 - v_1 = \frac{(1 + e)m_2(c_1 - c_2)}{m_1 + m_2}$$

$$c_2 - v_2 = -\frac{(1 + e)m_1(c_1 - c_2)}{m_1 + m_2}$$

$$\therefore \quad E = \frac{(1 + e)(c_1 - c_2)}{2(m_1 + m_2)} \{m_1 m_2(c_1 + v_1) - m_1 m_2(c_2 + v_2)\}$$

$$\therefore \quad E = \frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)(1 + e)(c_1 - c_2 + v_1 - v_2).$$

But, from (1) and (2),

$$v_1 - v_2 = -e(c_1 - c_2)$$

$$\therefore \quad E = \frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)(1 + e)(c_1 - c_2)(1 - e)$$

or

$$E = (1 - e^2)\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)^2. \tag{7}$$

When $e = 1$, or the elasticity is perfect, this loss of energy becomes zero.

When $e = 0$, or the bodies are totally inelastic, then the loss of energy becomes

$$\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)^2, \tag{8}$$

as has been already shown in § 107.

An interesting fact in this connection is, that since (8) is the work expended in producing compression, and (7) is the work lost in all, therefore the work restored by the elasticity of the body is

$$e^2\left\{\frac{m_1 m_2}{m_1 + m_2}(c_1 - c_2)^2\right\}; \tag{9}$$

so that e^2, or the square of the co-efficient of restitution, is the ratio of the work restored by the elasticity of the bodies, to the work expended in compressing the bodies up to the time of greatest compression.

§ 113. **Special Cases.** — (*a*) Let m_2 be at rest, therefore $c_2 = 0$. Then we shall have

$$v_1 = c_1 \left\{ 1 - \frac{m_2(1 + e)}{m_1 + m_2} \right\} = c_1 \frac{m_1 - em_2}{m_1 + m_2}, \qquad (1)$$

$$v_2 = (1 + e)c_1 \frac{m_1}{m_1 + m_2}, \qquad (2)$$

and for loss of energy

$$E = (1 - e^2) \frac{m_1 m_2}{2(m_1 + m_2)} c_1^2. \qquad (3)$$

(*b*) When $m_2 = \infty$, and $c_2 = 0$, we have

$$v_1 = -ec_1, \qquad (4)$$

$$v_2 = 0,$$

$$E = (1 - e^2) \frac{m_1 c_1^2}{2}. \qquad (5)$$

(*c*) When $m_1 c_1 = -m_2 c_2$, then

$$v_1 = -ec_1,$$

$$v_2 = -ec_2,$$

$$E = \frac{(1 - e^2)m_1 c_1(c_1 - c_2)}{2} = \frac{(1 - e^2)m_2 c_2(c_2 - c_1)}{2}$$

$$= (1 - e^2) \frac{m_1(m_1 + m_2)}{2m_2} c_1^2. \qquad (6)$$

§ 114. **Values of *e* as Determined by Experiment.** — Since we have

$$e = \frac{v - v_1}{c_1 - v},$$

we shall have, when

$$m_2 = \infty \quad \text{and} \quad c_2 = 0,$$

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2} = 0.$$

Hence

$$e = -\frac{v_1}{c_1}.$$

Now, if we let a round ball fall vertically upon a horizontal slab from the height H, we shall have for the velocity of approach

$$c_1 = \sqrt{2gH};$$

and if we measure the height h to which it rises on its rebound, we shall have

$$-v_1 = \sqrt{2gh}.$$

Hence

$$e = -\frac{v_1}{c_1} = \sqrt{\frac{h}{H}}.$$

In this way the value of e can be determined experimentally for different substances.

Newton found for values of e: for glass, $\frac{15}{16}$; for steel, $\frac{5}{9}$; and Coriolis gives for ivory from 0.5 to 0.6.

On the other hand, if we desired to adopt as our constant the ratio of the work restored, to the work spent in compression, we should have for our constant e^2, and hence the squares of the preceding numbers.

EXAMPLES.

1. If two trains of cars, weighing 120000 and 160000 lbs., come into collision when they are moving in opposite directions with velocities 20 and 15 feet per second respectively, what is the loss of mechanical effect expended in destroying the locomotives and cars?

2. Two perfectly inelastic balls approach each other with equal velocities, and are reduced to rest by the collision; what must be the ratio of their weights?

3. Two steel balls, weighing 10 lbs. each, are moving with velocities 5 and 10 feet per second respectively, and in the same direction: find their velocities after impact, the fastest ball being in the rear, and overtaking the other; also the loss of mechanical effect due to the impact, assuming $e = 0.55$.

§ 115. Oblique Impact.

Let m_1, m_2, be the masses of the colliding bodies;
c_1, c_2, their respective velocities before impact;
α_1, α_2, the angles made by c_1, c_2, with the line of centres;
v_1, v_2, the components of the velocities after impact;
$c_1 \cos \alpha_1$, $c_2 \cos \alpha_2$, the components of c_1, c_2, along the line of centres;
$c_1 \sin \alpha_1$, $c_2 \sin \alpha_2$, the components of c_1, c_2, at right angles to the line of centres;
v the common component of the velocity at the instant of greatest compression along line of centres;
v', v'', actual velocities after impact;
α', α'', angles they make with line of centres;
v_c', v_c'', actual velocities when compression is greatest;
α_c', α_c'', angles they make with line of centres.

Then we shall have, by proceeding in the same way as was done in § 112,

$$v_1 = c_1 \cos \alpha_1 - (1 + e)\frac{m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2), \quad (1)$$

$$v_2 = c_2 \cos \alpha_2 + (1 + e)\frac{m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2), \quad (2)$$

$$v' = \sqrt{v_1^2 + c_1^2 \sin^2 \alpha_1}, \qquad (3)$$

$$v'' = \sqrt{v_2^2 + c_2^2 \sin^2 \alpha_2}, \qquad (4)$$

$$\cos \alpha' = \frac{v_1}{v'}, \qquad (5)$$

$$\cos \alpha'' = \frac{v_2}{v''}, \qquad (6)$$

$$v = \frac{m_1 c_1 \cos \alpha_1 + m_2 c_2 \cos \alpha_2}{m_1 + m_2}, \qquad (7)$$

$$v_c' = \sqrt{v^2 + c_1^2 \sin^2 \alpha_1}, \qquad (8)$$

$$v_c'' = \sqrt{v^2 + c_2^2 \sin^2 \alpha_2}, \qquad (9)$$

$$\cos \alpha_c' = \frac{v}{v_c'}, \qquad (10)$$

$$\cos \alpha_c'' = \frac{v}{v_c''}. \qquad (11)$$

And for the energy lost in impact, we have

$$E = (1 - e^2)\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2)^2. \qquad (12)$$

When the bodies are perfectly elastic,

$$e = 1,$$

and equations (1), (2), and (12) become respectively

$$v_1 = c_1 \cos \alpha_1 - \frac{2m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$v_2 = c_2 \cos \alpha_2 + \frac{2m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$E = 0.$$

The rest remain the same in form.

When the bodies are totally inelastic,

$$e = 0,$$

and equations (1), (2), and (12) become respectively

$$v_1 = c_1 \cos \alpha_1 - \frac{m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$v_2 = c_2 \cos \alpha_2 + \frac{m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$E = \frac{m_1 m_2}{2(m_1 + m_2)}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2)^2.$$

The rest remain the same in form.

§ 116. **Impact of Revolving Bodies.** — Let the bodies A and B revolve about parallel axes, and impinge upon each other.

Draw a common normal at the point of contact. This common normal will be the line of impact.

Let ϵ_1 = angular velocity of A before impact,
ϵ_2 = angular velocity of B before impact,
ω_1 = angular velocity of A after impact,
ω_2 = angular velocity of B after impact,
a_1 = perpendicular from axis of A on line of impact,
a_2 = perpendicular from axis of B on line of impact,
I_1 = moment of inertia of A about its axis,
I_2 = moment of inertia of B about its axis.

Then we shall have

$a_1\epsilon_1 = c_1$ = linear velocity of A at point of contact before impact;
$a_2\epsilon_2 = c_2$ = linear velocity of B at point of contact before impact;
$a_1\omega_1 = v_1$ = linear velocity of A at point of contact after impact;
$a_2\omega_2 = v_2$ = linear velocity of B at point of contact after impact;

$\dfrac{I_1\epsilon_1^2}{2g} = \left(\dfrac{I_1}{a_1^2}\right)\dfrac{c_1^2}{2g}$ = actual energy of A before impact;

$\dfrac{I_2\epsilon_2^2}{2g} = \left(\dfrac{I_2}{a_2^2}\right)\dfrac{c_2^2}{2g}$ = actual energy of B before impact;

$\dfrac{I_1\omega_1^2}{2g} = \left(\dfrac{I_1}{a_1^2}\right)\dfrac{v_1^2}{2g}$ = actual energy of A after impact;

$\dfrac{I_2\omega_2^2}{2g} = \left(\dfrac{I_2}{a_2^2}\right)\dfrac{v_2^2}{2g}$ = actual energy of B after impact;

Hence it follows that we have the case explained in § 112 for imperfectly elastic impact, provided only we write

$$\frac{I_1}{a_1^2} \text{ instead of } m_1 g \quad \text{and} \quad \frac{I_2}{a_2^2} \text{ instead of } m_2 g.$$

Hence we shall have

$$\omega_1 = \epsilon_1 - a_1(a_1\epsilon_1 - a_2\epsilon_2)\frac{I_2}{I_1 a_1^2 + I_2 a_2^2}(1 + e), \qquad (1)$$

$$\omega_2 = \epsilon_2 + a_2(a_1\epsilon_1 - a_2\epsilon_2)\frac{I_1}{I_1 a_1^2 + I_2 a_2^2}(1 + e), \qquad (2)$$

The case of perfect elasticity is obtained by making $e = 1$.

The case of total lack of elasticity is obtained by making $e = 0$.

In the latter case the loss of energy is

$$\frac{(a_1\epsilon_1 - a_2\epsilon_2)^2}{2}\,\frac{I_1 I_2}{I_1 a_1^2 + I_2 a_2^2}, \qquad (3)$$

as can be seen by substituting the proper values in equation (8), § 112.

CHAPTER III.

ROOF-TRUSSES.

§ 117. **Definitions and Remarks.** — *The term "truss" may be applied to any framed structure intended to support a load.*

In the case of any truss, the external loads may be applied only at the joints, or some of the truss members may support loads at points other than the joints.

In the latter case those members are subjected, not merely to direct tension or compression, but also to a bending-action, the determination of which we shall defer until we have studied the mode of ascertaining the stresses in a loaded beam; and we shall at present confine ourselves to the consideration of the direct stresses of tension and compression.

For this purpose any loads applied between two adjacent joints must be resolved into two parallel components acting at those joints, and the truss is then to be considered as loaded at the joints. By this means we shall obtain the entire stresses in the members whenever the loads are concentrated at the joints; and, when certain members are loaded at other points, our results will be the direct tensions and compressions of these members, leaving the stresses due to bending yet to be determined.

A tie is a member suited to bear only tension.

A strut is a member suited to bear compression.

§ 118. **Frames of Two Bars.** — Frames of two bars may consist, (1) of two ties (Fig. 69), (2) of two struts (Fig. 70), (3) of a strut and a tie (Fig. 71).

CASE I. *Two Ties* (Fig. 69). — Let the load be represented graphically by $CF = W$. Then if we resolve it into two components, CD and CE, acting along CB and CA respectively, CD will represent graphically the pull or tension in the tie CB, and CE that in the tie CA.

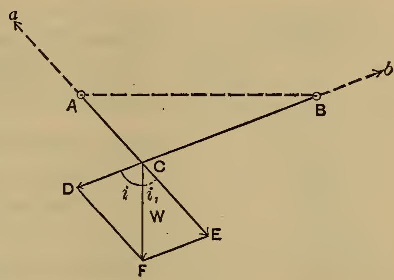


FIG. 69.

The force acting on CB at B is equal and opposite to CD, while that acting on CA at A is equal and opposite to CE.

To compute these stresses analytically, we have

$$CE = CF\frac{\sin CFE}{\sin CEF} = W\frac{\sin i}{\sin(i + i_1)},$$

$$CD = CF\frac{\sin CFD}{\sin CDF} = W\frac{\sin i_1}{\sin(i + i_1)}.$$

CASE II. *Two Struts* (Fig. 70). — Let the load be represented graphically by $CF = W$. Then will the components CD and CE represent the thrusts in the struts CB and CA respectively, and the re-actions of the supports at B and A will be equal and opposite to them. For analytical solution, we derive from the figure

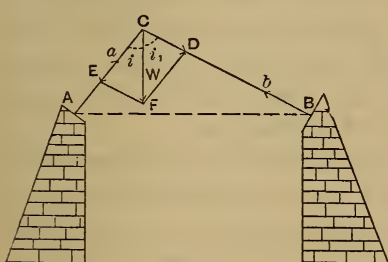


FIG. 70.

$$CE = W\frac{\sin i_1}{\sin(i + i_1)}, \qquad CD = W\frac{\sin i}{\sin(i + i_1)}.$$

CASE III. *A Strut and a Tie* (Fig. 71). — Let the load be represented graphically by $CF = W$. Resolve it, as before, into components along the members of the truss. Then will

CE represent the tension in the tie AC, and CD will represent the thrust in the strut BC; and we may deduce the analytical formulæ as before.

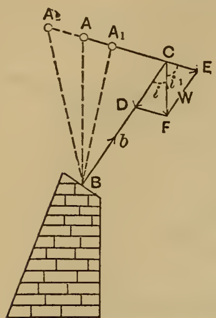


FIG. 71.

§ 119. **Stability for Lateral Deviations.** — In Case I, if the joint C be moved a little out of the plane of the paper, the load at C has such a direction that it will cause the truss to rotate around AB so as to return to its former position; hence such a frame is stable as regards lateral deviations.

In Case II the effect of the load, if C were moved a little out of the plane of the paper, would be to cause rotation in such a way as to overturn the truss; hence such a frame is unstable as regards lateral deviations.

In Case III the stability for lateral deviations will depend upon whether the load $CF = W$ is parallel to AB, is directed away from it or towards it. If the first is the case (i.e., if A is the point of suspension of the tie), the frame is neutral, as the load has no effect, either to restore the truss to its former position, or to overturn it; if the second is the case (i.e., if A_1 is the point of suspension of the tie), the truss is stable; and, if the third is the case (i.e., if A_2 is the point of suspension of the tie), it is unstable as regards lateral deviations.

§ 120. **General Methods for Determining the Stresses in Trusses.** — In the determination of the stresses as above, it would have been sufficient to construct only the triangle CFD by laying off $CF = W$ to scale, and then drawing CD parallel to CB, and FD parallel to CA, and the triangle CFD would have given us the complete solution of the problem. Moreover, the determination of the supporting forces of any truss, and of the stresses in the several members, is a question of equilibrium.

Adopting the following as definitions, viz., —

External forces are the loads and supporting forces,

Internal forces are the stresses in the members: we must have

1°. The external forces must form a balanced system; i.e., the supporting forces must balance the loads.

2°. The forces (external and internal) acting at each joint of the truss must form a balanced system; i.e., the external forces (if any) at the joint must be balanced by the stresses in the members which meet at that joint.

3°. If any section be made, dividing the truss into two parts, the external forces which act upon that part which lies on one side of the section, must be balanced by the forces (internal) exerted by that part of the truss which lies on the other side of the section, upon the first part.

The above three principles, the triangle, and polygon of forces, and the conditions of equilibrium for forces in a plane, enable us to determine the stresses in the different members of roof and bridge trusses.

§ 121. **Triangular Frame.** — Given the triangular frame ABC (Fig. 72), and given the load W at C in magnitude and direction, given also the direction of the supporting force at B, to find the magnitude of this supporting force, the magnitude and direction of the other supporting force, and the stresses in the members.

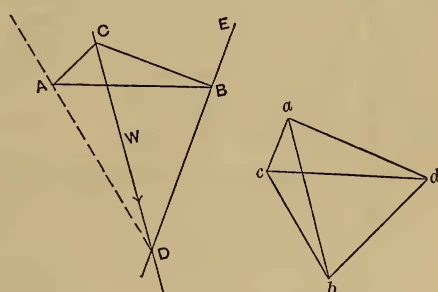


Fig. 72.

Solution. — Join A with D, the point of intersection of the line of direction of the load and the line BE. Then will DA be the direction of the other supporting force; for the three external forces, in order to form a balanced system, must meet in a point, except when they are parallel. Then draw ab to scale, parallel to CD and equal to W. From

a draw ac parallel to BD, and from b draw bc parallel to AD; then will the triangle $abca$ be the triangle of external forces, the sides ab, bc, and ca, taken in order, representing respectively the load W, the supporting force at A, and the supporting force at B.

Then from a draw ad parallel to BC, and from c draw cd parallel to AB; then will the triangle acd be the triangle of forces for the joint B, and the sides ca, ad, and dc, taken in order, will represent respectively the supporting force at B, the force exerted by the bar BC at the point B, and the force exerted by the bar AB at the point B.

Since, therefore, the force ad exerted by the bar CB at B is directed *away from* the bar, it follows that CB is in compression; and, since the force dc exerted by the bar AB at B is directed *towards* the bar, it follows that AB is in tension.

In the same way bdc is the triangle of forces for the point A; the sides bc, cd, and db representing respectively the supporting force at A, the force exerted by the bar AB at A, and the force exerted by the bar AC at A.

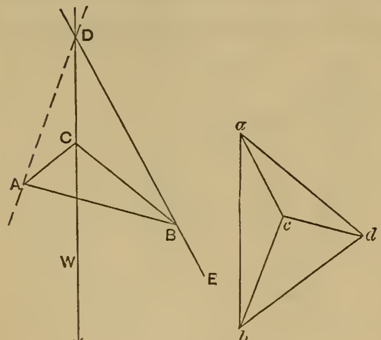


Fig. 73.

The bar AB is again seen to be in tension, as the force cd exerted by the bar AB at A is directed towards the bar.

So likewise the triangle abd is the triangle of forces for the point C.

Fig. 73 shows the case when the supporting forces meet the load-line above, instead of below, the truss.

§ 122. **Triangular Frame with Load and Supporting Forces Vertical.** — Fig. 74 shows the construction when the load and also the supporting forces are vertical. In this case

the diagram becomes very much simplified, the triangle of external forces *abd* becoming a straight line. The diagram is otherwise constructed just like the last one.

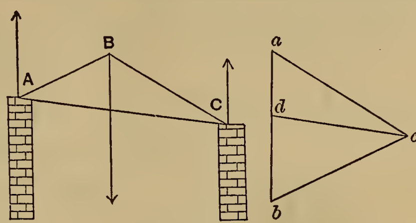


Fig. 74.

§ 123. **Bow's Notation.** — The notation devised by Robert H. Bow very much simplifies the construction of the stress diagrams of roof-trusses.

This notation is as follows: Let the radiating lines (Fig. 75) represent the lines of action of a system of forces in equilibrium, and let the polygon *abcdefa* be the polygon representing these forces in magnitude and direction; then denote the sides of the polygon in the ordinary way, by placing small letters at the vertices, but denote the radiating lines by capital letters placed in the angles. Thus the line *AB* is the line of direction of the force *ab*, etc. In applying the notation to roof-trusses, we letter the truss with capital letters in the spaces, and the stress diagram with small letters at the vertices. If, then, in drawing the polygon of equilibrium for any one joint of the truss, we take the forces always in the same order, proceeding always in right-handed or always in left-handed rotation, we shall be led to the simplest diagrams. Hereafter this notation will be used exclusively in determining the stresses in roof-trusses.

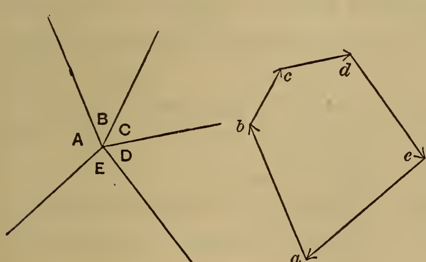


Fig. 75.

§ 124. **Isosceles Triangular Frame: Concentrated Load** (Fig. 76.) — Let the load W act at the apex, the supporting

forces being vertical; each will be equal to $\frac{1}{2}W$: hence the polygon of external forces will be the triangle *abc*, the sides of which, *ab*, *bc*, and *ca*, all lie in one straight line. Then begin at the left-hand support, and proceed again in right-handed rotation, and we have as the triangle of forces at this joint *cad*, the forces *ca*, *ad*, and *dc*, these being respectively the supporting force, the stress in *AD*, and that in *DC*; the directions of these forces being indicated by the order in which the letters follow each other: thus, *ca* is an upward force, *ad* is a downward force; and, this being the force exerted by the bar *AD* at the left-hand support, we conclude that the bar *AD* is in compression. Again: *dc* is directed towards the right, or towards the bar itself, and hence the bar *DC* is in tension. The triangle of forces for the other support is *bcd*, and that for the apex *abd*.

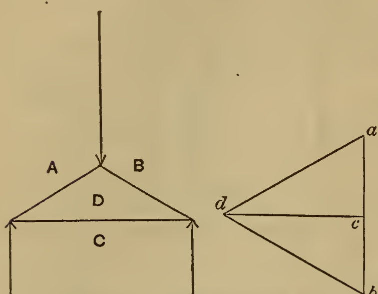


FIG. 76.

§ 125. **Isosceles Triangular Frame: Distributed Load.** — Let the load W be uniformly distributed over the two rafters *AF* and *FB* (Fig. 77); then will these two rafters be subjected to a direct stress, and also to a bending action: and if we resolve the load on each rafter into two components at the ends of the rafter, then, considering these components as the loads at the joints, we shall determine correctly by our diagram the direct stresses in all the bars of the truss.

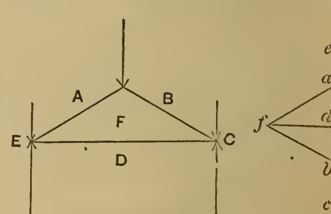


FIG. 77.

The load distributed over *AF* is $\frac{W}{2}$; and of this, one-half is

the component at the support, and one-half at the apex, and similarly for the other rafter. This gives as our loads, $\frac{W}{4}$ at each support, and $\frac{W}{2}$ at the apex. The polygon of external forces is *eabcde*, where the sides are as follows:—

$$ea = \frac{W}{4}, \quad ab = \frac{W}{2}, \quad bc = \frac{W}{4}, \quad cd = \frac{W}{2}, \quad de = \frac{W}{2}.$$

Then, beginning at the left-hand support, we shall have for the polygon of forces the quadrilateral *deafd*, where $de = \frac{W}{2} =$ supporting force, $ea = \frac{W}{4} =$ downward load at support, $af =$ stress in AF (compression), $fd =$ stress in FD (tension). The polygon for the apex is *abf*, and that for the right-hand support *cdfbc*.

§ 126. **Polygonal Frame.** — Given a polygonal frame (Fig. 78) formed of bars jointed together at the vertices of the angles, and free to turn on these joints, it is evident, that, in order that the frame may retain its form, it is necessary that the directions of, and the proportions between, the loads at the different joints, should be specially adapted to the given form: otherwise the frame will change its form. We will proceed to solve the following problem: Given the form of the frame, the magnitude of one load as AB, and the direction of all the external forces (loads and supporting forces) except one, we shall have sufficient data to determine the magnitudes of all,

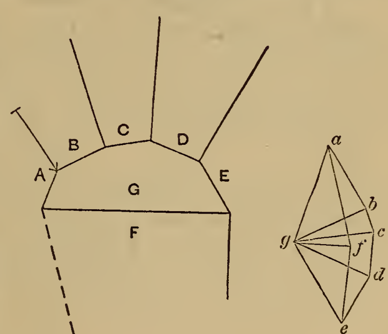


FIG. 78.

and the direction of the remaining external forces, and also the stresses in the bars

Let the direction of all the loads be given, and also that of the supporting force *EF*, that of the supporting force *AF* being thus far unknown; and let the magnitude of *AB* be given. Then, beginning at the joint *ABG*, we have for triangle of forces *abg* formed by drawing *ab* || and = *AB*, then drawing *ga* || *AG*, and *bg* || *BG*; *ga* and *bg* both being thrusts. Then, passing to the joint *BCG*, we have the thrust in *BG* already determined, and it will in this case be represented by *gb*. If, now, we draw *bc* || *BC*, and *gc* || *GC*, we shall have determined the load *BC* as *bc*, and we shall have *cg* and *gb* as the thrusts in *CG* and *GB* respectively. Continuing in the same way, we obtain the triangles *gcd*, *gde*, and *gfe*, thus determining the magnitudes of the loads *cd*, *de*, and of the supporting force *ef*; and then the triangle *gaf*, formed by joining *a* and *f*, gives us *af* for the magnitude and direction of the left-hand support. The polygon *abcdefa* of external forces is called the *Force Polygon*, while the frame itself is called the *Equilibrium Polygon*.

§ 127. **Polygonal Frame with Loads and Supporting Forces Vertical.** — In this case (Fig. 79) we may give the form of the frame and the magnitude of one of the loads, to determine the other loads and the supporting forces, and also the stresses in the bars; or we may give the form of the frame and the magnitude of the resultant of the loads, to find the loads and supporting forces. In the former case let the load *AB* be given. Then, proceeding in the same way as before, we find the diagram of Fig. 79; the polygon of external forces *abcdefa* falling all in one straight line.

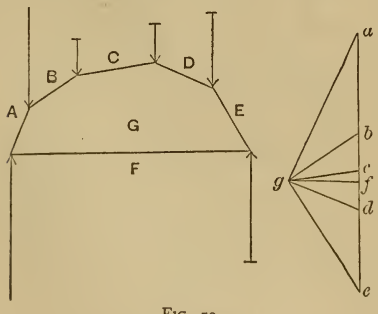


FIG. 79.

If, on the other hand, the whole load ae be given, we observe that this is borne by the stresses in the extreme bars AG and GE; hence, drawing $ag \parallel AG$, and $eg \parallel EG$, we find eg and ga as the stresses in EG and GA respectively. Then, proceeding to the joint ABG, we find, since ga is the force exerted by GA at this point, that, drawing $gb \parallel GB$, we shall have ab as the part of the load acting at the joint ABG, etc.

§ 128. **Funicular Polygon.** — If the frame of Fig. 79 be inverted, we shall have the case of Fig. 80, where all the bars, except FG, are subjected to tension; FG itself being subjected to compression. The construction of the diagram of stresses being entirely similar to that already explained for Fig. 79, the explanation will not be repeated here. If the compression piece be omitted, the case becomes that of a chain hung at the upper joints (the supporting forces then becoming identical with the tensions in the two extreme bars), the line gf would then be omitted from the diagram, and the polygon of external forces would become $abcdega$.

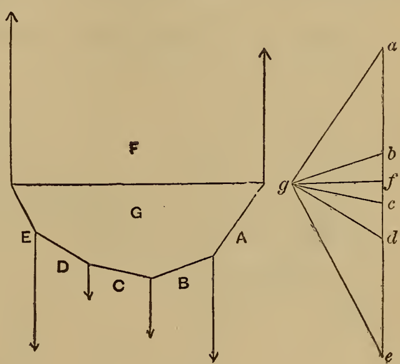


FIG. 80.

§ 129. **Triangular Truss: Wind Pressure.** — Inasmuch as the pressure of the wind upon a truss is assumed to be normal to the rafter on which it blows, we will next consider the case of a triangular truss with the load distributed over one rafter only, and normal to the rafter.

There may be three cases: —

1°. When there is a roller under one end, and the wind blows from the other side.

2°. When there is a roller under one end, and the wind blows from the side of the roller.

3°. When there is no roller under either end.

The last arrangement should always be avoided except in small and unimportant constructions; for the presence of a roller under one end is necessary to allow the truss to change its length with the changes of temperature, and to prevent the stresses that would occur if it were confined.

CASE I. — Using Bow's notation, we have (Fig. 81) the whole load represented in the diagram by *db*. Its resultant acts at the middle of the rafter *AE*, whereas the supporting force at the right-hand end is (in consequence of the presence of the roller) vertical. Hence, to find the line of action of the other supporting force, produce the line of action of the load till it meets a vertical line drawn through the roller, and join their point of intersection with the support where there is no roller. We thus obtain *CD* as the line of action of the left-hand support.

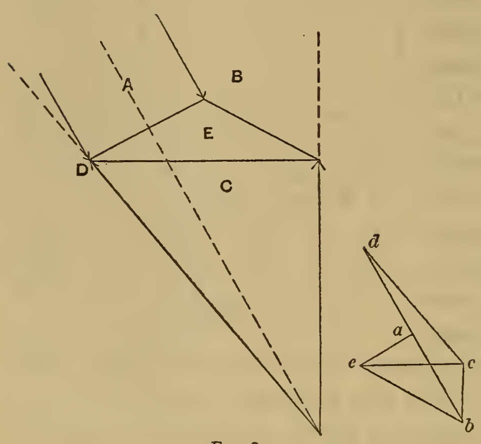


FIG. 81.

We can now determine the magnitude of the supporting forces *bc* and *cd* by constructing the triangle *bcdb* of external forces.

Now resolve the normal distributed force *db* into two single forces (equal to each other in this case), *da* and *ab* respectively, acting at the left-hand support and at the apex.

Now proceed to the left-hand support. We find four forces in equilibrium, of which two are entirely known; viz., *cd* and *da:* hence, constructing the quadrilateral *cdaec*, we have *ae* as the thrust in *AE*, and *ec* as the tension in *EC*.

Next proceed to the apex, and construct the triangle of equilibrium *abea*, and we obtain *be* as the thrust in *BE*.

The triangle *bceb* is then the triangle of equilibrium for the right-hand support.

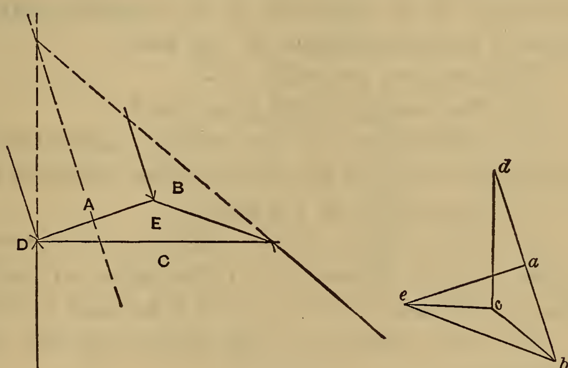


Fig. 82.

Case II. — In this case (Fig. 82) we follow the same method of procedure, only the point of intersection of the load and supporting forces is above, instead of below, the truss. The figure explains itself so fully that it is unnecessary to explain it here.

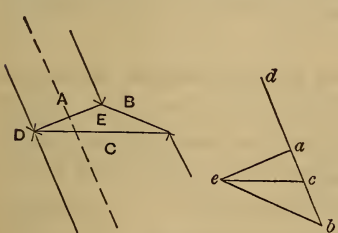


Fig. 83.

Case III. — In this case the supports are capable of exerting resistance in any direction whatever; so that, if any circumstance should determine the direction of one of them, that of the other would be determined also. When there is no such circumstance, it is customary to assume them parallel to the load (Fig. 83). Making this assumption, we begin by dividing the line *db*, which represents the load, into two parts, inversely

proportional to the two segments into which the line of action of the resultant of the load (the dotted line in the figure) divides the line EC. We thus obtain the supporting forces bc and cd, and $bcdb$ is the triangle of external forces. We then follow the same method as in the preceding cases.

§ 130. **General Determination of the Stresses in Roof-Trusses.** — In order to compute the stresses in the different members of a roof-truss, it is necessary first to know the amount and distribution of the load.

This consists generally of —

1°. The weight of the truss itself.

2°. The weight of the purlins, jack-rafters, and superincumbent roofing, as the planks, slate, shingles, felt, etc.

3°. The weight of the snow.

4°. The weight of the ceiling of the room immediately below if this is hung from the truss, or the weight of the floor of the loft, and its load, if it be used as a room.

5°. The pressure of the wind; and this may blow from either side.

6°. Any accidental load depending on the purposes for which the building is used. As an instance, we might have the case where a system of pulleys, by means of which heavy weights are lifted, is attached to the roof.

In regard to the first two items, and the fourth, whenever the construction is of importance, the actual weights should be determined and used. In so doing, we can first make an approximate computation of the weight of the truss, and use it in the computation of the stresses; the weights of the ceiling or of the floor below being accurately determined. After the stresses in the different members have been ascertained by the use of these loads, and the necessary dimensions of the members determined, we should compute the actual weight of the truss; and if our approximate value is sufficiently different from the true value to warrant it, we should compute again

the stresses. This second computation will, however, seldom be necessary.

In making these computations, the weights of a cubic foot of the materials used will be needed; and average values are given in the following table with sufficient accuracy for the purpose.

Weight of some Building Materials per Cubic Foot.	Pounds.
Timber.	
Chestnut	41
Hemlock	25
Maple	41
Oak, live	59
Oak, white	49
Pine, white	25 to 30
Pine, yellow, Southern	45
Spruce	25 to 30
Iron.	
Iron, cast	450
Iron, wrought	480
Steel	490
Other Substances.	
Asphaltum	80 to 90
Mortar, hardened	103
Snow, freshly fallen	5 to 12
Snow, compacted by rain	15 to 50
Slate	140 to 180

Weight of Slating per Square Foot. According to Trautwine.	Pounds.
$\frac{1}{8}$ inch thick on laths	4.75
$\frac{1}{8}$ " " " 1-inch boards	6.75
$\frac{1}{8}$ " " " $1\frac{1}{4}$ " "	7.30
$\frac{3}{16}$ " " " laths	7.00
$\frac{3}{16}$ " " " 1-inch boards,	9.00
$\frac{3}{16}$ " " " $1\frac{1}{4}$ " "	9.55
$\frac{1}{4}$ " " " laths	9.25
$\frac{1}{4}$ " " " 1-inch boards,	11.25
$\frac{1}{4}$ " " " $1\frac{1}{4}$ " "	11.80
With slating-felt add	$\frac{1}{4}$ lb.
With $\frac{1}{4}$-inch mortar add	3 lbs.

Number of Nails in One Pound.	No.
3-penny	450
4 "	340
6 "	150
8 "	100
10 "	60
12 "	40
20 "	25

As to the weight of the snow upon the roof, Stoney recommends the use of 20 pounds per square foot in moderate climates; and this would seem to the writer to be borne out by the experiments of Trautwine as recorded in his handbook,

although Trautwine himself considers 12 pounds per square foot as sufficient.

§ 131. **Wind Pressure.** — The wind pressure, as has been stated, is not a vertical force, but is always assumed as acting normal to the roof on that side from which the wind blows. In order to determine the magnitude of this pressure, it has been advocated by some, to resolve the force of the wind into two components, one acting along, and one at right angles to, the roof. This method is, however, incorrect, because the moving air, striking against other particles of air, is deviated from its course, and the air, after striking the roof, cannot escape freely, but meets with the resistance of the surrounding air. Hence we must resort to experiment to determine the amount of the pressure due to the wind. The experiments of Hutton, according to Greene, give, for the pressure upon a surface whose inclination to the horizon is i, the value

$$P \sin i^{1.84 \cos i - 1}$$

where P is the pressure which would be exerted on the surface if normal to the wind; and, taking the maximum force of the wind as 40 pounds per square foot, he gives for the normal pressures on surfaces inclined to the horizon the following: —

Inclination.	Normal Pressure.	Inclination.	Normal Pressure.
5°	5.2	35°	30.1
10°	9.6	40°	33.4
15°	14.0	45°	36.1
20°	18.3	50°	38.1
25°	22.5	55°	39.6
30°	26.5	60°	40

and for all steeper angles 40.

Later experiments show that these numbers are too small: stresses of 70 lbs. per square foot have been registered on vertical surfaces; but we have no systematic set of tests on the subject.

§ 132. **Approximate Estimation of the Load.** — In all important constructions, the estimates of the loads should be made as above described. For smaller constructions, and for the purposes of a preliminary computation in all cases, we only estimate the roof-weight roughly; and some authors even assume the wind pressure as a vertical force.

Trautwine recommends the use of the following figures for the total load per square foot, including wind and snow, when the span is 75 feet or less: —

Roof covered with corrugated iron, unboarded . . .	28 lbs.
Roof plastered below the rafters	38 "
Roof, corrugated iron on boards	31 "
Roof plastered below the rafters	41 "
Roof, slate, unboarded or on laths	33 "
Roof, slate, on boards $1\frac{1}{4}$ inches thick	35 "
Roof, slate, if plastered below the rafters	46 "
Roof, shingles on laths	30 "
Roof plastered below rafters or below tie-beam . . .	40 "

From 75 to 100 feet, add 4 lbs. to each.

§ 133. **Distribution of the Loads.** — The methods for determining the stresses, which will be used here, give correct results only when the loads are concentrated at joints, and are not distributed over any members of the truss.

In constructions of importance, this concentration of the loads at the joints should always be effected if possible; because, when this is the case, the stresses in the members of the truss can be, if properly fitted, obliged to resist only stresses of direct tension, or of direct compression; but, when there is a load distributed over any member of the truss, that member, in addition to the direct compression or direct tension, is subjected to a bending-stress. The effect of this bending

cannot be discussed until we have studied the theory of beams. Nevertheless, it is a fact that we have no experimental knowledge of the behavior of a piece under combined compression and bending; and we know that when certain pieces are to resist bending, in addition to tension, they must be made much larger in proportion than is necessary when they bear tension only.

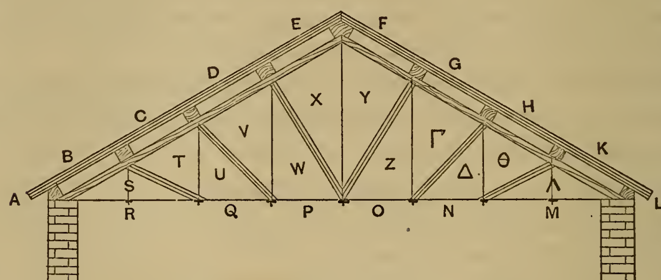


FIG. 84.

The manner in which this concentration of the loads is effected, is shown in Fig. 84, which is intended to represent one of a series of trusses that supports a roof, the rafters being the two lower ones in the figure. Resting on two consecutive trusses, and extending from one to the other, are beams called purlins, which should be placed only above the joints of the truss, and which are shown in cross-section in the figure. On these purlins are supported the jack-rafters parallel to the rafters, and at sufficiently frequent intervals to support suitably the plank and superincumbent roofing-materials.

By this means we secure that the entire bending-stress comes upon the jack-rafters and purlins, and that the rafters proper are subjected only to a direct compression. When, however, the load is distributed, i.e., when the roofing rests directly on the rafters, or when the purlins are placed at points other than the joints, the bending-stress should be taken into account; and while the methods to be developed here will give the stresses

in all the members that are not subjected to bending, the bending-stress may be largely in excess of the direct stress in those pieces that are subjected to bending. How to take this into account will be explained later.

Another important item to consider is, that, in constructions of importance, a roller should be placed under one end of the truss to allow it to move with the change of temperature; as otherwise some of the members will be either bent, or at least subjected to initial stresses. The presence of a roller obliges the supporting force at that point to be vertical, whether the load be vertical or inclined.

It is customary, and does not entail any appreciable error, to consider the weight of the truss itself, as well as that of the superincumbent load, as concentrated at the upper joints; i.e., those on the rafters.

In the case of a ceiling on the room below, or of a loft whose floor rests on the lower joints, we must, of course, account the proper load as resting on the lower joints.

§ 134. **Direct Determination of the Stresses.** — This, as we have seen, is merely a question of equilibrium of forces in a plane, where certain forces acting are known, and others are to be determined.

As to the methods of solution, we might adopt —

1°. A graphical solution, laying off the loads to scale, and constructing the diagram by the use of the propositions of the polygon, and the triangle of forces, and then determining the results by measuring the lines representing the stresses to the same scale.

2°. An analytical solution, imposing the analytical conditions of equilibrium, as given under the "Composition of Forces," between the known and unknown forces.

3°. A third method is to construct the diagram as in the graphical solution, but then, instead of determining the results by measuring the resulting lines to scale, to compute the un-

known from the known lines of the diagram by the ordinary methods of trigonometry.

The first, or purely graphical, method, is one which has received a very large amount of attention of late years, and in which a great deal of progress has been made. It is, doubtless, very convenient for a skilled draughtsman, and especially convenient for one who, though skilled in draughting, is not free with trigonometric work; but it seems to me, that, when the results are determined by computation from the diagram, there is less chance of a slight error in some unfavorable triangle vitiating all the results. I am therefore disposed to recommend for roof-trusses the third method.

In the case of bridge-trusses, on the other hand, I believe the graphical not to be as convenient as a purely analytic method.

§ 135. **Roof-Trusses.** — In what follows, the graphical solutions will be explained with very little reference to the trigonometric work, as that varies in each special case, and to one who has a reasonable familiarity with the solution of plane triangles, it will present no difficulty; whereas to deduce the formulæ for each case would complicate matters very much. Proceeding to special examples, let us take, first, the truss shown in Fig. 85, and let the vertical load upon it be W uniformly distributed over the top of the roof, the purlins being at the joints on the rafters.

The loads at the several joints will then be as follows, viz. (Fig. 85*a*), —

$$ab = kl = \frac{W}{16}, \quad bc = cd = de = ef = fg = gh = hk = \frac{W}{8}.$$

Then the supporting forces will be

$$lm = ma = \frac{W}{2}.$$

We thus have, as polygon of external forces, *abcdefghklma*.

Now proceed to either support, say, the left-hand one; and there we have the two forces *ab* and *ma* known, while *by* and *ym* are unknown. We then construct the quadrilateral *maby* in the figure, and thus determine *by* and *ym*. As to whether

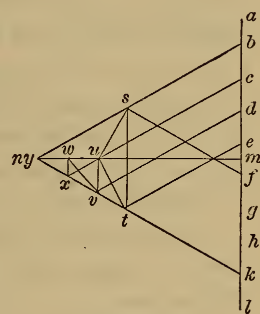


FIG. 85*a*.

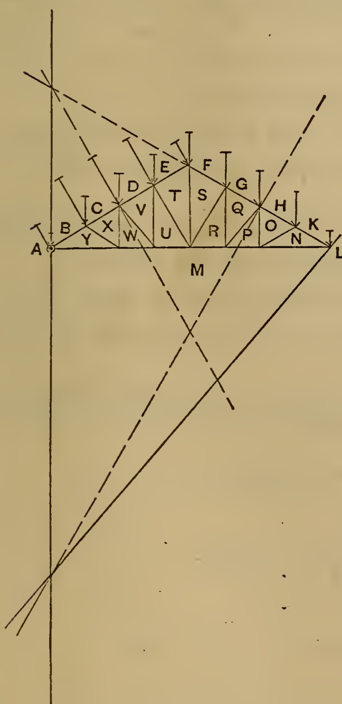


FIG. 85.

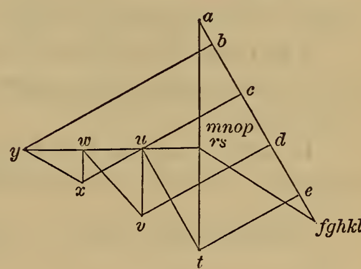


FIG. 85*b*.

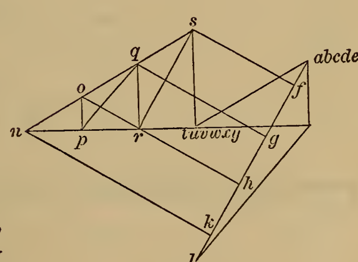


FIG. 85*c*.

these represent thrust or tension, we need only remember that they are the forces exerted by the respective bars at the joints: and, since *by* is directed away from the bar *BY*, this bar is in compression; whereas, *ym* being directed towards the bar *YM*, that bar is in tension.

Having determined these two stresses, we next proceed to another joint, where we have only two unknown forces. Take the joint at which the load bc acts, and we have as known quantities the load bc, and also the force exerted by the bar YB, which is in compression. This force is now directed away from the bar, and hence is represented by yb. The unknown forces are the stresses in CX and XY. Hence we construct the quadrilateral $cxybc$; and we thus determine the stresses in CX and XY as cx and xy, both being thrusts.

Next proceed to the joint YXW, and construct the quadrilateral $myxwm$, and thus determine the tension xw and the tension wm.

Next proceed to the joint where cd acts, and so on. We thus obtain the diagram (Fig. 85a) giving all the stresses.

The truss in the figure was constructed with an angle of 30° at the base, and hence gives special values in accordance with that angle.

In order to show how we may compute the stresses from the diagram, the computation will be given.

From triangle bmy, we have $bm = \frac{7}{16} W$

$$\therefore \quad ym = \frac{7}{16} W \cot 30° = \frac{7\sqrt{3}}{16} W$$

$$by = \frac{7}{16} W \operatorname{cosec} 30° = \frac{7}{8} W = ky.$$

From the triangle umc, we have $cm = \frac{5}{16} W$,

$$um = \frac{5\sqrt{3}}{16} W,$$

$$yw = \frac{1}{2} yu = \frac{1}{2}\left(\frac{2}{16} W\sqrt{3}\right) = \frac{\sqrt{3}}{16} W,$$

$$yx = yw \sec 30^\circ = \left(\frac{\sqrt{3}}{16} W\right)\frac{2}{\sqrt{3}} = \frac{W}{8} = xv = vt,$$

$$xw = \tfrac{1}{2} yx = \frac{W}{16}, \qquad uv = \frac{W}{8}, \qquad st = 2\left(\frac{3}{16} W\right) = \frac{3}{8} W,$$

$$wv = \sqrt{(wu)^2 + (uv)^2} = W\sqrt{\frac{3}{256} + \frac{4}{256}} = \frac{\sqrt{7}}{16} W,$$

$$ut = \sqrt{(wu)^2 + (\tfrac{1}{2}st)^2} = W\sqrt{\frac{3}{256} + \frac{9}{256}} = \frac{\sqrt{3}}{8} W,$$

$$cx = wm \sec 30^\circ = \left(\frac{3\sqrt{3}}{8} W\right)\frac{2}{\sqrt{3}} = \frac{3}{4} W,$$

$$vd = um \sec 30^\circ = \left(\frac{5\sqrt{3}}{16} W\right)\frac{2}{\sqrt{3}} = \frac{5}{8} W,$$

$$et = \left(\frac{\sqrt{3}}{4} W\right)\frac{2}{\sqrt{3}} = \frac{1}{2} W.$$

Hence we shall have for the stresses, —

RAFTERS (compression).

by	$= kn$	$= \frac{7}{8} W.$
cx	$= ho$	$= \frac{3}{4} W.$
dv	$= gq$	$= \frac{5}{8} W.$
ct	$= fs$	$= \frac{1}{2} W.$

HORIZONTAL TIES (tension).

my	$= mn$	$= \frac{7\sqrt{3}}{16} W.$
mw	$= mp$	$= \frac{3\sqrt{3}}{8} W.$
mu	$= mr$	$= \frac{5\sqrt{3}}{16} W.$

VERTICALS (tension).

xw	$= op$	$= \frac{W}{16}.$
vn	$= qr$	$= \frac{W}{8}.$
ts	$=$	$\frac{3}{8} W.$

DIAGONAL BRACES (compression).

xy	$= ou$	$= \frac{W}{8}.$
wv	$= qp$	$= \frac{\sqrt{7}}{16} W.$
tu	$= sr$	$= \frac{\sqrt{3}}{8} W.$

Next, as to the stresses due to wind pressure, we will suppose that there is a roller under the left-hand end of the truss, and none under the right-hand end; and we will proceed to determine the stresses due to wind pressure.

First, suppose the wind to blow from the left-hand side of the truss, and let the total wind pressure be (Fig. 85*b*) $af = W_1$. The resultant, of course, acts along the dotted line drawn perpendicular to the left-hand rafter at its middle point, as shown in Fig. 85.

The left-hand supporting force will be vertical: hence, producing the above-described dotted line, and a vertical through the roller to their intersection, and joining this point with the right-hand end of the truss, we have the direction of the right-hand supporting force. In this case, since the angle of the truss is 30°, the line of action of the right-hand supporting force coincides in direction with the right-hand rafter. We now construct the triangle of external forces *afm*, and we obtain the supporting forces *fm* and *ma*. We then have, as the loads at the joints,

$$ab = \frac{W_1}{8} = ef,$$

$$bc = \frac{W_1}{4} = cd = de.$$

Then proceed as before to the left-hand joint; and we find that two of the four forces acting there are known, viz., *ma* and *ab*, and two are unknown, viz., the stresses in *BY* and *YM*. Then construct the quadrilateral *mabym*, and we have the stresses *by* and *ym*; the first being compression and the second tension, as shown by reasoning similar to that previously adopted.

Then pass to the next joint on the rafter, and construct the quadrilateral *ybcxy*, where *yb* and *bc* are already known, and we obtain *cx* and *xy*; and so proceed as before from joint to joint,

remembering, that, in order to be able to construct the polygon of forces in each case, it is necessary that only two of the forces acting should be unknown.

When the wind blows from the other side, we shall obtain the diagram shown in Fig. 85*c*.

After having determined the stresses from the vertical load diagram and those from the two wind diagrams, we should, in order to obtain the greatest stress that can come on any one member of the truss, add to the stress due to the vertical load the greater of the stresses due to the wind pressure.

§ 136. **Roof-Truss with Loads at Lower Joints.** — In Fig. 86 is drawn a stress diagram for the truss shown in Fig. 84 on the supposition that there is also a load on the lower joints. In this case let W be the whole load of the truss, except the ceiling, and W_1 the weight of the ceiling below; the latter being supported at the lower joints and on the two extreme vertical suspension rods. Then will the loads at the joints be as follows; viz., —

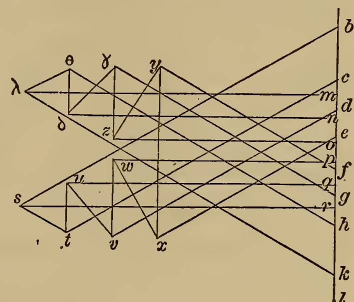


Fig. 86.

$$ab = \tfrac{1}{16}(W + W_1) = kl,$$

$$bc = \tfrac{1}{8}(W + W_1) = hk,$$

$$cd = \tfrac{1}{8}W = gh = de = fg = ef,$$

$$mn = \tfrac{1}{8}W_1 = rq = on = qp = op.$$

Observe that there is no joint at the lower end of either of the end suspension rods, but that whatever load is supported by these is hung directly from the upper joints, where bc and hk act.

We have also for each of the supporting forces lm and ra

$$\tfrac{1}{2}(W + W_1).$$

Hence we have, for the polygon of external forces,

$$abcdefghklmnopqra,$$

which is all in one straight line, and which laps over on itself.

In constructing the diagram, we then proceed in the same way as heretofore.

§ 137. **General Remarks.** — As to the course to be pursued in general, we may lay down the following directions: —

1°. *Determine all the external forces; in other words, the loads being known, determine the supporting forces.*

2°. *Construct the polygon of forces for each joint of the truss, beginning at some joint where only two of the forces acting at that joint are unknown.* This is usually the case at the support. Then proceed from joint to joint, bearing in mind that we can only determine the polygon of forces when the magnitudes of all but two sides are known.

3°. *Adopt a certain direction of rotation, and adhere to it throughout; i.e., if we proceed in right-handed rotation at one joint, we must do the same at all, and we shall thus obtain neat and convenient figures.*

4°. *Observe that the stresses obtained are the forces exerted by the bars under consideration, and that these are thrusts when they act away from the bars, and tensions when they are directed towards the bars.*

We will next take some examples of roof-trusses, and construct the diagrams of some of them, calling attention only to special peculiarities in those cases where they exist.

It will be assumed that the student can make the trigonometric computations from the diagram.

The scale of load and wind diagram will not always be the same; and the stress diagrams will in general be smaller than is advisable in using them, and very much too small if the

results were to be obtained by a purely graphical process without any computation.

The loads will in all cases be assumed to be distributed uniformly over the jack-rafters, or, in other words, concentrated at the joints.

Those cases where no stress diagram is drawn may be considered as problems to be solved.

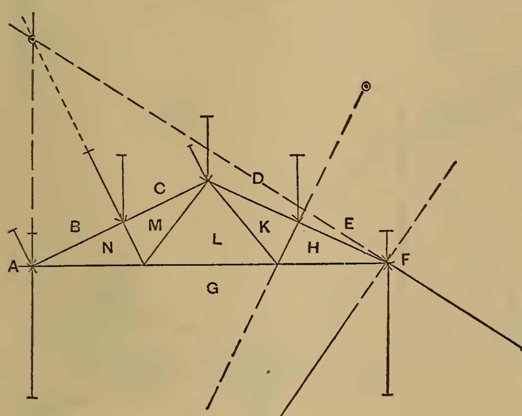


Fig. 87.

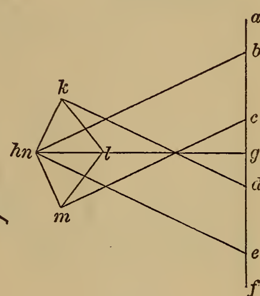


Fig. 87*a*.

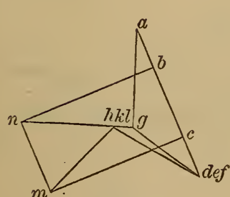


Fig. 87*b*.

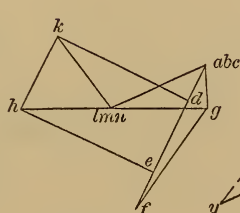


Fig. 87*c*.

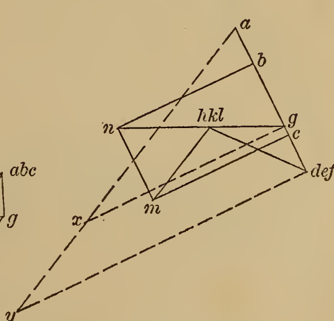


Fig. 87*d*.

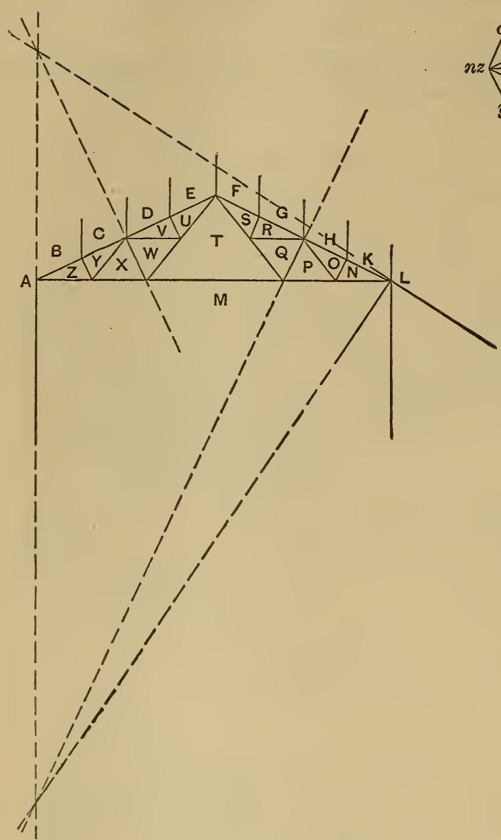


FIG. 88.

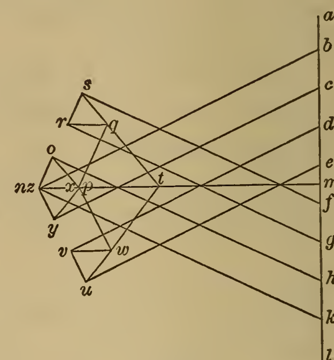


FIG. 88*a*.

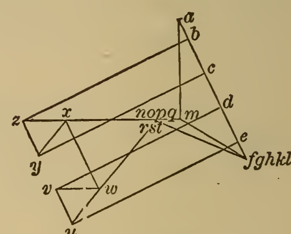


FIG. 88*b*.

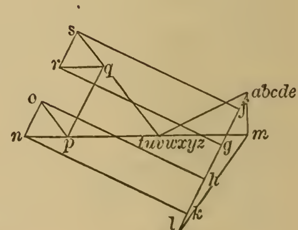


FIG. 88*c*.

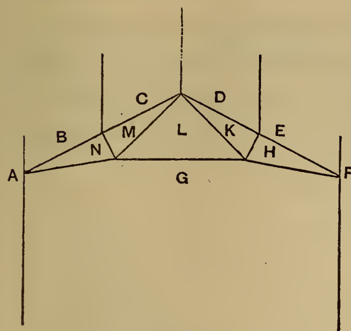


FIG. 89.

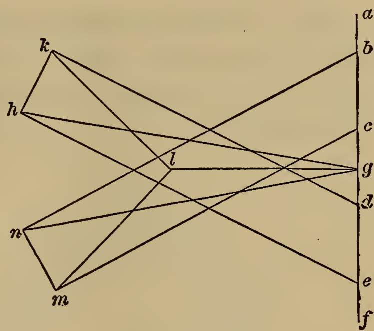


FIG. 89a.

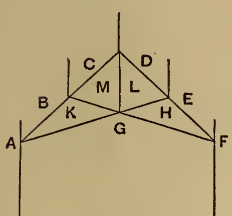


FIG. 90.

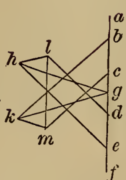


FIG. 90a.

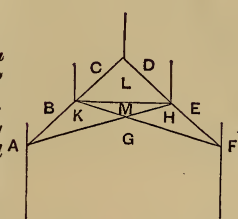


FIG. 91.

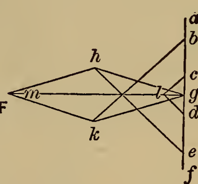


FIG. 91a.

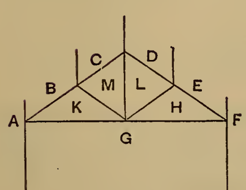


FIG. 92.

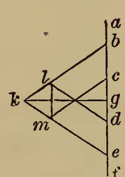


FIG. 92a.

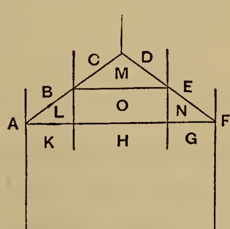


FIG. 93.

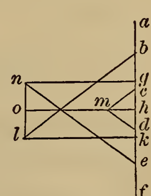


FIG. 93a.

§ 138. **Hammer-Beam Truss** (Fig. 94). — This form of truss is frequently used in constructions where architectural effect is the principal consideration rather than strength. It is not an advantageous form from the point of view of strength,

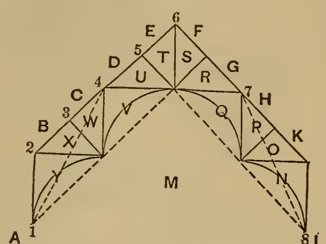


FIG. 94.

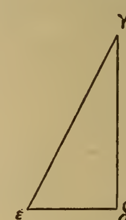


FIG. 94*a*.

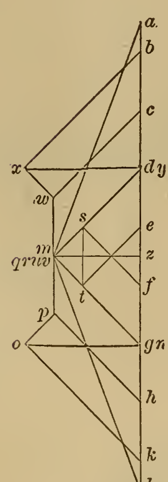


FIG. 94*b*.

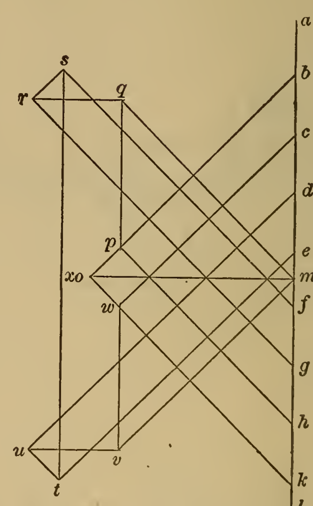


FIG. 94*c*.

for the absence of a tie-rod joining the two lower joints causes a tendency to spread out at the base, which tendency is usually counteracted by the horizontal thrust furnished by the buttresses against which it is supported.

When such a thrust is furnished (or were there a tie-rod), and the load is symmetrical and vertical, the bars are not all needed, and some of them are left without any stress. In the case in hand, it will be found that UV, VM, MQ, and QR are not needed. We must also observe that the effect of the curved members MY, MV, MQ, and MN on the other parts of the truss is just the same as though they were straight, as shown in the dotted lines. The curved piece, of course, has to be subjected to a bending-stress in order to resist the stress acting upon it. If, as is generally the case, the abutments are capable of furnishing all the horizontal thrust needed, it will first be necessary to ascertain how much they will be called upon to furnish. To do this, observe that we have really a truss similar to that shown in Fig. 92, supported on two inclined framed struts, of which the lines of resistance are the dotted lines (Fig. 94) 1 4 and 7 8, and that, under a symmetrical load, this polygonal frame will be in equilibrium, and, moreover, the curved pieces MV and MQ will be without stress, these only being of use to resist unsymmetrical loads, as the snow or wind.

Let the whole load, concentrated by means of the purlins at the joints of the rafters, be W. Then will the truss 4 6 7 have to bear $\frac{1}{2}W$, and this will give $\frac{W}{4}$ to be supported at each of the points 4 and 7. Moreover, on the space 2 4 is distributed $\frac{W}{4}$, which has, as far as overturning the strut is concerned, the same effect as $\frac{W}{8}$ at 2, and $\frac{W}{8}$ at 4. Hence the load to be supported at 4 by the inclined strut is a vertical load equal to $(\frac{1}{4} + \frac{1}{8})W = \frac{3}{8}W$. We may then find the force that must be furnished by the abutment, or by the tie-rod, in either of the two following ways: —

1°. By constructing the triangle $\gamma\delta\epsilon$ (Fig. 94*a*), with $\gamma\delta = \frac{3}{8}W$, $\gamma\epsilon \parallel 14$, and $\epsilon\delta$ parallel to the horizontal thrust of the abutment; then will $\gamma\delta\epsilon$ be the triangle of forces at 1, and $\epsilon\delta$ will be the thrust at 1.

2°. Multiply $\frac{3}{8}W$ by the perpendicular distance from 4 to 1 2, and divide by the height of 4 above 1 8 for the thrust of the abutment; in other words, take moments about the point 1.

Now, to construct the diagram of stresses, let, in Fig. 94*b*, the loads be

$$ab,\ bc,\ cd,\ de,\ ef,\ fg,\ gh,\ hk,\ \text{and}\ kl,$$

and let

$$lz = za = \tfrac{1}{2}W$$

be the vertical component of the supporting force; let zm be the thrust of the abutment: then will lm and ma be the real supporting forces; and we shall have, for polygon of external forces,

$$abcdefghklma.$$

Then, proceeding to the joint 1, we obtain, for polygon of forces,

$$maym;$$

and, proceeding from joint to joint, we obtain the stresses in all the members of the truss, as shown in Fig. 94*b*.

It will be noticed that UV and RQ are also free from stress.

If we had no horizontal thrust from the abutment, and the supporting forces were vertical, the members MV and MQ would be called into action, and MY and MN would be inactive. To exhibit this case, I have drawn diagram 94*c*, which shows the stresses that would then be developed. AY and NL would become merely part of the supports.

In this latter case the stresses are generally much greater than in the former, and a stress is developed in UV.

§ 139. **Hammer-Beam Truss: Wind Pressure.** — Fig. 95 shows the stress diagram of the hammer-beam truss for wind pressure when there is no roller under either end, and when the wind blows from the left. A similar diagram would give the stresses when it blows from the right.

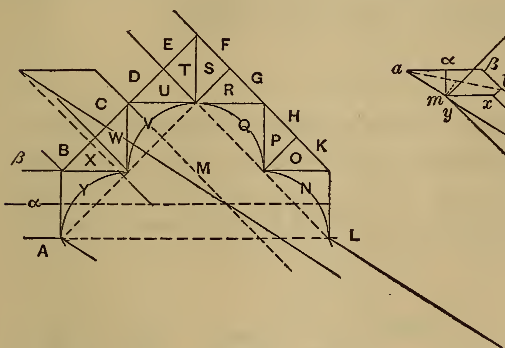


Fig. 95.

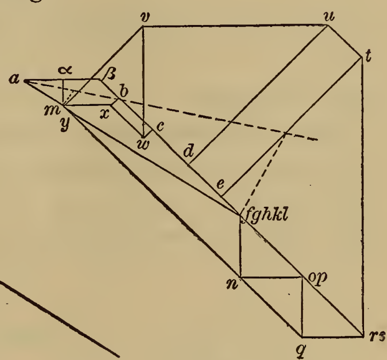


Fig. 95*a*.

The cases when there is a roller are not drawn: the student may construct them for himself.

§ 140. **Scissor-Beam Truss.** — We have already discussed two forms of scissor-beam truss in Figs. 90 and 91. These trusses having the right number of parts, their diagrams present no difficulty. Another form of the scissor-beam truss is shown in Fig. 96, and its diagram presents no difficulty.

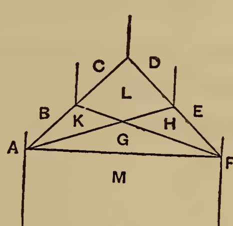


Fig. 96.

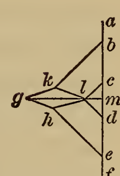


Fig. 96*a*.

The only peculiarity to be noticed is, that, after having constructed the polygon of external forces,

abcdefma,

we cannot proceed to construct the polygon of equilibrium for one of the supports, because there are three unknown forces

there. We therefore begin at the apex *CD*, and construct the triangle of forces *cdl* for this point; then proceed to joint *CB*, and construct the quadrilateral

bclkb;

then proceed to the left-hand support, and obtain

mabkgm;

and so continue.

§ 141. **Scissor-Beam Truss without Horizontal Tie.** — Very often the scissor-beam truss is constructed without any horizontal tie, in which case it has the appearance of Fig. 97, where there is sometimes a pin at *GKLH* and sometimes not.

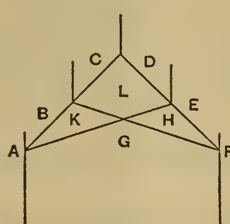


FIG. 97.

FIG. 97*a*.

FIG. 97*b*.

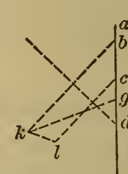


FIG. 97*c*.

In this case, if the abutments are capable of furnishing horizontal thrust to take the place of the horizontal tie of Fig. 96, we are reduced back to that case. If the abutments are not capable of furnishing horizontal thrust, we are then relying on the stiffness of the rafters to prevent the deformation of the truss; for, were the points *BC* and *DE* really joints, with pins, the deformation would take place, as shown in Fig. 97*a* or Fig. 97*b*, according as the two inclined ties were each made in one piece or in two (i.e., according as they are not pinned together at *KH*, or as they are pinned). This necessity of depending on the stiffness of the rafters, and the liability to deformation if they had joints at their middle points, become apparent as soon as we attempt to draw the diagram. Such an attempt is

made in Fig. 97*c*, where *abcdefga* is the polygon of external forces, *gabkg* the polygon of stresses for the left-hand support, *kbclk* that for joint *BC*. Then, on proceeding to draw the triangle of stresses for the vertex, we find that the line joining *d* and *l* is not parallel to *DL*, and hence that the truss is not stable. We ought, however, in this latter case, when the supporting forces are vertical, and when we rely upon the stiffness of the rafters to prevent deformation, to be able to determine the direct stresses in the bars; and for this we will employ an analytical instead of a graphical method, as being the most convenient in this case.

Let us assume that there is no pin at the intersection of the two ties, and that the two rafters are inclined at an angle of 45° to the horizon.

We then have, if $W =$ the entire load, and $\alpha =$ angle between *BK* and *KG*,

$$ab = ef = \frac{W}{8}, \quad bc = cd = de = \frac{W}{4},$$

$$\tan\alpha = \tfrac{1}{2}, \quad \sin\alpha = \frac{1}{\sqrt{5}}, \quad \cos\alpha = \frac{2}{\sqrt{5}},$$

Let x be the stress in each tie, and let $y = cl = dl =$ thrust in each upper half of the rafters.

Then we must observe that the rafter has, in addition to its direct stresses, a tendency to bend, due to a normal load at the middle, this normal load being equal to the sum of the normal components of *bc* and of x, when these are resolved along and normal to the rafter. Hence

$$\text{normal load} = x\cos\alpha + \frac{W}{4}\sin 45^\circ.$$

This, resolved into components acting at each end of the rafter, gives a normal downward force at each end equal to

$$\tfrac{1}{2}x\cos\alpha + \tfrac{1}{8}W\sin 45^\circ.$$

Hence, resolving all the forces acting at the left-hand support into components along and at right angles to the rafter, and imposing the condition of equilibrium that the algebraic sum of their normal components shall equal zero, we have, if we call upward forces positive,

$$\tfrac{3}{8}W\sin 45^\circ - (\tfrac{1}{2}x\cos\alpha + \tfrac{1}{8}W\sin 45^\circ) - x\sin\alpha = 0; \quad (1)$$

but, since

$$x\cos\alpha = 2x\sin\alpha,$$

we have from (1)

$$2x\sin\alpha = \frac{W}{4}\sin 45^\circ$$

$$\therefore \quad x\sin\alpha = \frac{W}{8}\sin 45^\circ$$

$$\therefore \quad x = \tfrac{1}{8}W\frac{\sin 45^\circ}{\sin\alpha}. \quad (2)$$

Then, proceeding to the apex of the roof, we have that the load

$$cd = \frac{W}{4}$$

gives, when resolved along the two rafters, a stress in each equal to

$$\frac{W}{4}\sin 45^\circ.$$

Hence the load to be supported in a direction normal to the rafter at the apex is

$$\frac{W}{4}\sin 45^\circ + (\tfrac{1}{2}x\cos\alpha + \frac{W}{8}\sin 45^\circ).$$

Hence, substituting for x its value, we have

$$y = cl = dl = \frac{W}{2}\sin 45^\circ. \quad (3)$$

Then, proceeding to the left-hand support, and equating to zero the algebraic sum of the components along the rafter, we have

$$bk = (ga - ab)\cos 45^\circ + x\cos\alpha$$
$$= \tfrac{3}{8}W\sin 45^\circ + \tfrac{1}{4}W\sin 45^\circ = \tfrac{5}{8}W\sin 45^\circ. \quad (4)$$

We have thus determined in (2), (3), and (4) the values of x, y, and $bk = eh$.

By way of verification, proceed to the middle of the left-hand rafter, and we find the algebraic sum of the components of bc and x along the rafter to be

$$\tfrac{1}{4}W\cos 45^\circ - x\sin\alpha = \tfrac{1}{8}W\sin 45^\circ;$$

and this is the difference between bk and cl, as it should be.

We have thus obtained the direct stresses; and we have, in addition, that the rafter itself is also subjected to a bending-moment from a normal load at the centre, this load being equal to

$$x\cos\alpha + \frac{W}{4}\sin 45^\circ = \frac{W}{2}\sin 45^\circ.$$

How to take this into account will be explained under the "Theory of Beams."

§ 142. **Examples.** — The following figures of roof-trusses may be considered as a set of examples, for which the stress diagrams are to be worked out.

Observe, that, wherever there is a joint, the truss is to be supposed perfectly flexible, i.e., free to turn around a pin.

FIG. 98.

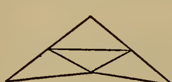


FIG. 99.

FIG. 100.

FIG. 101.

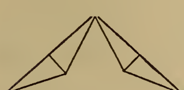


FIG. 102.

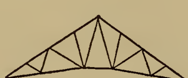


FIG. 103.

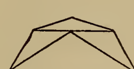


FIG. 104.

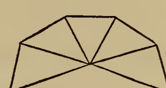


FIG. 105.

FIG. 106.

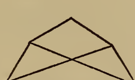


FIG. 107.

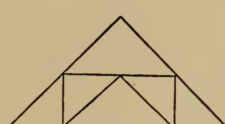


FIG. 108

CHAPTER IV.

BRIDGE-TRUSSES.

§ 143. **Method of Sections.** — It is perfectly possible to determine the stresses in the members of a bridge-truss graphically, or by any methods that are used for roof-trusses.

In this work an analytical method will be used; i.e., a method of sections. This method involves the use of the analytical conditions of equilibrium for forces in a plane explained in § 63. These are as follows; viz., —

If a set of forces in a plane, which are in equilibrium, be resolved into components in two directions at right angles to each other, then —

1°. The algebraic sum of the components in one of these directions must be zero.

2°. The algebraic sum of the components in the other of these directions must be zero.

3°. The algebraic sum of the moments of the forces about any axis perpendicular to the plane of the forces must be zero.

Assume, now, a bridge-truss (Figs. 109, 110, 111, 112, pages 176 and 177) loaded at a part or all of the joints. Conceive a vertical section *ab* cutting the horizontal members 6–8 and 7–9 and the diagonal 7–8, and dividing the truss into two parts. Then the forces acting on either part must be in equilibrium, in other words, the external forces, loads, and supporting forces, acting on one part, must be balanced by the stresses in the members cut by the section; i.e., by the forces exerted by the other part of the truss on the part under consideration. Hence we must have the three following conditions; viz., —

1°. The algebraic sum of the vertical components of the above-mentioned forces must be zero.

2°. The algebraic sum of the horizontal components of these forces must be zero.

3°. The algebraic sum of the moments of these forces about any axis perpendicular to the plane of the truss must be zero.

§ 144. **Shearing-Force and Bending-Moment.** — Assuming all the loads and supporting forces to be vertical, we shall have the following as definitions.

The *Shearing-Force* at any section is the force with which the part of the girder on one side of the section tends to slide by the part on the other side.

In a girder free at one end, it is equal to the sum of the loads between the section and the free end.

In a girder supported at both ends, it is equal in magnitude to the difference between the supporting force at either end, and the sum of the loads between the section and that supporting force.

The *Bending-Moment* at any section is the resultant moment of the external forces acting on the part of the girder to one side of the section, tending to rotate that part of the girder around a horizontal axis lying in the plane of the section.

In a girder free at one end, it is equal to the sum of the moments of the loads between the section and the free end, about a horizontal axis in the section.

In a girder supported at both ends, it is the difference between the moment of either supporting force, and the sum of the moments of the loads between the section and that support; all the moments being taken about a horizontal axis in the section.

§ 145. **Use of Shearing-Force and Bending-Moment.** — The three conditions stated in § 143 may be expressed as follows: —

1°. The algebraic sum of the horizontal components of the stresses in the members cut by the section must be zero.

2°. The algebraic sum of the vertical components of the stresses in the members cut by the section must balance the shearing-force.

3°. The algebraic sum of the moments of the stresses in the members cut by the section, about any axis perpendicular to the plane of the truss, and lying in the plane of the section, must balance the bending-moment at the section.

As the conditions of equilibrium are three in number, they will enable us to determine the stresses in the members, provided the section does not cut more than three; and this determination will require the solution of three simultaneous equations of the first degree with three unknown quantities (the stresses in the three members).

By a little care, however, in choosing the section, we can very much simplify the operations, and reduce our work to the solution of one equation with only one unknown quantity; the proper choice of the section taking the place of the elimination.

§ 146. **Examples of Bridge-Trusses.** — Figs. 109–112 represent two common kinds of bridge-trusses: in the first two the braces are all diagonal, in the last two they are partly vertical and partly diagonal.

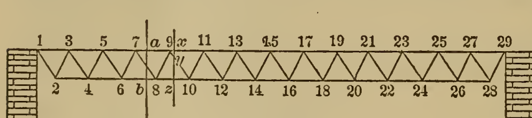


Fig. 109.

The first two are called Warren girders, or half-lattice girders; since there is only one system of bracing, as in the figures. When, on the other hand, there are more than one system, so that the diagonals cross each other, they are called lattice girders.

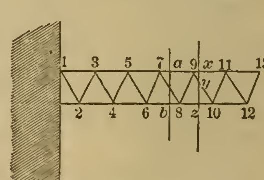


Fig. 110.

§ 147. **General Outline of the Steps to be taken in determining the Stresses in a Bridge-Truss under a Fixed Load.**

1°. If the truss is supported at both ends, find the supporting forces.

2°. Assume, in all cases, a section, in such a manner as not to cut more than three members if possible, or, rather, three of those that are brought into action by the loads on the truss; and it will save labor if we assume the section so as to cut two of the three very near their point of intersection.

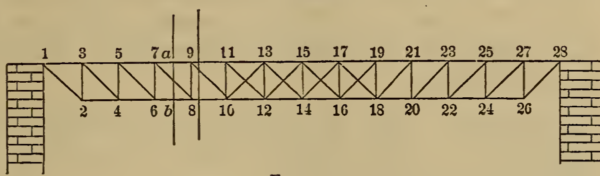


Fig. 111.

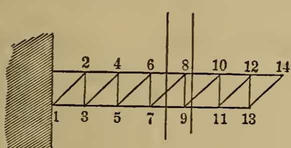


Fig. 112.

3°. Find the shearing-force at the section.

4°. Find the bending-moment at the section.

5°. Impose the analytical conditions of equilibrium on *all* the forces acting on the part of the girder to one side of the section, — the part between the section and the free end when the girder is free at one end, or either part when it is supported at both ends.

In the cases shown in Figs. 109 and 110, we may describe the process as follows; viz., —

(*a*) Find the stress in the diagonal from the fact, that (since the stress in the diagonal is the only one that has a vertical component at the section) the vertical component of the stress in the diagonal must balance the shearing-force.

(*b*) Take moments about the point of intersection of the diagonal and horizontal chord near which the section is taken; then the stresses in those members will have no moment, so that the moment of the stress in the other horizontal must balance the bending-moment at the section. Hence the stress in the horizontal will be found by dividing the bending-moment at the section by the height of the girder.

The above will be best illustrated by some examples.

EXAMPLE I. — Given the semi-girder shown in Fig. 110, loaded at joint 13 with 4000 pounds, and at each of the joints 1, 3, 5, 7, 9, and 11 with 8000 pounds. Suppose the length of each chord and each diagonal to be 5 feet. Required the stress in each member.

Solution. — For the purpose of explaining the method of procedure, we will suppose that we desire to find first the stresses in 8–10 and 9–10.

Assume a vertical section very near the joint 9, but to the right of it, so that it shall cut both 8–10 and 9–10.

If, now, the truss were actually separated into two parts at this section, the right-hand part would, in consequence of the loads acting on it, separate from the other part. This tendency to separate is counteracted by the following three forces: —

1°. The pull exerted by the part $9-x$ of the bar 9–11 on the part $x-11$ of the same bar.

2°. The thrust exerted by the part $8-z$ of the bar 8–10 on the part $z-10$ of the same bar.

3°. The pull exerted by the part $9-y$ of the bar 9–10 on the part $y-10$ of the same bar.

The shearing-force at this section is

$$8000 + 4000 = 12000 \text{ lbs.},$$

and this is equal to the vertical component of the stress in the diagonal. Hence

$$\text{Stress in } 9\text{–}10 = \frac{12000}{\sin 60^\circ} = 12000(1.1547) = 13856 \text{ lbs.}$$

This stress is a pull, as may be seen from the fact, that, in order to prevent the part of the girder to the right of the section from sliding downwards under the action of the load, the part $9-y$ of the diagonal 9–10 must pull the part $y-10$ of the same diagonal.

Next take moments about 9: and, since the moment of the stresses in 9–11 and 9–10 about 9 is zero, we must have that the moment of the stress in 8–10; i.e., the product of this stress by the height of the girder, must equal the bending-moment.

The bending-moment about 9 is

$$8000 \times 5 + 4000 \times 10 = 80000 \text{ foot-lbs.}$$

Hence

$$\text{Stress in } 8\text{–}10 = \frac{80000}{4.33} = 80000(0.23094) = 18475 \text{ lbs.}$$

Proceed in a similar way for all the other members. The work may be arranged as in the following table; the diagonal stresses being deduced from the shearing-forces by multiplying by 1.1547, and the chord stresses from the bending-moments by multiplying by 0.23094.

Section just to the right of	Shearing-Force in lbs.	Stresses in Diagonals cut by Section, in lbs.		Bending-Moment, in foot-lbs.	Stresses in Chords opposite the respective Joints.	
		Tension.	Compression.		Tension.	Compression.
1	44000	50806		720000		166277
2	44000		50806	610000	140873	
3	36000	41569		500000		115470
4	36000		41569	410000	94685	
5	28000	32331		320000		73901
6	28000		32331	250000	57735	
7	20000	23094		180000		41569
8	20000		23094	130000	30022	
9	12000	13856		80000		18475
10	12000		13856	50000	11547	
11	4000	4619		20000		4618
12	4000		4619	10000	2309	

EXAMPLE II. — Given the truss (Fig. 109) loaded at each of the lower joints with 10000 lbs.: find the stresses in the members. The length of chord is equal to the length of diagonal = 10 ft.

In future, tensions will be written with the minus, and compressions with the plus sign.

Solution. — Total load = 14(10000) = 140000 lbs.
Each supporting force = 70000 "

The entire work is shown in the following tables: —

Section taken just to the — Right of	Left of	Shearing-Force at Section, in lbs.	Bending-Moment about Joint, in foot-lbs.
1	29	70000 = 70000	0
2	28	70000 − 10000 = 60000	70000 × 5 = 350000
3	27	70000 − 10000 = 60000	70000 × 10 − 10000 × 5 = 650000
4	26	70000 − 20000 = 50000	70000 × 15 − 10000 × 10 = 950000
5	25	70000 − 20000 = 50000	70000 × 20 − 10000 (5 + 15) = 1200000
6	24	70000 − 30000 = 40000	70000 × 25 − 10000 (10 + 20) = 1450000
7	23	70000 − 30000 = 40000	70000 × 30 − 10000 (5 + 15 + 25) = 1650000
8	22	70000 − 40000 = 30000	70000 × 35 − 10000 (10 + 20 + 30) = 1850000
9	21	70000 − 40000 = 30000	70000 × 40 − 10000 (5 + 15 + 25 + 35) = 2000000
10	20	70000 − 50000 = 20000	70000 × 45 − 10000 (10 + 20 + 30 + 40) = 2150000
11	19	70000 − 50000 = 20000	70000 × 50 − 10000 (5 + 15 + 25 + 35 + 45) = 2250000
12	18	70000 − 60000 = 10000	70000 × 55 − 10000 (10 + 20 + 30 + 40 + 50) = 2350000
13	17	70000 − 60000 = 10000	70000 × 60 − 10000 (5 + 15 + 25 + 35 + 45 + 55) = 2400000
14	16	70000 − 70000 = 0	70000 × 65 − 10000 (10 + 20 + 30 + 40 + 50 + 60) = 2450000
15	-	70000 − 70000 = 0	70000 × 70 − 10000 (5 + 15 + 25 + 35 + 45 + 55 + 65) = 2450000

Numbers of Diagonals.		Stresses in Diagonals, in lbs.
1- 2	28-29	$-70000 \times 1.1547 = -80829$
2- 3	27-28	$+60000 \times 1.1547 = +69282$
3- 4	26-27	$-60000 \times 1.1547 = -69282$
4- 5	25-26	$+50000 \times 1.1547 = +57735$
5- 6	24-25	$-50000 \times 1.1547 = -57735$
6- 7	23-24	$+40000 \times 1.1547 = +46188$
7- 8	22-23	$-40000 \times 1.1547 = -46188$
8- 9	21-22	$+30000 \times 1.1547 = +34641$
9-10	20-21	$-30000 \times 1.1547 = -34641$
10-11	19-20	$+20000 \times 1.1547 = +23094$
11-12	18-19	$-20000 \times 1.1547 = -23094$
12-13	17-18	$+10000 \times 1.1547 = +11547$
13-14	16-17	$-10000 \times 1.1547 = -11547$
14-15	15-16	$+0$ 0

UPPER CHORDS.

Numbers of Chords.		Stresses in Chords, in lbs.
1- 3	27-29	$650000 \times 0.11547 = + 75056$
3- 5	25-27	$1200000 \times 0.11547 = +138564$
5- 7	23-25	$1650000 \times 0.11547 = +190526$
7- 9	21-23	$2000000 \times 0.11547 = +230940$
9-11	19-21	$2250000 \times 0.11547 = +259808$
11-13	17-19	$2450000 \times 0.11547 = +282902$
13-15	15-17	$2450000 \times 0.11547 = +282902$

LOWER CHORDS.

Numbers of Chords.		Stresses in Chords, in lbs.
2– 4	26–28	– 350000 × 0.11547 = – 40415
4– 6	24–26	– 950000 × 0.11547 = –109697
6– 8	22–24	–1450000 × 0.11547 = –167432
8–10	20–22	–1850000 × 0.11547 = –213620
10–12	18–20	–2150000 × 0.11547 = –248261
12–14	16–18	–2350000 × 0.11547 = –267355
14–16		–2450000 × 0.11547 = –282902

EXAMPLE III. — Given the same truss as in Example II., loaded at 2, 4, 6, 8, 10, and 12 with 10000 lbs. at each point, the remaining lower joints being loaded with 50000 lbs. at each joint: find the stresses in the members.

EXAMPLE IV. — Given a semi-girder, free at one end (Fig. 112), loaded at 2, 4, and 6 with 10000 lbs., and at 8, 10, and 12 with 5000 lbs.: find the stresses in the members.

TRAVELLING-LOAD.

§ 148. **Half-Lattice Girder: Travelling-Load.** — When a girder is used for a bridge, it is not subjected all the time to the same set of loads.

The load in this case consists of two parts, — one, the dead load, including the bridge weight, together with any permanent load that may rest upon the bridge; and the other, the moving or variable load, also called the travelling-load, such as the weight of the whole or part of a railroad train if it is a railroad bridge, or the weight of the passing teams, etc., if it is a common-road bridge. Hence it is necessary that we should be able to determine the amount and distribution of the loads upon the bridge which will produce the greatest tension or the greatest

compression in every member, and the consequent stress produced.

§ 149. **Greatest Stresses in Semi-Girder.** — Wherever the section be assumed in a semi-girder, it is evident that any load placed on the truss at any point between the section and the free end increases both the shearing-force and the bending-moment at that section, and that any load placed between the section and the fixed end has no effect whatever on either the shearing-force or the bending-moment at that section.

Hence every member of a semi-girder will have a greater stress upon it when the entire load is on, than with any partial load.

§ 150. **Greatest Chord Stresses in Girder supported at Both Ends.** — Every load which is placed upon the truss, no matter where it is placed, will produce at any section whatever a *bending-moment* tending to turn the two parts of the truss on the two sides of the section upwards from the supports; i.e., so as to render the truss concave upwards.

Hence every load that is placed upon the truss causes compression in every horizontal upper chord, and tension in every horizontal lower chord. Hence, in order to obtain the greatest chord stresses, we assume the whole of the moving load to be upon the bridge.

§ 151. **Greatest Diagonal Stresses in Girder supported at Both Ends.** — To determine the distribution of the load that will produce the greatest stress of a certain kind (tension or compression) in any given diagonal, let us suppose the diagonal in question to be 7–8 (Fig. 109), through which we take our section *ab*. Now, it is evident that any load placed on the truss between *ab* and the left-hand (nearer) support will cause a shearing-force at that section which will tend to slide the part of the girder to the left of the section downwards with reference to the other part, and hence will cause a compressive stress in 7–8; while any load between the section and the right-

hand (farther) support will cause a shearing-force of the opposite kind, and hence a tension in the bar 7–8.

Now, the bridge weight itself brings an equal load upon each joint; hence, when the bridge weight is the only load upon the truss, the bar 7–8 is in tension.

Hence, any load placed upon the truss between the section and the farther support tends to increase the shearing-force at that section due to the dead load (provided this is equally distributed among the joints); whereas any load placed between the section and the nearer support tends to decrease the shearing-force at the section due to the dead load, or to produce a shearing-force of the opposite kind to that produced by the dead load at that section.

Hence, if we assume the dead load to be equally distributed among the joints, we shall have the two following propositions true: —

(*a*) In order to determine the greatest stress in any diagonal which is of the same kind as that produced by the dead load, we must assume the moving load to cover all the panel points between the section and the farther abutment, and no other panel points.

(*b*) In order to determine the greatest stress in any diagonal of the opposite kind to that produced by the dead load, we must assume the moving load to cover all the panel points between the section and the nearer abutment, and no others.

This will be made clear by an example.

Example I. — Given the truss shown in Fig. 113. Length of chord = length of diagonal = 10 feet. Dead load = 8000 lbs. applied at each upper panel point. Moving load = 30000 lbs. applied at each upper panel point. Find the greatest stresses in the members.

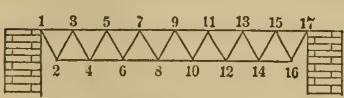


Fig. 113.

Solution. (*a*) *Chord Stresses.* — Assume the whole load to be upon the bridge: this will give 38000 lbs. at each upper panel point; i.e., omitting 1 and 17, where the load acts directly on the support, and not on the truss.

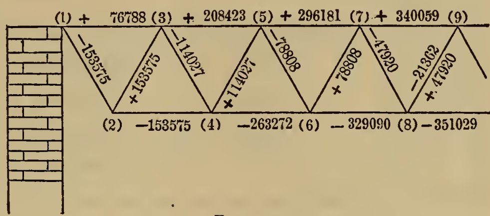


FIG. 114.

Hence, considering the bridge so loaded, we shall have the following results for the chord stresses: —

$$\text{Each supporting force} = 38000\left(\frac{7}{2}\right) = 133000.$$

Section at		Bending-Moment, in foot-lbs.
2	16	$133000 \times 5 = 665000$
3	15	$133000 \times 10 = 1330000$
4	14	$133000 \times 15 - 38000 \times 5 = 1805000$
5	13	$133000 \times 20 - 38000 \times 10 = 2280000$
6	12	$133000 \times 25 - 38000(5 + 15) = 2565000$
7	11	$133000 \times 30 - 38000(10 + 20) = 2850000$
8	10	$133000 \times 35 - 38000(5 + 15 + 25) = 2945000$
9		$133000 \times 40 - 38000(10 + 20 + 30) = 3040000$

Numbers of Chords.		Stresses in Upper Chords.
1–3	15–17	$665000 \times 0.11547 = +76788$
3–5	13–15	$1805000 \times 0.11547 = +208423$
5–7	11–13	$2565000 \times 0.11547 = +296181$
7–9	9–11	$2945000 \times 0.11547 = +340059$

Numbers of Chords.		Stresses in Lower Chords.
2– 4	14–16	$-1330000 \times 0.11547 = -153575$
4– 6	12–14	$-2280000 \times 0.11547 = -263272$
6– 8	10–12	$-2850000 \times 0.11547 = -329090$
8–10		$-3040000 \times 0.11547 = -351029$

Next, as to the diagonals, take, for instance, the diagonal 7–8. When the dead load alone is on the bridge, the diagonal 7–8 is in tension. From the preceding, we see that the greatest tension is produced in this bar when the moving load is on the points 9, 11, 13, and 15, and the dead load only on the points 3, 5, 7. Now, a load of 38000 lbs. at 13, for instance, causes a shearing-force of $\frac{4}{16}(38000) = 9500$ lbs. at any section to the left of 13; and this shearing-force tends to cause the part to the left of the section to slide upwards, and that to the right downwards.

On the other hand, with the same load at the same place, there is produced a shearing-force of $\frac{12}{16}(38000) = 28500$ lbs. at any section to the right of 13; and this shearing-force tends to cause the part to the left to slide downwards, and that to the right upwards. Paying attention to this fact, we shall have, when the loads are distributed as above described, a shearing-force at the bar 7–8 causing tension in this bar; the magnitude of this shearing-force being

$$\frac{38000}{16}(2 + 4 + 6 + 8) - \frac{8000}{16}(2 + 4 + 6) = 41500.$$

Hence, we may arrange the work as follows: —

Numbers of Diagonals.		Greatest Shearing-Forces producing Stresses of Same Kind as Dead Load.	Greatest Stresses in Diagonals of Same Kind as those due to Dead Load.
1–2	17–16	$\frac{38000}{16}(2+4+6+8+10+12+14) = 133000$	-153575
2–3	16–15	$\frac{38000}{16}(2+4+6+8+10+12+14) = 133000$	$+153575$
3–4	15–14	$\frac{38000}{16}(2+4+6+8+10+12) - \frac{8000}{16}(2) = 98750$	-114027
4–5	14–13	$\frac{38000}{16}(2+4+6+8+10+12) - \frac{8000}{16}(2) = 98750$	$+114027$
5–6	13–12	$\frac{38000}{16}(2+4+6+8+10) - \frac{8000}{16}(2+4) = 68250$	-78808
6–7	12–11	$\frac{38000}{16}(2+4+6+8+10) - \frac{8000}{16}(2+4) = 68250$	$+78808$
7–8	11–10	$\frac{38000}{16}(2+4+6+8) - \frac{8000}{16}(2+4+6) = 41500$	-47920
8–9	10–9	$\frac{38000}{16}(2+4+6+8) - \frac{8000}{16}(2+4+6) = 41500$	$+47920$

Numbers of Diagonals.		Greatest Shearing-Forces producing Stresses of Kind Opposite from Dead Load.	Greatest Stresses in Diagonals of Kind Opposite from Dead Load.
8–9	10–9	$\frac{38000}{16}(2+4+6) - \frac{8000}{16}(2+4+6+8) = 18500$	-21362
7–8	11–10	$\frac{38000}{16}(2+4) - \frac{8000}{16}(2+4+6+8+10) = -750$	None.

Hence, the diagonals 8–9 and 9–10 are the only ones that, under any circumstances, can have a stress of the kind opposite to that to which they are subjected under the dead load alone.

Fig. 114 exhibits the manner of writing the stresses on the diagram.

§ 152. **General Application of this Method.** — It is plain that the method used above will apply to any single system of bridge-truss with horizontal chords and diagonal bracing, whatever be the inclination of the braces.

When seeking the stress in a diagonal, the section must be so taken as to cut that diagonal; and, as far as this stress alone is concerned, it may be equally well taken at any point, as well as near a joint, provided only it cuts that diagonal which is in action under the load that produces the greatest stress in this one, and no other.

On the other hand, when we seek the stress in a horizontal chord, the section might very properly be taken through the joint opposite that chord.

Taking it very near the joint, only serves to make one section answer both purposes simultaneously.

§ 153. **Bridge-Trusses with Vertical and Diagonal Bracing.** — When, as in Figs. 111 and 112, there are both vertical and diagonal braces, and also horizontal chords, we may determine the stresses in the diagonals and in the chords just as before; only we must take the section just to one side of a joint, and never through the joint.

As to the verticals, in order to determine the stress in any vertical, we must impose the conditions of equilibrium between the vertical components of the forces acting at one end of that vertical: thus, if the loads are at the upper joints in Fig. 111, then the stress in vertical 3–2 must be equal and opposite to the vertical component of the stress in diagonal 1–2, as these stresses are the only vertical forces acting at joint 2.

Vertical 5–4 has for its stress the vertical component of the stress in 3–4, etc. Thus

Stress in 3–2 = shearing-force in panel 1–3,
Stress in 5–4 = shearing-force in panel 3–5, etc.

On the other hand, if the loads be applied at the lower joints, then

Stress in 3–2 = shearing-force in panel 3–5,
Stress in 5–4 = shearing-force in panel 5–7, etc.

EXAMPLE. — Given the truss shown in Fig. 111. Given panel length = height of truss = 10 feet, dead load per panel point = 12000 lbs., moving load per panel point = 23000 lbs.; load applied at upper joints.

Solution. (*a*) *Chord Stresses.* — Assume the entire load on the bridge, i.e., 35000 lbs. per panel point. Hence

Total load on truss = 13 (35000) = 455000 lbs.,
Each supporting force = 227500 lbs.

Joint near which Section is taken.		Bending-Moment at the Section very near the Joint, on Either Side of the Joint.	
1	28	0	
3	27	227500×10	$= 2275000$
5	25	$227500 \times 20 - 35000 \times 10$	$= 4200000$
7	23	$227500 \times 30 - 35000(10 + 20)$	$= 5775000$
9	21	$227500 \times 40 - 35000(10 + 20 + 30)$	$= 7000000$
11	19	$227500 \times 50 - 35000(10 + 20 + 30 + 40)$	$= 7875000$
13	17	$227500 \times 60 - 35000(10 + 20 + 30 + 40 + 50)$	$= 8400000$
15	–	$227500 \times 70 - 35000(10 + 20 + 30 + 40 + 50 + 60)$	$= 8575000$

To find any chord stress, divide the bending-moment at a section cutting the chord, and passing close to the opposite joint, by the height of the girder, which in this case is 10. Hence we have for the chord stresses (denoting, as before, compression by +, and tension by —): —

Stresses in Upper Chords.			Stresses in Lower Chords.		
1–3	27–28	+227500	2–4	24–26	−227500
3–5	25–27	+420000	4–6	22–24	−420000
5–7	23–25	+577500	6–8	20–22	−577500
7–9	21–23	+700000	8–10	18–20	−700000
9–11	19–21	+787500	10–12	16–18	−787500
11–13	17–19	+840000	12–14	14–16	−840000
13–15	15–17	+857500			

Diagonals. — It is evident, that, for the diagonals, the same rule holds as in the case of the Warren girder: i.e., the greatest stress of the same kind as that produced by the dead load occurs when the moving load is on all the joints between the diagonal in question and the farther abutment; whereas the greatest stress of the opposite kind occurs when the moving load covers all the joints between the diagonal in question and the nearer abutment.

The work of determining the greatest shearing-forces may be arranged as in tables on p. 191.

Counterbraces. — If the truss were constructed with those diagonals only that slope downwards towards the centre, and which may be called the main braces, the diagonals 11–12, 13–14, 14–17, and 16–19 would sometimes be called upon to bear a thrust, and the verticals 12–13 and 17–16 a pull: this would necessitate making these diagonals sufficiently strong to resist the greatest thrust to which they are liable, and fixing the verticals in such a way as to enable them to bear a pull.

In order to avoid this, the diagonals 10–13, 12–15, 15–16, and 17–18 are inserted, which are called counterbraces, and which come into action only when the corresponding main

braces would otherwise be subjected to thrust. They also prevent any tension in the verticals.

Diagonals.		Greatest Shearing-Forces of the Same Kind as those produced by Dead Load.
1– 2	28–26	$\frac{35000}{14}(1+2+3+\ldots+13) = 227500$
3– 4	27–24	$\frac{35000}{14}(1+2+3+\ldots+12) - \frac{12000}{14}(1) = 194143$
5– 6	25–22	$\frac{35000}{14}(1+2+3+\ldots+11) - \frac{12000}{14}(1+2) = 162429$
7– 8	23–20	$\frac{35000}{14}(1+2+3+\ldots+10) - \frac{12000}{14}(1+2+3) = 132357$
9–10	21–18	$\frac{35000}{14}(1+2+3+\ldots+9) - \frac{12000}{14}(1+2+\ldots+4) = 103929$
11–12	19–16	$\frac{35000}{14}(1+2+3+\ldots+8) - \frac{12000}{14}(1+2+\ldots+5) = 77143$
13–14	17–14	$\frac{35000}{14}(1+2+3+\ldots+7) - \frac{12000}{14}(1+2+\ldots+6) = 52000$

Diagonals.		Greatest Shearing-Forces of the Opposite Kind to those produced by Dead Load.
13–14	17–14	$\frac{35000}{14}(1+2+3+\ldots+6) - \frac{12000}{14}(1+2+\ldots+7) = 28500$
11–12	19–16	$\frac{35000}{14}(1+2+\ldots+5) - \frac{12000}{14}(1+2+\ldots+8) = 6643$
9–10	21–18	$\frac{35000}{14}(1+2+\ldots+4) - \frac{12000}{14}(1+2+\ldots+9) = -13571$

The main braces and counterbraces of a panel are never in action simultaneously. Hence we have, for the greatest stresses in the diagonals, the following results, obtained by multiplying the corresponding shearing-forces by $\frac{1}{\cos 45^\circ} = 1.414$.

In the following I have used this number to three decimal places, as being sufficiently accurate for practical purposes.

Stresses in Main Braces.			Stresses in Counterbraces.		
1– 2	28–26	−321685	15–12	15–16	−40299
3– 4	27–24	−274518	13–10	17–18	− 9393
5– 6	25–22	−229675			
7– 8	23–20	−187153			
9–10	21–18	−146956			
11–12	19–16	−109080			
13–14	17–14	− 73528			

Vertical Posts. — Since the loads are applied at the upper joints, the conditions of equilibrium at the lower joints require that the thrust in any vertical post shall be equal to the vertical component of the tension in that diagonal which, being in action at the time, meets it at its lower end.

Hence it is equal to the shearing-force in that panel where the acting diagonal meets it at its lower end.

The post 15–14 is a counterbrace, and comes into action only when the diagonal counterbraces are in action.

We therefore have, for the posts, the following as the greatest thrusts: —

STRESSES IN VERTICALS.

3– 2	27–26	+227500
5– 4	25–24	+194143
7– 6	23–22	+162429
9– 8	21–20	+132357
11–10	19–18	+103929
13–12	17–16	+ 77143
15–14		+ 28500

Fig. 115 shows the stresses marked on the diagram.

§ 154. **Manner of Concentrating the Load at the Joints.** — In using the methods given above, we are assuming that all the loads are concentrated at the joints, and that none are distributed over any of the pieces. As far as the moving load is concerned, and also all of the dead load except the weight of the truss itself, this always is, or ought to be, effected; and it is accomplished in a manner similar to that adopted in the case of roof-trusses. This method is shown in the figure (Fig. 116); floor-beams being laid across from girder to girder at the joints, on top of which are laid longitudinal beams, and on these the sleepers if it is a railroad bridge, or the floor if it is a road bridge. The weight of the *truss* itself is so small a part of what the bridge is called upon to bear, that it can, without appreciable error, be considered as concentrated at the joints either of the upper chord, of the lower chord, or of both, according to the manner in which the rest of the load is distributed.

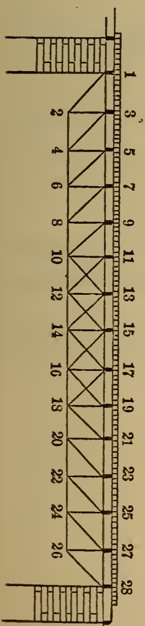


Fig. 116.

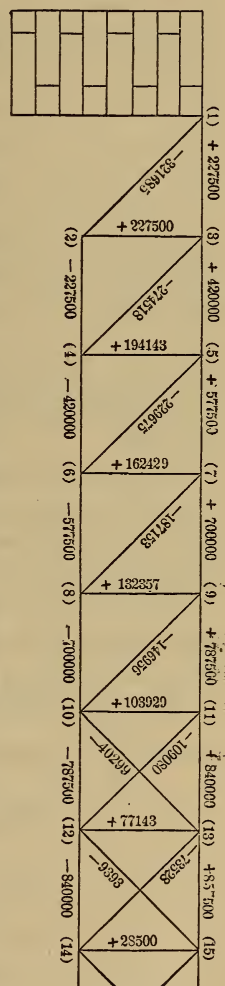


Fig. 115.

§ 155. **Closer Approximation to Actual Shearing-Force.** — In our computations of greatest shearing-force, we make an approximation which is generally considered to be

sufficiently close, and which is always on the safe side. To illustrate it, take the case of panel 3–5 of the last example. In determining its greatest shearing-force, we considered a load of 35000 lbs. per panel point to rest on all the joints from the right-hand support to joint 5, inclusive, and the dead load to rest on all the other joints of the truss. Now, it is impossible, if the load is distributed uniformly on the floor of the bridge, to have a load of 35000 lbs. on 5 and 12000 on 3 simultaneously; for, if the moving load extended on the bridge floor only up to 5, the load on 5 would be only $12000 + \frac{1}{2}(23000) = 23500$ lbs., and that on 3 would then be 12000 lbs. If, on the other hand, the moving load extends beyond 5 at all, as it must if the load on 5 is to be greater than 23500 lbs., then part of it will rest on 3, and the load on 3 will then be greater than 12000 lbs.; for whatever load there is between 3 and 5 is supported at 3 and 5.

Moreover, we know that the effect of increasing the load on 5 is to increase the shearing-force, provided we do not at the same time increase that on 3 so much as to destroy the effect of increasing that on 5.

Hence, there must be some point between 3 and 5 to which the moving load must extend in order to render the shearing-force in panel 3–5 a maximum.

Let the distance of this point from 5 be x; then, if we let

$w = \dfrac{23000}{10}$ = moving load per foot of length,

$$\text{Moving load on panel} = wx,$$

$$\text{Part supported at 3} = \frac{wx^2}{20},$$

$$\text{Part supported at 5} = wx - \frac{wx^2}{20}.$$

Hence, portion of shearing-force due to the moving load on panel 3–5 equals

$$\frac{12}{14}\left(wx - \frac{wx^2}{20}\right) - \frac{1}{14}\frac{wx^2}{20} = \frac{w}{14}\left(12x - \frac{13x^2}{20}\right).$$

This becomes a maximum when its first differential co-efficient becomes zero, i.e., when

$$12 - \tfrac{13}{10}x = 0;$$

therefore

$$x = 9'.23.$$

Hence, when the moving load extends to a distance of 9.23 feet from 5, then the shearing-force in panel 3–5, and hence the stress in diagonal 3–4, is a maximum.

Diagonals in Panel where Shearing-Force is taken.		Portion of Shearing-Force due to Moving Load on Panel.	Value of x, in feet.	Portion of Load at Joints named below.		Portion of Load at Joints named below.	
1– 3	27–28	$\frac{w}{14}\left(13x - \frac{13x^2}{20}\right)$	10.00	1	11500	3	11500
3– 5	25–27	$\frac{w}{14}\left(12x - \frac{13x^2}{20}\right)$	9.23	3	9797	5	11432
5– 7	23–25	$\frac{w}{14}\left(11x - \frac{13x^2}{20}\right)$	8.46	5	8230	7	11227
7– 9	21–23	$\frac{w}{14}\left(10x - \frac{13x^2}{20}\right)$	7.69	7	6801	9	10886
9–11	19–21	$\frac{w}{14}\left(9x - \frac{13x^2}{20}\right)$	6.92	9	5507	11	10409
11–13	17–19	$\frac{w}{14}\left(8x - \frac{13x^2}{20}\right)$	6.15	11	4350	13	9795
13–15	15–17	$\frac{w}{14}\left(7x - \frac{13x^2}{20}\right)$	5.38	13	3329	15	9045

To show how the adoption of this method would affect the resulting stresses in the diagonals and verticals, I have given the work above, and shown the difference between these and

the former results. In this table x = distance covered by load from end of panel nearest the centre.

Panels.		Greatest Shearing-Force of Same Kind as that due to Dead Load.	
1– 3	27–28	$\frac{35000}{14}(1+\ldots+13)$	$=227500$
3– 5	25–27	$\frac{35000}{14}(1+\ldots+11)+\frac{12}{14}(23432)-\frac{1}{14}(21797)$	$=183528$
5– 7	23–25	$\frac{35000}{14}(1+\ldots+10)+\frac{11}{14}(23227)-\frac{2}{14}(20230)-\frac{1}{14}(12000)$	$=152003$
7– 9	21–23	$\frac{35000}{14}(1+\ldots+9)+\frac{10}{14}(22886)-\frac{3}{14}(18801)-\frac{12000}{14}(1+2)$	$=122247$
9–11	19–21	$\frac{35000}{14}(1+\ldots+8)+\frac{9}{14}(22409)-\frac{4}{14}(17507)-\frac{12000}{14}(1+2+3)$	$=94261$
11–13	17–19	$\frac{35000}{14}(1+\ldots+7)+\frac{8}{14}(21795)-\frac{5}{14}(16350)-\frac{12000}{14}(1+\ldots+4)$	$=68043$
13–15	15–17	$\frac{35000}{14}(1+\ldots+6)+\frac{7}{14}(21664)-\frac{6}{14}(14710)+\frac{12000}{14}(1+\ldots+5)$	$=44171$

Hence, for stresses in main braces, we have

Diagonals.		Stresses.
1– 2	28–26	−321685
3– 4	27–24	−259509
5– 6	25–22	−214932
7– 8	23–20	−172857
9–10	21–18	−133285
11–12	19–16	− 96213
13–14	17–14	− 62458

Moreover, for the shearing-forces of opposite kind from

those due to dead load, we have, if $x =$ distance from end of panel nearest support which is covered by moving load, —

Panels.		Portion of Shear due to Moving Load on Panel.	Value of x.	Portion of Load at Joints named below.		Portion of Load at Joints named below.	
13–15	17–15	$\frac{w}{14}\left(6x - \frac{13x^2}{20}\right)$	4.62	15	2455	13	8171
11–13	19–17	$\frac{w}{14}\left(5x - \frac{13x^2}{20}\right)$	3.84	13	1695	11	7137

Panels.		Greatest Shearing-Forces of Opposite Kind from those due to Dead Load.
13–15	17–15	$\frac{35000}{14}(1+\ldots+5)+\frac{6}{14}(20171)-\frac{7}{14}(14455)-\frac{12000}{14}(1+\ldots+6) = 20917$
11–13	19–17	$\frac{35000}{14}(1+\ldots+4)+\frac{5}{14}(19137)-\frac{6}{14}(13695)-\frac{12000}{14}(1+\ldots+7) = 1965$

Hence we have the following as the stresses in the counter-braces: —

Counter-Braces.		Stresses.
15–12	15–16	−29577
13–10	17–18	− 2778

And, for the verticals, we have the new, instead of the old, shearing-forces.

The following table compares the results: —

Diagonals.		Stress, Ordinary Method.	Stress, New Method.	Difference.
1– 2	28–26	−321685	−321685	
3– 4	27–24	−274518	−259509	15009
5– 6	25–22	−229675	−214932	14743
7– 8	23–20	−187153	−172857	14296
9–10	21–18	−146596	−133285	13671
11–12	19–16	−109080	− 96213	12867
13–14	17–14	− 73528	− 62458	11070
15–12	15–16	− 40299	− 29577	10722
13–10	17–18	− 9393	− 2783	7610

Verticals.		Stress, Ordinary Method.	Stress, New Method.	Difference.
3– 2	27–26	+227500	+227500	10615
5– 4	25–24	+194143	+183528	10426
7– 6	23–22	+162429	+152003	10110
9– 8	21–20	+132357	+122247	9668
11–10	19–18	+103929	+ 94261	9100
13–12	17–16	+ 77143	+ 68043	7583
15–14		+ 28500	+ 20917	

§ 156. **Compound Bridge-Trusses.** — The trusses already discussed have contained but a single system of latticing, or

at least only one system that comes in play at one time; so that a vertical section never cuts more than three bars that are in action simultaneously, the main brace having no stress upon it when the counterbrace is in action, and *vice versa*.

We may, however, have bridge-trusses with more than one system of lattices; and, in determining the stresses in their members, we must resolve them into their component systems, and determine the greatest stress in each system separately, and then, for bars which are common to the two systems, add together the stresses brought about by each.

In some cases, the design is such that it is possible to resolve the truss into systems in more than one way, and then there arises an uncertainty as to which course the stresses will actually pursue.

In such cases, the only safe way is to determine the greatest stress in each piece with every possible mode of resolution of the systems, and then to design each piece in such a way as to be able to resist that stress.

Generally, however, such ambiguity is an indication of a waste of material; as it is most economical to put in the bridge only those pieces that are absolutely necessary to bear the stresses, as other pieces only add so much weight to the structure, and are useless to bear the load.

The mode of proceeding can be best explained by some examples.

EXAMPLE I. — Given the lattice-girder shown in Fig. 117, loaded at the lower panel points only. Dead load $= 7200$ lbs. per panel point, moving load $= 18000$ lbs per panel point; let the entire length of bridge be 60 feet; let the angle made by braces with horizontal $= 60°$.

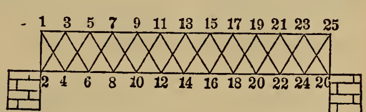


FIG. 117.

This truss evidently consists of the two single trusses shown in Figs. 117*a and* 117*b;* and we can compute the greatest stress of each kind in each member of these trusses, and thus obtain at once all the diagonal stresses, and then, by addition, the greatest chord stresses.

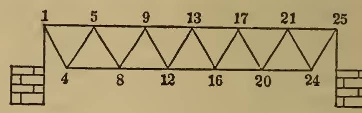


Fig. 117*a*.

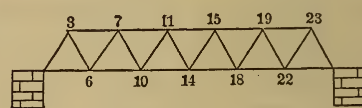


Fig. 117*b*.

Thus the stress in 1–3 (Fig. 117) is the same as the stress in 1–5 (Fig. 117*a*).

The stress in 3–5 = stress in 1–5 (Fig. 117*a*) + stress in 3–7 (Fig. 117*b*).

The stress in 5–7 = stress in 5–9 (Fig. 117*a*) + stress in 3–7 (Fig. 117*b*).

The results are given on the diagram (Fig. 117*c*); the work being left for the student, as it is similar to that done heretofore.

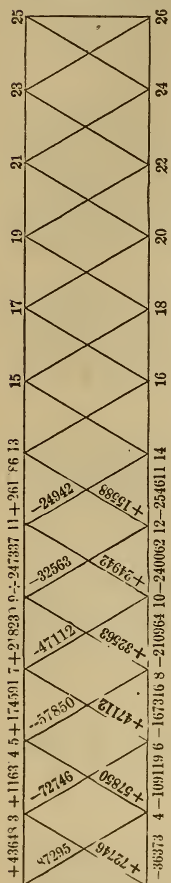


Fig. 117*c*.

Example II. — Given the lattice-girder shown in Fig. 118. Given, as before, Dead load = 7200 lbs. per panel point, moving load = 18000 lbs. per panel point, entire length of bridge = 25 feet; load applied at lower panel points.

Solution. — In this case, there are two possible modes of resolving it into systems. The first is shown in Figs. 118*a* and 118*b:* and this is necessarily the mode of division that must hold whenever the load is unevenly distributed, or when the

travelling-load covers only a part of the bridge; for a single load at 6 is necessarily put in communication with the support at 2 by means of the diagonals 6–3 and 3–2, and with the support at 3 by means of the diagonals 6–7, 7–10, 10–11, and the vertical 11–12, and can cause no stress in the other diagonals.

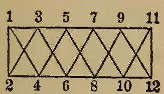


FIG. 118.

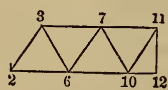


FIG. 118*a*.

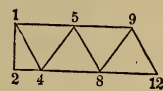


FIG. 118*b*.

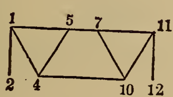


FIG. 118*c*.

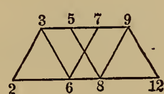


FIG. 118*d*.

When, however, the whole travelling-load is on the bridge, it is perfectly possible to divide it into the two trusses shown in Figs. 118*c* and 118*d*, the diagonals 4–5, 7–10, 6–7, and 5–8 having no stress upon them.

When the load is unevenly distributed, we have certainly the first method of division; and when evenly, we are not sure which will hold.

Hence we must compute the greatest stresses with each mode of division, and use for each member the greatest; for thus only shall we be sure that the truss is made strong enough.

We shall thus have the following results:—

FIRST MODE OF DIVISION (FIGS. 118*a* AND 118*b*).

Diagonals.		Greatest Shearing-Force of One Kind.	Greatest Shearing-Force of Opposite Kind.	Corresponding Stresses.	
Fig. 118*a*.	Fig. 118*b*.				
2– 3	12–9	$\frac{25200}{5}(3+1) = 20160$	0	+23279	− 0
3– 6	9–8	$\frac{25200}{5}(3+1) = 20160$	0	−23279	+ 0
6– 7	8–5	$\frac{25200}{5} = 5040$	$\frac{25200}{5}(2) = 10080$	+ 5820	−11639
7–10	5–4	$\frac{25200}{5} = 5040$	$\frac{25200}{5}(2) = 10080$	− 5820	+11639
10–11	4–1	$0 = 0$	$\frac{25200}{5}(2+4) = 30240$	0	−34918

Chords.

Supporting force at 2 (Fig. 118*a*) or 12 (Fig. 118*b*)

$$= \frac{25200}{5}(3+1) = 20160,$$

Supporting force at 12 (Fig. 118*a*) or 2 (Fig. 118*b*)

$$= \frac{25200}{5}(2+4) = 30240.$$

Section.		Bending-Moment.	Chords.		Maximum Stresses in Separate Trusses.	Chords.		Components of Stresses.	Greatest Resultant Stresses.
Fig. 118*a*.	Fig. 118*b*.		Fig. 118*a*.	Fig. 118*b*.					
3	9	20160× 5=100800	2– 6	8–12	−11639	1–3	9–11	0+1–5	+17459
6	8	20160×10=201600	3– 7	5– 9	+23279	3–5	7– 9	3– 7+1–5	+40738
7	5	30240×10=302400	6–10	4– 8	−34918	5–7		3– 7+5–9	+46558
10	4	30240× 5=151200	7–11	1– 5	+17459	2–4	10–12	2– 6+2–4	−11639
			10–12	2– 4	0	4–6	8–10	2– 6+4–8	−46557
						6–8		6–10+4–8	−69836

SECOND METHOD OF DIVISION (FIGS. 118*c* AND 118*d*).

Diagonals (Fig. 118*c*).

Diagonals.		Maximum Shear.	Corresponding Stresses.
1–4	10–11	25200	−29098
4–5	7–10	0	0

Fig. 118*d*.

Diagonals.		Maximum Shear.	Corresponding Stresses.
2–3	9–12	25200	+29098
3–6	8– 9	25200	−29098
6–7	5– 8	0	0

Chords.

Each supporting force in either figure = 25200.

Fig. 118*c*.

Bending-moment anywhere between 4 and 10 = (25200) (5) = 126000;

$\therefore$ Stress in 1–11 = +14549,
$\therefore$ Stress in 4–10 = −14549.

Fig. 118*d*.

Bending-moment at 3 or 9 = 126000,

Bending-moment anywhere between 6 and 8 = 252000;

$\therefore$ Stress in 3–9 = +29098,
Stress in 2–6 or 8–12 = −14549,
Stress in 6–8 = −29098.

Hence we have for chord stresses, with this second division, —

Chords.			Stresses.
1–3	9–11	1–11 + 0	+14549
3–5	7– 9	1–11 + 3–9	+43647
5–7	–	1–11 + 3–9	+43647
2–4	10–12	0 + 2–6	−14549
4–6	8–10	4–10 + 2–6	−29098
6–8	–	4–10 + 6–8	−43647

Hence, selecting for each bar the greatest, we shall have, as the stresses which the truss must be able to resist, —

1–4	10–11	+ 0	−34918	1–3	9–11	+17459
2–3	12– 9	+29098	− 0	3–5	7– 9	+43647
3–6	9– 8	+ 0	−29098	5–7	–	+46558
4–5	10– 7	+11639	− 5820	2–4	10–12	−14549
5–8	7– 6	+ 5820	−11639	4–6	8–10	−46557
				6–8		−69836

These results are recorded in Fig. 118*e*.

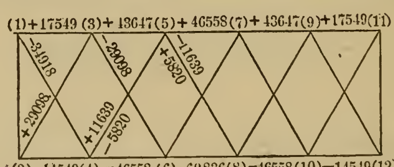


FIG. 118*e*.

§ 157. **Other Trusses.** — In Figs. 119, 120, and 121, we have examples of the double-panel system with the load placed

at the lower panel points only. When, as in 119 and 120, the number of panels is odd, the same ambiguity arises as took place in Fig. 118. When, on the other hand, the number of panels is even, as shown in Fig. 121, there is only one mode of division into systems possible. The diagrams speak for themselves, and need no explanation.

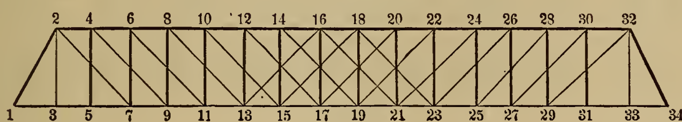


Fig. 119.

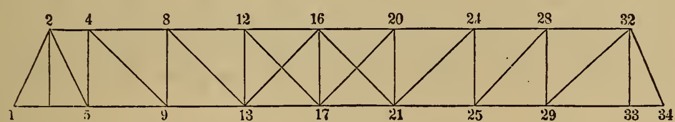


Fig. 119*a*.

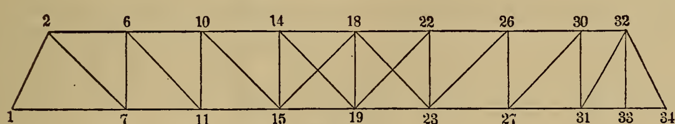


Fig. 119*b*.

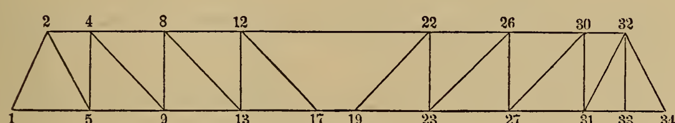


Fig. 119*c*.

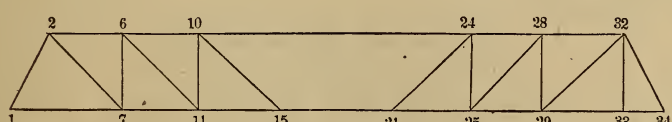


Fig. 119*d*.

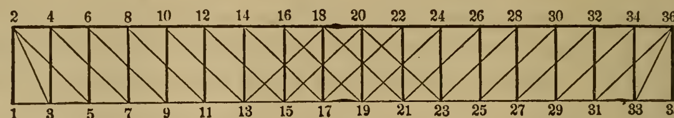


FIG. 120.

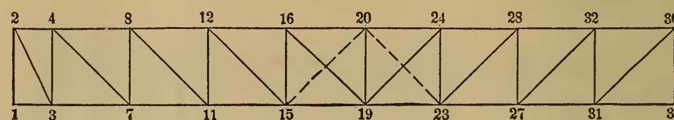


FIG. 120*a*.

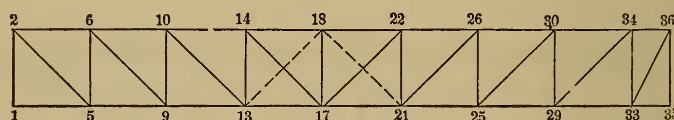


FIG. 120*b*.

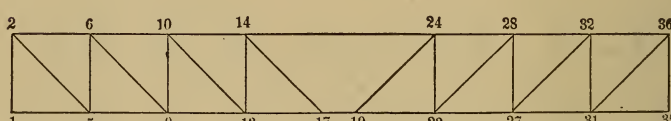


FIG. 120*c*.

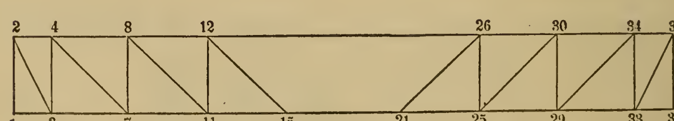


FIG. 120*d*.

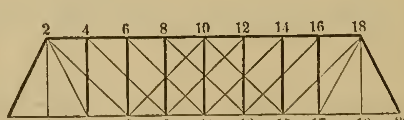


FIG. 120*e*.

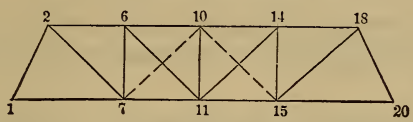


FIG. 121*a*.

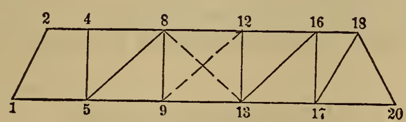


FIG. 121*b*.

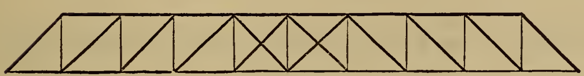


FIG. 122.

The trusses given above may be considered as examples, to be solved by the student by assuming the dead and the moving load per panel point respectively.

§ 158. **Fink's Truss.** — The description of this truss will be evident from the figure. There is, first, the primary truss 1–8–16; then on each side of 9–8 (the middle post of this truss) is a secondary truss (1–4–9 on the left, and 9–12–16 on the right). Each of these secondary trusses contains a pair of smaller secondary trusses, and the division might be continued if the segments into which the upper chord is thus divided were too long.

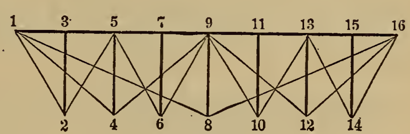


FIG. 123.

Of the inclined ties, there is none in which any load tends to produce compression; in other words, every load either increases the tension in the tie, or else does not affect it. Hence

the greatest stresses in all the members will be attained when the entire travelling-load is on the truss, and we need only consider that case.

The determination of the stress in any one member can readily be obtained by determining, by means of the triangle of forces, the stress in that member due to the presence of the total load per panel point, at each point, and then adding the results. This will be illustrated by a few diagonals.

$$\text{Let angle } 8\text{–}1\text{–}9 = i,$$
$$\text{Let angle } 4\text{–}1\text{–}5 = i_1,$$
$$\text{Let angle } 2\text{–}1\text{–}3 = i_2;$$

we shall have, if $w + w_1$ = entire load per panel point, —

Designation of Ties.	Effect of Loads at							Resultant Tensions.
	3	5	7	9	11	13	15	
1–2	$\frac{w+w_1}{2\sin i_2}$	0	0	0	0	0	0	$\frac{w+w_1}{2\sin i_2}$
2–5	$\frac{w+w_1}{2\sin i_2}$	0	0	0	0	0	0	$\frac{w+w_1}{2\sin i_2}$
5–6	0	0	$\frac{w+w_1}{2\sin i_2}$	0	0	0	0	$\frac{w+w_1}{2\sin i_2}$
6–9	0	0	$\frac{w+w_1}{2\sin i_2}$	0	0	0	0	$\frac{w+w_1}{2\sin i_2}$
1–4	$\frac{w+w_1}{4\sin i_1}$	$\frac{w+w_1}{2\sin i_1}$	$\frac{w+w_1}{4\sin i_1}$	0	0	0	0	$\frac{w+w_1}{\sin i_1}$
4–9	$\frac{w+w_1}{4\sin i_1}$	$\frac{w+w_1}{2\sin i_1}$	$\frac{w+w_1}{4\sin i_1}$	0	0	0	0	$\frac{w+w_1}{\sin i_1}$
1–8	$\frac{w+w_1}{8\sin i}$	$\frac{w+w_1}{4\sin i}$	$\frac{w+w_1}{8\sin i}$	$\frac{w+w_1}{2\sin i}$	$\frac{w+w_1}{8\sin i}$	$\frac{w+w_1}{4\sin i}$	$\frac{w+w_1}{8\sin i}$	$\frac{3(w+w_1)}{2\sin i}$

The stresses in all the other members may be found in a similar manner.

§ 159. **Bollman's Truss.** — The description of this truss is made sufficiently clear by the figure. The upper chord is made in separate pieces; and the short diagonals 2–5, 3–4, 4–7, 5–6, 7–8, 6–9, 8–11, and 9–10 are only needed to prevent a bending of the upper chord at the joints.

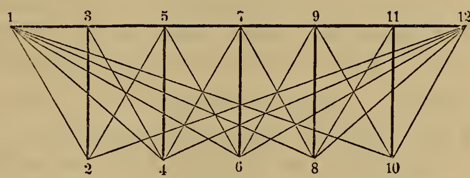


FIG. 124.

When this is their only object, the stress upon them cannot be calculated: indeed, it is zero until bending takes place; and then it is the less, the less the bending. Hence, in this case, the stress is wholly taken up by the principal ties; and these have their greatest stress when the whole load is on the bridge.

The computation of the stresses is made in a similar manner to that used in the Fink.

§ 160. **General Remarks.** — The methods already explained are intended to enable the student to solve any case of a bridge-truss where there is no ambiguity as to the course pursued by the stresses.

In cases where a large number of trusses of one given type are to be computed, it would, as a rule, be a saving of labor to determine formulæ for the stresses in the members, and then substitute in these formulæ.

Such formulæ may be deduced by using letters to denote the load and dimensions, instead of inserting directly their numerical values; and then, having deduced the formulæ for the type of truss, we can apply it to any case by merely substituting for the letters their numerical values corresponding to that case.

Such sets of formulæ would apply merely to specific styles of trusses, and any variation in these styles would require the formulæ to be changed.

In order to show how such formulæ are deduced, a few will be deduced for such a bridge as is shown in Fig. 111.

Let the load be applied at the upper panel points only; let dead load per panel point $= w$, moving load per panel point $= w_1$. Let the whole number of panels be N, N being an even number. Let the length of one panel = height of truss $= l$. Then length of entire span $= Nl$.

Consider the n^{th} panel from the middle.

The stress in the main tie is greatest when the moving load is on all the panel points from the farther abutment up to the panel in question.

Hence, for the n^{th} panel from the middle, the greatest shearing-force that causes tension in the main tie

$$= \frac{w+w_1}{N}\left\{1+2+3+\ldots+\frac{N}{2}+n\right\} - \frac{w}{N}\left\{1+2+3+\ldots+\frac{N}{2}-n-1\right\}$$

$$= \frac{1}{2N}\left\{w_1\left[\left(\frac{N}{2}+n\right)^2+\frac{N}{2}+n\right]+\frac{w}{2}(2n+1)N\right\}.$$

Hence stress in main tie

$$= \frac{\sqrt{2}}{2N}\left\{w_1\left[\left(\frac{N}{2}+n\right)^2+\frac{N}{2}+n\right]+\frac{w}{2}(2n+1)N\right\}. \quad (1)$$

For the counterbrace, we should obtain, in a similar way, the formula

$$\frac{\sqrt{2}}{2N}\left\{w_1\left[\left(\frac{N}{2}-n\right)^2-\frac{N}{2}+n\right]-\frac{wN}{2}(2n+1)\right\},$$

which represents tension when it is positive. Proceed in a similar way for the other members.

When there is more than one system, we must divide the truss into its component systems; and when there is ambiguity, we must use, in determining the dimensions of each member, the greatest stress that can possibly come upon it.

CHAPTER V.

CENTRE OF GRAVITY.

§ 161. The centre of gravity of a body or system of bodies, is that point through which the resultant of the system of parallel forces that constitutes the weight of the body or system of bodies always passes, whatever be the position in which the body is placed with reference to the direction of the forces.

§ 162. **Centre of Gravity of a System of Bodies.** — If we have a system of bodies whose weights are W_1, W_2, W_3, etc., the co-ordinates of their individual centres of gravity being (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc., respectively, and if we denote by x_0, y_0, z_0, the co-ordinates of the centre of gravity of the system, we should obtain, just as in the determination of the centre of any system of parallel forces, —

1°. By turning all the forces parallel to OZ, and taking moments about OY,

$$(W_1 + W_2 + W_3 + \text{etc.})x_0 = W_1x_1 + W_2x_2 + W_3x_3 + \text{etc.},$$

or

$$x_0\Sigma W = \Sigma Wx;$$

and, taking moments about OX,

$$(W_1 + W_2 + W_3 + \text{etc.})y_0 = W_1y_1 + W_2y_2 + W_3y_3 + \text{etc.},$$

or

$$y_0\Sigma W = \Sigma Wy.$$

2°. By turning all the forces parallel to OX, and taking moments about OY,

$$(W_1 + W_2 + W_3 + \text{etc.})z_0 = W_1 z_1 + W_2 z_2 + W_3 z_3 + \text{etc.},$$

or

$$z_0 \Sigma W = \Sigma Wz.$$

Hence we have, for the co-ordinates of the centre of gravity of the system,

$$x_0 = \frac{\Sigma Wx}{\Sigma W}, \quad y_0 = \frac{\Sigma Wy}{\Sigma W}, \quad z_0 = \frac{\Sigma Wz}{\Sigma W}.$$

EXAMPLES.

1. Suppose a rectangular, homogeneous plate of brass (Fig. 125), where $AD = 12$ inches, $AB = 5$ inches, and whose weight is 2 lbs., to have weights attached at the points A, B, C, and D respectively, equal to 8, 6, 5, and 3 lbs.; find the centre of gravity of the system.

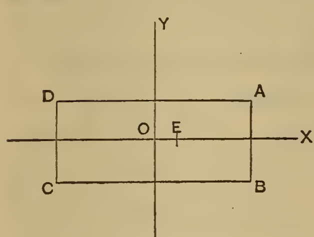


Fig. 125.

Solution.

Assume the origin of co-ordinates at the centre of the rectangle, and we have

$$\begin{array}{lllll} W_1 = 2, & W_2 = 8, & W_3 = 6, & W_4 = 5, & W_5 = 3, \\ x_1 = 0, & x_2 = 6, & x_3 = 6, & x_4 = -6, & x_5 = -6, \\ y_1 = 0, & y_2 = \frac{5}{2}, & y_3 = -\frac{5}{2}, & y_4 = -\frac{5}{2}, & y_5 = \frac{5}{2}; \end{array}$$

$$\begin{aligned} \therefore \quad \Sigma Wx &= 0 + 48 + 36 - 30.0 - 18.0 = 36, \\ \Sigma Wy &= 0 + 20 - 15 - 12.5 + 7.5 = 0, \\ \Sigma W &= 2 + 8 + 6 + 5.0 + 3.0 = 24; \end{aligned}$$

$$\therefore \quad x_0 = \frac{36}{24} = 1.5, \quad y_0 = \frac{0}{24} = 0.$$

Hence the centre of gravity is situated at a point E on the line OX, where $OE = 1.5$.

2. Given a uniform circular plate of radius 8, and weight 3 lbs. (Fig. 126). At the points A, B, C, and D, weights are attached equal to 10, 15, 25, and 23 lbs. respectively, also given $AB = 45°$, $BC = 105°$, $CD = 120°$; find the centre of gravity of the system.

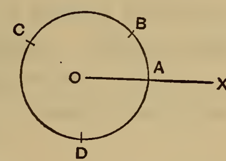


FIG. 126.

§ 163. **Centre of Gravity of Homogeneous Bodies.** — For the case of a single homogeneous body, the formulæ have been already deduced in § 44. They are

$$x_0 = \frac{\int x dV}{\int dV}, \quad y_0 = \frac{\int y dV}{\int dV}, \quad z_0 = \frac{\int z dV}{\int dV};$$

and for the weight of the body,

$$W = w \int dV,$$

where x_0, y_0, z_0, are the co-ordinates of the centre of gravity of the body, W its weight, and w its weight per unit of volume.

From these formulæ we can readily deduce those for any special cases; thus, —

(*a*) *For a volume* referred to rectangular co-ordinate axes, $dV = dxdydz$.

$$x_0 = \frac{\int\int\int x dxdydz}{\int\int\int dxdydz}, \quad y_0 = \frac{\int\int\int y dxdydz}{\int\int\int dxdydz}, \quad z_0 = \frac{\int\int\int z dxdydz}{\int\int\int dxdydz}.$$

(*b*) *For a flat plate of uniform thickness*, t, the centre of gravity is in the middle layer; hence only two co-ordinates are required to determine it. If it be referred to a system of rectangular axes in the middle plane, $dV = tdxdy$,

$$x_0 = \frac{\int\int x dxdy}{\int\int dxdy}, \quad y_0 = \frac{\int\int y dxdy}{\int\int dxdy}.$$

The centre of gravity of such a thin plate is also called the centre of gravity of the plane area that constitutes the middle plane section; hence —

(*c*) *For a plane area* referred to rectangular co-ordinate axes in its own plane,

$$x_0 = \frac{\int\int x dx dy}{\int\int dx dy}, \quad y_0 = \frac{\int\int y dx dy}{\int\int dx dy}.$$

(*d*) *For a slender rod of uniform sectional area, a,* if $x, y, z,$ be the co-ordinates of points on the axis (straight or curved) of the rod, we shall have $dV = ads = a\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$,

$$x_0 = \frac{\int x ds}{\int ds} = \frac{\int x \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx}{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx},$$

$$y_0 = \frac{\int y ds}{\int ds} = \frac{\int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx}{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx},$$

$$z_0 = \frac{\int z ds}{\int ds} = \frac{\int z \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx}{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx}.$$

(*e*) *For a slender rod whose axis lies wholly in one plane,* the centre of gravity lies, of course, in the same plane; and if our co-ordinate axes be taken in this plane, we shall have $z = 0$ $\therefore \frac{dz}{dx} = 0$, and also $z_0 = 0$. Hence we need only two co-

ordinates to determine the centre of gravity, hence $dV = ads$ $= a\sqrt{(dx)^2 + (dy)^2}$.

$$x_0 = \frac{\int xds}{\int ds} = \frac{\int x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx},$$

$$y_0 = \frac{\int yds}{\int ds} = \frac{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}.$$

(f) *For a line*, straight or curved, which lies entirely in one plane, we shall have, again,

$$x_0 = \frac{\int xds}{\int ds} = \frac{\int x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx},$$

$$y_0 = \frac{\int yds}{\int ds} = \frac{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}.$$

Whenever the body of which we wish to determine the centre of gravity is made up of simple figures, of which we already know the positions of the centres of gravity, the method explained in § 162 should be used, and not the formulæ that involve integration ; i.e., taking moments about any given line will give us the perpendicular distance of the centre of gravity from that line.

In the case of the determination of the strength and stiffness of beams, it is necessary to know the distance of a horizontal line passing through the centre of gravity of the section,

from the top or the bottom of the section; but it is of no practical importance to know the position of the centre of gravity on this line. In most of the examples that follow, therefore, the results given are these distances. These examples should be worked out by the student.

In the case of wrought-iron beams of various sections, on account of the thinness of the iron, a sufficiently close approximation is often obtained by considering the cross-section as composed of its central lines; the area of any given portion being found by multiplying the thickness of the iron by the corresponding length of line, the several areas being assumed to be concentrated in single lines.

EXAMPLES.

1. *Straight Line* AB (Fig. 127). — The centre of gravity is evidently at the middle of the line, as this is a point of symmetry.

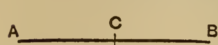


FIG. 127.

2. *Combination of Two Straight Lines.* — The centre of gravity in each case lies on the line OO_1, Figs. 128, 129, 130, and 131.

(*a*) *Angle-Iron of Unequal Arms* (Fig. 128). — Length $AB = b$, length $BC = h$, area $AB = A$, area $BC = B$;

$$\therefore \quad BE = DE = \frac{1}{2}\frac{bh}{\sqrt{b^2 + h^2}}.$$

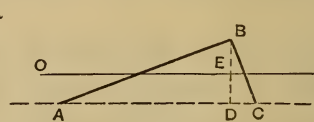


FIG. 128.

(*b*) *Angle-Iron of Equal Arms* (Fig. 129). — Length $AB = BC = b$;

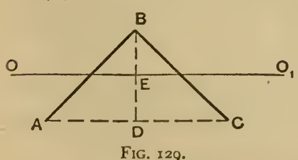


FIG. 129.

$$\therefore \quad BE = DE = \frac{b}{2\sqrt{2}} = \frac{b}{4}\sqrt{2}.$$

(*c*) *Cross of Equal Arms* (Fig. 130).—$AB = OO_1 = h$;

$$\therefore \quad AC = BC = \frac{h}{2}.$$

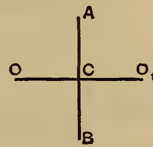


FIG 130.

(*d*) T-*Iron* (Fig. 131).—Area $AB = A$, area $CE = B$, length $CE = h$;

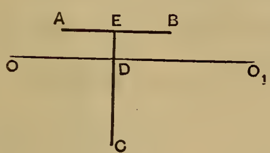


FIG. 131.

$$\therefore \quad DE = \frac{Bh}{2(A+B)},$$

$$CD = \frac{2A+B}{2(A+B)}h.$$

3. *Combination of Three Lines.*— OO_1 = line passing through the centre of gravity.

(*a*) *Thin Isosceles Triangular Cell* (Fig. 132).—Length $AB = BC = a$, length $AC = b$, area $AB = BC = A$, area $AC = B$;

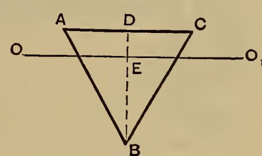


FIG. 132.

$$\therefore \quad DB = \sqrt{a^2 - \frac{b^2}{4}}$$

$$= \tfrac{1}{2}\sqrt{(2a-b)(2a+b)}$$

$$\therefore \quad DE = \frac{A}{2(2A+B)}\sqrt{(2a-b)(2a+b)},$$

$$BE = \frac{A+B}{2(2A+B)}\sqrt{(2a-b)(2a+b)}.$$

(*b*) *Same in Different Position* (Fig. 133).

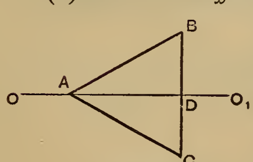


FIG. 133.

$$BD = DC = \frac{b}{2}.$$

(*c*) *Channel-Iron* (Fig. 134).—Area of flanges $= A$, area of web $= B$, depth of flanges $+ \frac{1}{2}$ thickness of web $= h$;

FIG. 134.

$$\therefore \quad CE = \tfrac{1}{2}\frac{Ah}{A+B},$$

$$DE = \frac{h}{2}\frac{A+2B}{A+B}.$$

(*d*) I-*Beam* (Fig. 135).—Area of upper flange $= A_1$, area of lower flange $= A_2$, area of web $= B$, height $= h$.

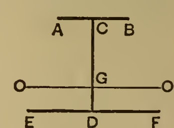


FIG. 135.

$$CG = \frac{h}{2}\frac{2A_2 + B}{A_1 + A_2 + B},$$

$$GD = \frac{h}{2}\frac{2A_1 + B}{A_1 + A_2 + B}.$$

4. *Combination of Four Lines.*— OO_1 = line passing through the centre of gravity.

(*a*) *Thin Rectangular Cell* (Fig. 136).—Length $AB = h$;

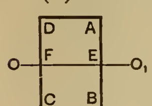


FIG. 136.

$$\therefore \quad AE = BE = \frac{h}{2}.$$

(*b*) *Thin Square Cell* (Fig. 137).—$AB = BC = h$;

$$\therefore \quad BE = CE = \frac{h}{2}.$$

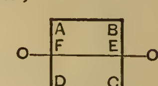


FIG. 137.

5. *Circular Arcs.*

(*a*) *Circular Arc* AB (Fig. 138).—Angle $AOB = \theta_1$, radius $= r$. Use formula

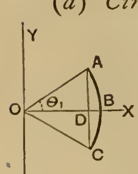


FIG. 138.

$$x_0 = \frac{\int x ds}{\int ds}, \quad y_0 = \frac{\int y ds}{\int ds};$$

but use polar co-ordinates, where

$$ds = r d\theta, \quad x = r\cos\theta, \quad y = r\sin\theta,$$

$$\therefore \quad x_0 = \frac{r^2 \int_0^{\theta_1} \cos\theta d\theta}{r\int_0^{\theta_1} d\theta} = r\frac{(\sin\theta_1)}{\theta_1},$$

$$y_0 = \frac{r^2 \int_0^{\theta_1} \sin\theta d\theta}{r\int_0^{\theta_1} d\theta} = r\frac{(1 - \cos\theta_1)}{\theta_1} = 2r\frac{\sin^2\frac{1}{2}\theta_1}{\theta_1}.$$

(*b*) *Circular Arc* AC (same figure).

$$x_0 = \frac{r\sin\theta_1}{\theta_1}, \quad y_0 = 0.$$

(*c*) *Quarter-Arc of Circle* AB, *Radius* r (Fig. 139).

$$x_0 = \frac{r^2 \int_0^{\frac{\pi}{2}} \cos\theta d\theta}{r\int_0^{\frac{\pi}{2}} d\theta} = \frac{2r}{\pi}.$$

$$y_0 = \frac{2r}{\pi}.$$

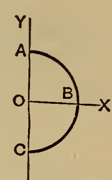


FIG. 139.

(*d*) *Semi-circumference* ABC (same figure).

$$x_0 = \frac{2r}{\pi}, \quad y_0 = 0.$$

6. *Combination of Circles and Straight Lines.*

Barlow Rail (Fig. 140). — Two quadrants, radius r, and web, whose area $= \frac{3}{11}$ the united area of the quadrants. Let united area of quadrants $= A$, area of web $= \frac{3}{11}A$; let $EF = x_0$:

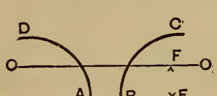


FIG. 140.

$$\pi = \frac{22}{7},$$

$$\therefore \quad \frac{14}{11}Ax_0 = A\left(\frac{2r}{\pi}\right) = \frac{7}{11}Ar \qquad \therefore \quad x_0 = \frac{r}{2} = EF_1.$$

7. *Areas.*

(*a*) *T-Section* (Fig. 141).—Let length $AB = B$, $EF = b$, entire height $= H$, $GE = h$. Let distance of centre of gravity below $AB = x_1$; therefore, taking moments about AB as an axis,

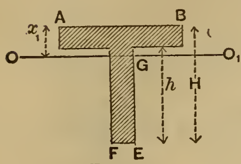


FIG. 141

$$x_1\{BH - h(B - b)\}$$
$$= \tfrac{1}{2}BH^2 - h(B - b)\left(H - \frac{h}{2}\right),$$

whence we can readily derive x_1.

(*b*) *I-Section* (Fig. 142).—Let $AB = B$, $GH = b$, $MN = b_1$, entire height $= H$, $BC = H - h$, $EH = h_1$; and let $x_1 =$ distance of centre of gravity below AB.

Hence, taking moments about AB, we have

$$x_1\{B(H - h) + b_1(h - h_1) + bh_1\}$$
$$= \frac{B}{2}(H - h)^2 + bh_1\left(H - \frac{h_1}{2}\right)$$
$$+ b_1(h - h_1)\left(H - h + \frac{h - h_1}{2}\right),$$

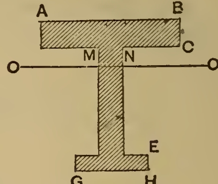


FIG. 142.

whence we can deduce x_1.

(*c*) *Triangle* (Fig. 143).—If we consider the triangle OBC as composed of an indefinite number of narrow strips parallel to the side CB, of which $FLHK$ is one, the centre of gravity of each one of these strips will be on the line OD drawn from O to the middle point of the side CB; hence the centre of gravity of the entire triangle must be on the line OD. For a similar reason, it must be on the median line CE; hence the centre of gravity must be at the intersection of the median lines, and hence

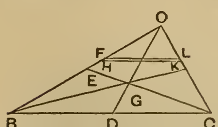


FIG. 143.

$$x_0 = OG = \tfrac{2}{3}OD. \quad \text{Moreover, area} = \frac{BC \,.\, OD \sin ODC}{2}.$$

(d) *Trapezoid* (Fig. 144).

Graphical Solution.—Bisect AB in O, and CE in D; join O and D; consider the trapezoid to be divided into two triangles, ABC and CEB; let g_1 be the centre of gravity of CEB, and g_2 that of ABC. Then will

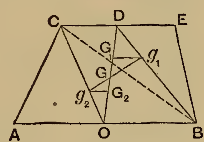


FIG. 144.

$$DG_1 = G_1G_2 = OG_2 = \tfrac{1}{3}OD,$$

where $G_1g_1 \parallel G_2g_2 \parallel AB$. Then will G, the centre of gravity of the trapezoid, be on the line g_1g_2, at such a point as shall make

$$\frac{Gg_1}{Gg_2} = \frac{ABC}{CEB}.$$

But a similar reasoning to that used in the case of the triangle will show that it must be on the line OD; hence it must be at the intersection of OD and g_1g_2. This gives a graphical construction, and from this we can deduce the value of OG as follows:—

From the similarity of GG_1g_1 and GG_2g_2 we have

$$\frac{GG_1}{GG_2} = \frac{Gg_1}{Gg_2} = \frac{ABC}{BEC} = \frac{AB}{CE} = \frac{B}{b},$$

since the triangles ABC and CEB have the same altitude: hence, from the proportion $\frac{GG_1}{GG_2} = \frac{B}{b}$, we have

$$\frac{GG_2}{G_1G_2} = \frac{b}{B+b};$$

but $G_1G_2 = \frac{OD}{3}$, and hence

$$OG = OG_2 + GG_2 = \frac{OD}{3} + GG_2 = \frac{OD}{3}\left(1 + \frac{b}{B+b}\right).$$

(*e*) *Parabolic Half-Segment* OAB (Fig. 145). — Let $OA = x_1$, $AB = y_1$; let x_0, y_0, be the co-ordinates of the centre of gravity; let the equation of the parabola be $y^2 = 4ax$:

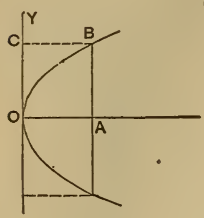


FIG. 145.

$$\therefore \quad x_0 = \frac{\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} xdxdy}{\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} dxdy} = \frac{2a^{\frac{1}{2}}\int_0^{x_1} x^{\frac{3}{2}}dx}{2a^{\frac{1}{2}}\int_0^{x_1} x^{\frac{1}{2}}dx}$$

$$= \frac{\frac{2}{5}x_1^{\frac{5}{2}}}{\frac{2}{3}x_1^{\frac{3}{2}}} = \frac{3}{5}x_1,$$

$$y_0 = \frac{\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} ydxdy}{\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} dxdy} = \frac{3}{4}a^{\frac{1}{2}}x_1^{\frac{1}{2}} = \frac{3}{8}y_1,$$

$$\text{Area} = \int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} dxdy = \frac{4}{3}a^{\frac{1}{2}}x_1^{\frac{3}{2}} = \frac{2}{3}(2a^{\frac{1}{2}}x_1^{\frac{1}{2}})x_1 = \frac{2}{3}x_1y_1.$$

(*f*) *Parabolic Spandril* OBC (Fig. 145). — Let x_0, y_0, be co-ordinates of centre of gravity of the spandril.

$$x_0 = \frac{\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1}\int_0^{x_1} xdxdy}{\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1}\int_0^{x_1} dxdy} = \frac{3}{10}x_1,$$

$$y_0 = \frac{\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1}\int_0^{x_1} ydxdy}{\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1}\int_0^{x_1} dxdy} = \frac{3}{4}y_1,$$

$$\text{Area} = x_1y_1 - \frac{2}{3}x_1y_1 \quad = \frac{1}{3}x_1y_1.$$

(*g*) *Circular Sector* OAC (Fig. 146).—Let $OA = r$, $AOX = \theta_1$, x_0, y_0, be the co-ordinates of the centre of gravity;

$$\therefore \quad y_0 = 0,$$

$$x_0 = \frac{\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{r\cos\theta_1}^{r} xdxdy + \int_{-x\tan\theta_1}^{x\tan\theta_1} \int_0^{r\cos\theta_1} xdxdy}{\frac{1}{2}r(2r\theta_1)}$$

$$= \tfrac{2}{3}r\frac{\sin\theta_1}{\theta_1},$$

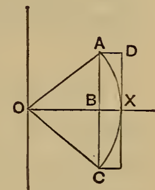


FIG. 146.

$$\text{Area} = \tfrac{1}{2}r(2r\theta_1) = r^2\theta_1.$$

Second Solution.

Consider the sector to be made up of an indefinite number of narrow rings; let ρ be the variable radius, and $d\rho$ the thickness;

$$\therefore \quad \text{Elementary area} = 2\rho\theta_1 d\rho,$$

and centre of gravity of this elementary area is on OX, at a distance from O equal to $\rho\dfrac{\sin\theta_1}{\theta_1}$ [see Example 5 (*b*)];

$$\therefore \quad x_0 = \frac{\int_0^r \left\{\rho\frac{\sin\theta_1}{\theta_1}\right\}\left\{2\rho\theta_1 d\rho\right\}}{\int_0^r 2\rho\theta_1 d\rho} = \frac{2\sin\theta_1\int_0^r \rho^2 d\rho}{2\theta_1\int_0^r \rho d\rho} = \tfrac{2}{3}r\frac{\sin\theta_1}{\theta_1}.$$

(*h*) *Circular Half-Segment* ABX (Fig. 146).

$$x_0 = \frac{\int_0^{\sqrt{r^2-x^2}}\int_{r\cos\theta_1}^{r} xdxdy}{\text{Sector minus triangle}} = \frac{\int_{r\cos\theta_1}^{r} x\sqrt{r^2 - x^2}dx}{\frac{1}{2}r^2\theta_1 - \frac{1}{2}r^2\sin\theta_1\cos\theta_1}$$

$$= \tfrac{2}{3}r\frac{\sin^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1},$$

$$y_0 = \frac{\int_0^{\sqrt{r^2-x^2}}\int_{r\cos\theta_1}^{r} ydxdy}{\frac{1}{2}r^2(\theta_1 - \sin\theta_1\cos\theta_1)} = \tfrac{1}{3}r\frac{4\sin^2\frac{1}{2}\theta_1 - \sin^2\theta_1\cos\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1}.$$

§ 164. **Pappus's Theorems.** — The following two theorems serve often to simplify the determination of the centres of gravity of lines and areas. They are as follows: —

THEOREM I. — If a plane curve lies wholly on one side of a straight line in its own plane, and, revolving about that line, generates thereby a surface of revolution, the area of the surface is equal to the product of the length of the revolving line, and of the path described by its centre of gravity.

Proof. — Let the curve lie in the xy plane, and let the axis of y be the line about which it revolves. We have, from what precedes, § 163 (*e*), $x_0 = \frac{\int xds}{\int ds}$;

$$\therefore \quad x_0 \int ds = \int xds,$$

where x_0 equals the perpendicular distance of the centre of gravity of the curve from OY, ds = elementary arc,

$$\therefore \quad 2\pi x_0 \int ds = \int (2\pi x) ds;$$

or, reversing the equation,

$$\int (2\pi x) ds = (2\pi x_0) s.$$

But $\int (2\pi x) ds$ = surface described in one revolution, while s = length of arc, and $2\pi x_0$ = path described by the centre of gravity in one revolution. Hence follows the proposition.

THEOREM II. — If a plane area lying wholly on the same side of a straight line in its own plane revolves about that line, and thereby generates a solid of revolution, the volume of the solid thus generated is equal to the product of the revolving area, and of the path described by the centre of gravity of the plane area during the revolution.

Proof. — Let the area lie in the xy plane, and let the axis OY be the axis of revolution. We then have, from what has preceded, if x_0 = perpendicular distance of the centre of gravity of the plane area from OY, the equation, § 163 (b),

$$x_0 = \frac{\int\int x dx dy}{\int\int dx dy}.$$

Hence

$$x_0 \int\int dx dy = \int\int x dx dy;$$

$$\therefore \quad (2\pi x_0)\int\int dx dy = \int\int (2\pi x) dx dy$$

or

$$\int\int (2\pi x) dx dy = 2\pi x_0 \int dx dy.$$

But $\int\int (2\pi x) dx dy$ = volume described in one revolution, and $2\pi x_0$ = path described by the centre of gravity in one revolution. Hence follows the proposition.

The same propositions hold true for any part of a revolution, as well as for an entire revolution, since we might have multiplied through by the circular measure θ, instead of by 2π.

It is evident that the first of these two theorems may be used to determine the centre of gravity of a line, when the length of the line, and the surface described by revolving it about the axis, are known; and so also that the second theorem may be used to determine the centre of gravity of a plane area whenever the area is known, and also the volume described by revolving it around the axis.

EXAMPLES.

1. *Circular Arc* AC (Fig. 138). — Length of arc = $s = 2r\theta$, surface of zone described by revolving it about OY = circumference of a great circle multiplied by the altitude = $(2\pi r)(2r \sin \theta_1)$;

$$\therefore \quad (2\pi x_0)(2r\theta_1) = (2\pi r)(2r \sin \theta_1) \qquad \therefore \quad x_0\theta_1 = r \sin \theta_1$$

$$\therefore \quad x = r\frac{\sin \theta_1}{\theta_1}.$$

2. *Semicircular Arc* (Fig. 139).—Length of arc $= \pi r$, surface of sphere described $= 4\pi r^2$;

$$\therefore \quad 2\pi x_0(\pi r) = 4\pi r^2 \qquad \therefore \quad x_0 = \frac{2r}{\pi}.$$

3. *Trapezoid* (Fig. 147).—Let $AD = b$, $BC = b$; let it revolve around AD: it generates two cones and a cylinder.

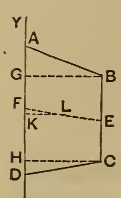


Fig. 147.

$$\text{Area of trapezoid} = \frac{AD + BC}{2} BG,$$

$$\text{Volume} = \frac{\pi(GB)^2}{3}(AG + HD) + \pi(GB)^2 . BC$$

$$= \frac{\pi(GB)^2}{3}(AG + HD + 3BC)$$

$$= \frac{\pi(GB)^2}{3}(AD + BC + BC)$$

$$\therefore \quad (2\pi x_0)\left(\frac{AD + BC}{2}\right) . GB = \frac{\pi(GB)^2}{3}\{(AD + BC) + BC\}$$

$$\therefore \quad x_0 = \frac{GB}{3}\left(1 + \frac{BC}{AD + BC}\right) = \frac{GB}{3}\left(1 + \frac{b}{B + b}\right) = KL$$

$$\therefore \quad FL = \frac{FE}{3}\left(1 + \frac{b}{B + b}\right).$$

4. *Circular Sector* ACO (Fig. 146).—Area of sector $= r^2\theta_1$, volume described $= \frac{1}{3}r$(surface of zone) $= \frac{1}{3}r(2\pi r)(2r \sin \theta_1) = \frac{4}{3}\pi r^3 \sin \theta_1$;

$$\therefore \quad (2\pi x_0)(r^2\theta_1) = \tfrac{4}{3}\pi r^3 \sin \theta_1 \qquad \therefore \quad x_0 = \tfrac{2}{3}r\frac{\sin \theta_1}{\theta_1}.$$

§ 165. **Centre of Gravity of Solid Bodies.**—The general formulæ furnish, in most cases, a very complicated solution, and hence we generally have recourse to some simpler method. A few examples will be given in this and the next section.

Tetrahedron ABCD (Fig. 148). — The plane ABE, containing the edge AB and the middle point E of the edge CD, bisects all lines drawn parallel to CD, and terminating in the faces ABD and ABC: hence a similar reasoning to that used in the case of the triangle will show that the centre of gravity of the pyramid must be in the plane ABE; in the same way it may be shown that it must lie in the plane ACF. Hence it must lie in their intersection, or in the line AG joining the vertex A with the centre of gravity (intersection of the medians) of the opposite face. In the same way it can be shown that the centre of gravity of the triangular pyramid must lie in the line drawn from the vertex B to the centre of gravity of the face ACD. Hence the centre of gravity of the tetrahedron will be found on the line AG at a distance from G equal to $\frac{1}{4}AG$.

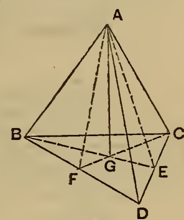


FIG. 148.

§ 166. **Centre of Gravity of Bodies which are Symmetrical with Respect to an Axis.** — Such solids may be generated by the motion of a plane figure, as $ABCD$ (Fig. 149), of variable dimensions, and of any form whose centre G remains upon the axis OX; its plane being always perpendicular to OX, and its variable area X being a function of x, its distance from the origin.

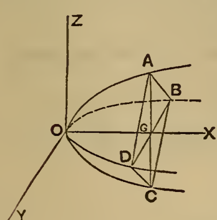


FIG. 149.

Here the centre of gravity will evidently lie on the axis OX, and the elementary volume will be the volume of a thin plate whose area is X and thickness Δx; hence the elementary volume will be $X\Delta x$.

Take moments about OY, and we shall have

$$x_0 \int X dx = \int X x dx \quad \text{and} \quad \text{Volume} = \int X dx,$$

or

$$x_0 = \frac{\int X x dx}{\int X dx}, \qquad V = \int X dx.$$

EXAMPLES.

1. *Ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Fig. 150).—Find centre of gravity of the half to the right of the x plane. Let $OK = x$. Now if, in the equation of the ellipsoid, we make $y = 0$, we have $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$;

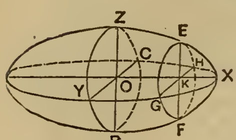


FIG. 150.

$$\therefore \quad z = \frac{c}{a}\sqrt{a^2 - x^2},$$

where $z = EK$.

Make $z = 0$ in the equation of the ellipsoid, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

$$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2},$$

where $y = KG$;

$$\therefore \quad EK = \frac{c}{a}\sqrt{a^2 - x^2}, \qquad KG = \frac{b}{a}\sqrt{a^2 - x^2},$$

are the semi-axes of the variable ellipse $EGFH$, which, by moving along OX, generates the ellipsoid. Hence

$$\text{Area } EGFH = \pi(EK \cdot GK) = \frac{\pi bc}{a^2}(a^2 - x^2) = X;$$

hence

$$\text{Elementary volume} = \frac{\pi bc}{a^2}(a^2 - x^2)\Delta x$$

$$\therefore \quad x_0 = \frac{\frac{\pi bc}{a^2}\int_0^a (a^2x - x^3)dx}{\frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)dx} = \frac{\left\{\frac{a^2x^2}{2} - \frac{x^4}{4}\right\}_0^a}{\left\{a^2x - \frac{x^3}{3}\right\}_0^a} = \tfrac{3}{8}a.$$

$$V = \frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)dx = \tfrac{2}{3}\pi abc.$$

2. *Hemisphere.*—Make $a = b = c$, and $x_0 = \frac{3}{8}a$, $V = \frac{2}{3}\pi a^3$.

If the section X were oblique to OX, making an angle θ with it, the elementary volume would not be Xdx, but $Xdx\sin\theta$, and we should have

$$x_0 = \frac{\sin\theta\int Xxdx}{\sin\theta\int Xdx} = \frac{\int Xxdx}{\int Xdx} \quad \text{and} \quad V = \sin\theta\int Xdx.$$

3. *Oblique Cone* (Fig. 151).—Let $OA = h$; let area of base be A, and let the angle made by OX with the base be θ;

$$\therefore\ \frac{X}{A} = \frac{x^2}{h^2} \qquad \therefore\ X = \frac{A}{h^2}x^2$$

$$\therefore\ x_0 = \frac{\frac{A}{h^2}\int_0^h x^3dx}{\frac{A}{h^2}\int_0^h x^2dx} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \tfrac{3}{4}h.$$

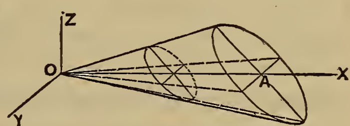


FIG. 151.

$$V = \sin\theta\int Xdx = \frac{A}{h^2}\sin\theta\int_0^h x^2dx = \tfrac{1}{3}Ah\sin\theta.$$

4. *Truncated Cone* (Fig. 151).—Let height of entire cone be $h = OA$; let height of portion cut off be h_1;

$$\therefore\ x_0 = \frac{\frac{A}{h^2}\int_{h_1}^h x^3dx}{\frac{A}{h^2}\int_{h_1}^h x^2dx} = \frac{\frac{h^4 - h_1^4}{4}}{\frac{h^3 - h_1^3}{3}} = \frac{3}{4}\frac{h^4 - h_1^4}{h^3 - h_1^3}.$$

$$V\frac{A}{h^2}\sin\theta\int_{h_1}^h x^2dx = \frac{Ah\sin\theta}{3h^3}(h^3 - h_1^3) = \tfrac{1}{3}Ah\sin\theta\left(1 - \frac{h_1^3}{h^3}\right).$$

CHAPTER VI.

STRENGTH OF MATERIALS.

§ 167. **Stress, Strain, and Modulus of Elasticity.** — When a body is subjected to the action of external forces, if we imagine a plane section dividing the body into two parts, the force with which one part of the body acts upon the other at this plane is called the *stress* on the plane; and, in order to know it completely, we must know its distribution and its direction at each point of the plane. If we consider a small area lying in this plane, including the point O, and represent the stress on this area by p, whereas the area itself is represented by a, then will the limit of $\frac{p}{a}$ as a approaches zero be the intensity of the stress on the plane under consideration at the point O.

When a body is subjected to the action of external forces, and, in consequence of this, undergoes a change of form, it will be found that lines drawn within the body are changed, by the action of these external forces, in length, in direction, or in both; and the entire change of form of the body may be correctly described by describing a sufficient number of these changes.

If we join two points, A and B, of a body before the external forces are applied, and find, that, after the application of the external forces, the line joining the same two points of the body has undergone a change of length $\Delta(AB)$, then is the

limit of the ratio $\frac{\Delta(AB)}{AB}$ as AB approaches zero called the strain of the body at the point A in the direction AB.

If $AB + \Delta(AB) > AB$, the strain is one of tension.
If $AB + \Delta(AB) < AB$, the strain is one of compression.

Suppose a straight rod of uniform section A to be subjected to a pull P in the direction of its length, and that this pull is uniformly distributed over the cross-section : then will the intensity of the stress on the cross-section be

$$p = \frac{P}{A}.$$

If P be measured in pounds, and A in square inches, then will p be measured in pounds per square inch.

If the length of the rod before the load is applied be l, and its length after the load is applied be $l + e$, then is e the elongation of the rod; and if this elongation is uniform throughout the length of the rod, then is $\frac{e}{l}$ the elongation of the rod per unit of length, or the strain.

Hence, if α represent the strain due to the stress p per unit of area, we shall have

$$\alpha = \frac{e}{l}.$$

The *Modulus of Elasticity* is the quotient obtained by dividing the stress per unit of area by the strain, or

$$E = \frac{p}{\alpha};$$

and this is expressed in units of weight per unit of area, as in pounds per square inch.

The modulus of elasticity is sometimes defined as the weight that would be required to stretch a rod one square inch in section to double its length, if Hooke's law, "The stress is proportional to the strain," held up to that point, and the rod did not break. Of course the modulus of elasticity is a constant for any given material just as far as, and no farther than, Hooke's law holds.

EXAMPLES.

1. A wrought-iron rod 10 feet long and 1 inch in diameter is loaded in the direction of its length with 8000 lbs.; find (1) the intensity of the stress, (2) the elongation of the rod; assuming the modulus of the iron to be 28000000 lbs. per square inch.

2. What would be the elongation of a similar rod of cast-iron under the same load, assuming the modulus of elasticity of cast-iron to be 17000000 lbs. per square inch?

3. Given a steel bar, area of section being 4 square inches, the length of a certain portion under a load of 25000 lbs. being 10 feet, and its length under a load of 100000 lbs. being $10' \, 0''.075$; find the modulus of elasticity of the material.

4. What load will be required to stretch the rod in the first example $\frac{1}{10}$ inch?

§ 168. **Resistance to Stretching and Tearing.** — The most-used criterion of safety against injury for a loaded piece is, that the greatest intensity of the stress to which any part of it is subjected shall nowhere exceed a certain fixed amount, called the working-strength of the material; this working-strength being a certain fraction of the breaking-strength determined by practical considerations.

The more correct but less used criterion is, that the greatest strain in any part of the structure shall nowhere exceed the working-strain; the greatest allowable amount of strain being a fixed quantity determined by practical considerations.

This is equivalent to limiting the allowable elongation or compression to a certain fraction of its length, or the deflection of a beam to a certain fraction of the span.

If the stress on a plane surface be uniformly distributed, its resultant will evidently act at the centre of gravity of the surface, as has been already shown in § 42 to be the case with any uniformly distributed force.

If a straight rod of uniform section and material be subjected to a pull in the direction of its length, and if the resultant of the pull acts along a line passing through the centres of gravity of the sections of the rod, it is assumed in practice that the stress is uniformly distributed throughout the rod, and hence that for any section we shall obtain the stress per square inch by dividing the total pull by the number of square inches in the section.

If, on the other hand, the resultant of the pull does not act through the centres of gravity of the sections, the pull is not uniformly distributed; and while

$$p = \frac{P}{A}$$

will express the mean stress per square inch, the actual intensity of the stress will vary at different points of the section, being greater than $\frac{P}{A}$ at some points and less at others. How to determine its greatest intensity in such cases will be shown later.

With good workmanship and well-fitting joints, the first case, or that of a uniformly distributed stress, can be practically realized; but with ill-fitting joints or poor workmanship, or with a material that is not homogeneous, the resultant of the pull is liable to be thrown to one side of the line passing through the centres of gravity of the sections, and thus there

is set up a bending-action in addition to the direct tension, and therefore an unevenly distributed stress.

It is of the greatest importance in practice to take cognizance of any such irregularities, and determine the greatest intensity of the stress to which the piece is subjected: though it is too often taken account of merely by means of a factor of safety; in other words, by guess.

Leaving, then, this latter case until we have studied the stresses due to bending, we will confine ourselves to the case of the uniformly distributed stress.

If the total pull on the rod in the direction of its length be P, and the area of its cross-section A, we shall have, for the intensity of the pull,

$$p = \frac{P}{A}.$$

On the other hand, if the working-strength of the material per unit of area be f, we shall have, for the greatest admissible load to be applied,

$$P = fA.$$

If f be the working-strength of the material per square inch, and E the modulus of elasticity, then is the greatest admissible strain equal to

$$\alpha = \frac{f}{E}.$$

Thus, assuming 12000 lbs. per square inch as the working tensile strength of wrought-iron, and 28000000 lbs. per square inch as its modulus of elasticity, its working-strain would be

$$\alpha = \frac{12000}{28000000} = \frac{3}{7000}.$$

Hence the greatest safe elongation of the bar would be $\frac{3}{7000}$ of its length. Hence a rod 10 feet long could safely be stretched $\frac{3}{700}$ of a foot $= 0.0514''$.

§ 169. **Approximate Values of Breaking and Working Strength, and of Modulus of Elasticity.** — In a later part of this book the attempt will be made to give an account of the experiments that have been made to determine the strength and elasticity of the materials ordinarily used in construction, in such a way as to enable the student to decide for himself, in any special case, upon the proper values of the constants that he ought to use.

For the present, however, the following will be given as a rough approximation to some of these quantities, which we may make use of in our work until we reach the above-mentioned account.

(*a*) *Cast-Iron.*

Breaking tensile strength per square inch, of common qualities, 14000 to 20000 lbs.; of gun iron, 30000 to 33000 lbs.

Modulus of elasticity for tension and for compression, about 17000000 lbs. per square inch.

(*b*) *Wrought-Iron.*

Breaking tensile strength per square inch, from 40000 to 60000 lbs.

Modulus of elasticity for tension and for compression, about 28000000.

(*c*) *Mild Steel.*

Breaking tensile strength per square inch, 55000 to 70000 lbs.

Modulus of elasticity for tension and for compression, from 28000000 to 30000000 lbs. per square inch.

(*d*) *Wood.*

Breaking compressive strength per square inch: —

Oak, green	3000 lbs.
Oak, dry	3000 to 6000 lbs.
Yellow pine, green	3000 to 4000 lbs.
Yellow pine, dry.	4000 to 7000 lbs.

Modulus of elasticity for compression (average values): —

Oak . . . . 1300000 lbs. per square inch.
Yellow pine . . . . . . 1600000 lbs. per square inch.

§ 170. **Sudden Application of the Load.** — If a wrought-iron rod 10 feet long and 1 square inch in section be loaded with 12000 pounds in the direction of its length, and if the modulus of elasticity of the iron be 28000000, it will stretch $0.0514''$ provided the load be gradually applied: thus, the rod begins to stretch as soon as a small load is applied; and, as the load gradually increases, the stretch increases, until it reaches $0.0514''$.

If, on the other hand, the load of 12000 lbs. be suddenly applied (i.e., put on all at once) without being allowed to fall through any height beforehand, it would cause a greater stretch at first, the rod undergoing a series of oscillations, finally settling down to an elongation of $0.0514''$.

To ascertain what suddenly applied load will produce at most the elongation $0.0514''$, observe, that, in the case of the gradually applied load, we have a load gradually increasing from

0 to 12000 lbs.

Its mean value is, therefore, $\frac{1}{2}(12000) = 6000$ lbs.; and this force descends through a distance of

$0.0514''$.

Hence the amount of mechanical work done on the rod by the gradually applied load in producing this elongation is

$$(6000)(0.0514) = 308.4 \text{ inch-lbs.}$$

Hence, if we are to perform upon the rod 308.4 inch-lbs. of work with a constant force, and if the stretch is to be $0.0514''$, the magnitude of the force must be

$$\frac{308.4}{0.0514''} = 6000 \text{ lbs.}$$

Hence a suddenly applied load will produce double the strain that would be produced by the same load gradually applied; and, moreover, a suddenly applied load should be only half as great as one gradually applied if it is to produce the same strain.

§ 171. **Resilience of a Tension-Bar.** — The resilience of a tension-rod is the mechanical work done in stretching it to the same amount that it would stretch under the greatest allowable gradually applied load, and is found by multiplying the greatest allowable load by half the corresponding elongation.

Thus, suppose a load of 100 lbs. to be dropped upon the rod described above in such a way as to cause an elongation not greater than 0.0514″, it would be necessary to drop it from a height not greater than 3.08″.

EXAMPLES.

1. A wrought-iron rod is 12 feet long and 1 inch in diameter, and is loaded in the direction of its length; the working-strength of the iron being 12000 lbs. per square inch, and the modulus of elasticity 28000000 lbs. per square inch.

Find the working-strain.
Find the working-load.
Find the working-elongation.
Find the working-resilience.

From what height can a 50-pound weight be dropped so as to produce tension, without stretching it more than the working-elongation?

2. Do the same for a cast-iron rod, where the working-strength is 5000 pounds per square inch, and the modulus of elasticity 17000000; the dimensions of the rod being the same.

§ 172. **Results of Wöhler's Experiments on Tensile Strength.** — According to the experiments of Wöhler, of which an account will be given later, the breaking-strength of a piece

depends, not only on whether the load is gradually or suddenly applied, but also on the extreme variations of load that the piece is called upon to undergo, and the number of changes to which it is to be submitted during its life.

For a piece which is always in tension, he determines the following two constants; viz., t, the carrying-strength per square inch, or the greatest quiescent stress that the piece will bear, and u, the primitive safe strength, or the greatest stress per square inch of which the piece will bear an indefinite number of repetitions, the stress being entirely removed in the intervals.

This primitive safe strength, u, is used as the breaking-strength when the stress varies from 0 to u every time. Then, by means of Launhardt's formula, we are able to determine the ultimate strength per square inch for any different limits of stress, as for a piece that is to be alternately subjected to 80000 and 6000 pounds.

Thus, for Phœnix Company's axle iron, Wöhler finds

$$t = 3290 \text{ kil. per sq. cent.} = 46800 \text{ lbs. per sq. in.,}$$
$$u = 2100 \text{ kil. per sq. cent.} = 30000 \text{ lbs. per sq. in.}$$

Launhardt's formula for the ultimate strength per unit of area is

$$a = u\left\{1 + \frac{t-u}{u}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}.$$

Hence, with these values of t and u, we should have, for the ultimate strength per square inch,

$$a = 2100\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{kil. per sq. cent.,}$$

or

$$a = 30000\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.}$$

Thus, if least stress = 6000, and greatest = 80000, we should have

$$a = 30000\{1 + \tfrac{1}{2} \cdot \tfrac{6}{80}\} = 30000\{1 + \tfrac{3}{80}\} = 31125;$$

if least stress = 60000, and greatest = 80000,

$$a = 30000\{1 + \tfrac{1}{2} \cdot \tfrac{6}{8}\} = 30000\{1 + \tfrac{3}{8}\} = 41250;$$

if least stress = greatest stress = 80000,

$$a = 30000\{1 + \tfrac{1}{2}\} = 45000 = \text{carrying-strength.}$$

Hence, instead of using, as breaking-strength per square inch in all cases, 45000, we should use a set of values varying from 45000 down to 30000, according to the variation of stress which the piece is to undergo.

For working-strength, Weyrauch divides this by 3: thus obtaining, for working-strength per square inch,

$$b = 10000\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.};$$

for Krupp's cast-steel,

$$t = 7340 \text{ kil. per sq. cent.} = 104400 \text{ lbs. per sq. in.},$$
$$u = 3300 \text{ kil. per sq. cent.} = 46900 \text{ lbs. per sq. in.};$$

$$\therefore\ a = 3300\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{kil. per sq. cent.},$$

or

$$a = 46900\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.},$$

$$\therefore\ b = 15633\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.}$$

EXAMPLES.

Find the breaking-strength per square inch for a wrought-iron tension-rod.

1. Extreme loads are 75000 and 6000 lbs.
2. Extreme loads are 120000 and 100000 lbs.
3. Extreme loads are 300000 and 10000 lbs.

Find the safe section for the rod in each case.

§ 173. **Suspension-Rod of Uniform Strength.** — In the case of a long suspension-rod, the weight of the rod itself sometimes becomes an important item. The upper section must, of course, be large enough to bear the weight that is hung from the rod plus the weight of the rod itself; but it is sometimes desirable to diminish the sections as they descend. This is often accomplished in mines by making the rod in sections, each section being calculated to bear the weight below it plus its own weight.

Were the sections gradually diminished, so that each section would be just large enough to support the weight below it, we should, of course, have a curvilinear form; and the equation of this curve could be found as follows, or, rather, the area of any section at a distance from the bottom of the rod.

Let $W =$ weight hung at O (Fig. 152),

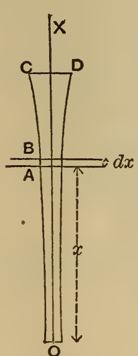


FIG. 152.

Let w $=$ weight per unit of volume of the rod,

Let x $=$ distance AO,

Let S $=$ area of section A,

Let $x + dx$ $=$ distance BO,

Let $S + dS$ $=$ area of section at B,

Let f $=$ working-strength of the material per square inch.

1°. The section at O must be just large enough to sustain the load W;

$$\therefore \quad S_0 = \frac{W}{f}.$$

2°. The area in dS must be just enough to sustain the weight of the portion of the rod between A and B.

The weight of this portion is $wSdx$;

$$\therefore \quad dS = \frac{wSdx}{f}$$

$$\therefore \quad \frac{dS}{S} = \frac{w}{f}dx \qquad \therefore \quad \log_e S = \frac{w}{f}x + \text{a constant.}$$

When $x = 0$, $S = \dfrac{W}{f}$;

$$\therefore \quad \log_e \frac{W}{f} = \text{the constant} \qquad \therefore \quad \log_e S - \log_e \left(\frac{W}{f}\right) = \frac{w}{f}x$$

$$\therefore \quad \frac{S}{\left(\frac{W}{f}\right)} = e^{\frac{wx}{f}} \qquad \therefore \quad S = \frac{W}{f}e^{\frac{wx}{f}}.$$

This gives us the means of determining the area at any distance x from O.

EXAMPLES.

1. A wrought-iron tension-rod 200 feet long is to sustain a load of 2000 lbs. with a factor of safety of 4, and is to be made in 4 sections, each 50 feet long; find the diameter of each section, the weight of the wrought iron being 480 lbs. per cubic foot.

2. Find the diameter needed if the rod were made of uniform section, also the weight of the extra iron necessary to use in this case.

3. Find the equation of the longitudinal section of the rod, assuming a square cross-section, if it were one of uniform strength, instead of being made in 4 sections.

§ 174. **Thin Hollow Cylinders subjected to an Internal Normal Pressure.** — Let p denote the uniform intensity of the pressure exerted by a fluid which is confined within a hollow cylinder of radius r and of thickness t (Fig. 153), the thickness being small compared with the radius. Let us consider a unit of length of the cylinder, and let us also consider the forces acting on the upper half-ring CED.

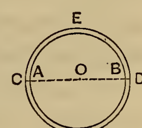


FIG. 153.

The total upward force acting on this half-ring, in consequence of the internal normal pressure, will be the same as that acting on a section of the cylinder made by a plane passing through its axis, and the diameter CD. The area of this

section will be $2r \times 1 = 2r$: hence the total upward force will be $2r \times p = 2pr$; and the tendency of this upward force is to cause the cylinder to give way at A and B, the upper part separating from the lower.

This tendency is resisted by the tension in the metal at the sections AC and BD; hence at each of these sections, there has to be resisted a tensile stress equal to $\frac{1}{2}(2pr) = pr$. This stress is really not distributed uniformly throughout the cross-section of the metal; but, inasmuch as the metal is thin, no serious error will be made if it be accounted as distributed uniformly. The area of each section, however, is $t \times 1 = t$; therefore, if T denote the intensity of the tension in the metal in a tangential direction (i.e., the intensity of the hoop tension), we shall have

$$T = \frac{pr}{t}.$$

Hence, to insure safety, T must not be greater than f, the working-strength of the material for tension; hence, putting

$$f = \frac{pr}{t},$$

we shall have

$$t = \frac{pr}{f}$$

as the proper thickness, when p = normal pressure per square inch, and radius = r.

The above are the formulæ in common use for the determination of the thickness of the shell of a steam-boiler; for in that case the steam-pressure is so great that the tension induced by any shocks that are likely to occur, or by the weight of the boiler, is very small in comparison with that induced by the steam-pressure. On the other hand, in the case of an ordinary water-pipe, the reverse is the case.

To provide for this case, Weisbach directs us to add to the thickness we should obtain by the above formulæ, a constant minimum thickness.

The following are his formulæ, d being the diameter in inches, p the internal normal pressure in pounds per square inch, and t the thickness in inches. For tubes made of

Sheet-iron	$t = 0.00086\,pd + 0.12$
Cast-iron	$t = 0.00238\,pd + 0.34$
Copper	$t = 0.00148\,pd + 0.16$
Lead	$t = 0.00507\,pd + 0.21$
Zinc	$t = 0.00242\,pd + 0.16$
Wood	$t = 0.03230\,pd + 1.07$
Natural stone	$t = 0.03690\,pd + 1.18$
Artificial stone	$t = 0.05380\,pd + 1.58$

§ 175. **Resistance to Direct Compression.** — When a piece is subjected to compression, the distribution of the compressive stress on any cross-section depends, first, upon whether the resultant of the pressure acts along the line containing the centres of gravity of the sections, and, secondly, upon the dimensions of the piece; thus determining whether it will bend or not.

In the case of an eccentric load, or of a piece of such length that it yields by bending, the stress is not uniformly distributed; and, in order to proportion the piece, we must determine the greatest intensity of the stress upon it, and so proportion it that this shall be kept within the working-strength of the material for compression.

Either of these cases is not a case of direct compression.

In the case of direct compression (i.e., where the stress over each section is uniformly distributed), the intensity of the stress is found by dividing the total compression by the area of the

section; so that, if P be the total compression, and A the area of the section, and p the intensity of the compressive stress,

$$p = \frac{P}{A}.$$

On the other hand, if f is the compressive working-strength of the material per square inch, and A the area of the section in square inches, then the greatest allowable load on the piece subjected to compression is

$$P = fA.$$

The same remarks as were made in regard to a suddenly applied load and resilience, in the case of direct tension, apply in the case of direct compression.

§ 176. **Results of Wöhler's Experiments on Compressive Strength.** — Wöhler also made experiments in regard to pieces subjected to alternate tension and compression, taking, in the experiments themselves, the case where the metal is subjected to alternate tensions and compressions of equal amount.

The greatest stress of which the piece would bear an indefinite number of changes under these conditions, is called the vibration safe strength, and is denoted by s.

Weyrauch deduces a formula similar to that of Launhardt for the greatest allowable stress per unit of area on the piece when it is subjected to alternate tensions and compressions of different amounts.

Thus, for Phœnix Company's axle iron, Wöhler deduces

t = 3290 kil. per sq. cent. = 46800 lbs. per sq. in.,
u = 2100 kil. per sq. cent. = 30000 lbs. per sq. in.,
s = 1170 kil. per sq. cent. = 16600 lbs. per sq. in.

Weyrauch's formula for the ultimate strength per unit of area is

$$a = u\left\{1 - \frac{u - s}{u}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\};$$

and, with these values of u and s, it gives

$$a = 2100\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{kil. per sq. cent.,}$$

or

$$a = 30000\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.}$$

With a factor of safety of 3, we should have, for the greatest admissible stress per square inch,

$$b = 10000\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs.}$$

For Krupp's cast-steel,

t = 7340 kil. per sq. cent. = 104400 lbs. per sq. in.,
u = 3300 kil. per sq. cent. = 46900 lbs. per sq. in. approximately,
s = 2050 kil. per sq. cent. = 29150 lbs. per sq. in. approximately.

We have, therefore, for the breaking-strength per unit of area, according to Weyrauch's formula,

$$a = 3300\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{kil. per sq. cent.,}$$

or

$$a = 46900\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.;}$$

and, using a factor of safety of 3, we have, for the greatest admissible stress per square inch,

$$b = 15630\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.}$$

The principles respecting an eccentric compressive load, and those respecting the giving-way of long columns so far as they are known, can only be treated after we have studied the resistance of beams to bending; hence this subject will be deferred until that time.

EXAMPLES.

Find the proper working and breaking strength per square inch to be used for a wrought-iron rod, the extreme stresses being —

1. 80000 lbs. tension and 6000 lbs. compression.
2. 100000 lbs. tension and 100000 lbs. compression.
3. 70000 lbs. tension and 60000 lbs. compression.

Do the same for a steel rod.

§ 177. **Resistance to Shearing.** — One of the principal cases where the resistance to shearing comes into practical use is that where the members of a structure, which are themselves subjected to direct tension or compression or bending, are united by such pieces as bolts, rivets, pins, or keys, which are subjected to shearing. Sometimes the shearing is combined with tension or with bending; and whenever this is the case, it is necessary to take account of this fact in designing the pieces. It is important that the pins, keys, etc., should be equally strong with the pieces they connect.

Probably one of the most important modes of connection is by means of rivets. In order that there may be only a shearing action, without any bending of the rivets, the latter must fit very tightly. The manner in which the riveting is done will necessarily affect very essentially the strength of the joints;

hence the only way to discuss fully the strength of riveted joints is to take into account the manner of effecting the riveting, and hence the results of experiments. These will be spoken of later; but the ordinary theories by which the strength and proportions of riveted joints are determined will be given, which theories are necessary also in discussing the results of experiments thereon.

The principle on which the theory is based is that of making the resistance of the joint to yielding equal in all the ways in which it is possible for it to yield.

A single-riveted lap-joint is one with a single row of rivets, as shown in Fig. 154.

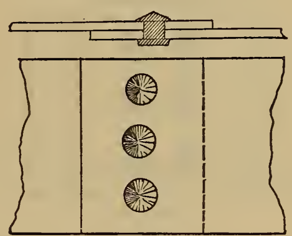


FIG. 154.

A single-riveted butt-joint with one covering plate is shown in Fig. 155.

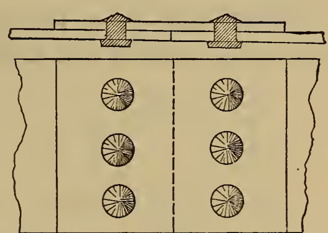


FIG. 155.

A single-riveted butt-joint with two covering plates is shown in Fig. 156.

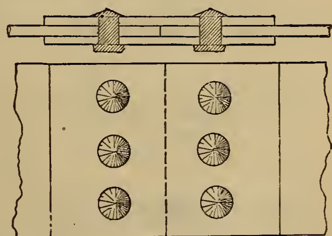


FIG. 156.

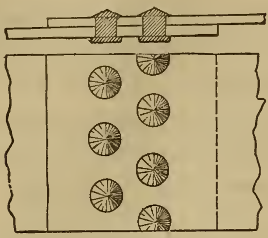


FIG. 157.

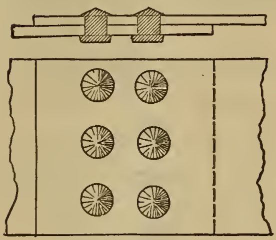


FIG. 158.

A double-riveted lap-joint with the rivets staggered is shown in Fig. 157; one with chain riveting, in Fig. 158.

Taking the case of the single-riveted lap-joint shown in Fig. 154, it may yield in one of four ways: —

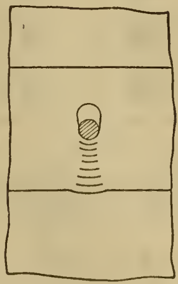


FIG. 159.

1°. By the crushing of the plate in front of the rivet (Fig. 159).

FIG. 160.

2°. By the shearing of the rivet (Fig. 160).

3°. By the tearing of the plate between the rivet-holes (Fig. 161).

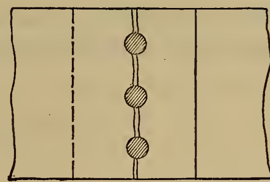


FIG. 161.

4°. By the rivet breaking through the plate (Fig. 162).

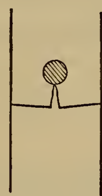


FIG. 162.

Let us call

d the diameter of a rivet.

c the pitch of the rivets; i.e., their distance apart from centre to centre.

t the thickness of the plate.

l the lap of the plate; i.e., the distance from the outer edge of a rivet-hole to the outer edge of the plate.

f_t the ultimate tensile strength of the iron.

f_s the ultimate shearing-strength of the rivet iron.

f_c the ultimate crushing-strength of the iron.

We shall then have —

1°. Resistance of plate in front of rivet to crushing $= f_c td.$

2°. Resistance of one rivet to shearing $= f_s\left(\frac{\pi d^2}{4}\right).$

3°. Resistance of plate between two rivet-holes to tearing $= f_t t(c - d).$

4° Resistance of plate to being broken through $= a\frac{tl^2}{d}$, where a is a constant depending on the material. This may be taken as empirical for the present.

An average value of this constant, as given by Robert Wilson, is 100000 lbs., where all the dimensions are measured in

inches. Assuming that we know the thickness of the plate to start with, we obtain, by equating the first two resistances,

$$f_c td = f_s \frac{\pi d^2}{4} \qquad \therefore \quad d = \frac{4t}{\pi} \frac{f_c}{f_s},$$

which determines the diameter of the rivet.

Equating 3° and 2°, we obtain

$$f_t t(c - d) = f_s \frac{\pi d^2}{4} \qquad \therefore \quad c = d + \frac{f_s}{f_t} \frac{\pi d^2}{4t},$$

which gives the pitch of the rivets in terms of the diameter of the rivet, and the thickness of the plate.

Equating, next, 4° and 1°, we have

$$a\frac{tl^2}{d} = f_c td \qquad \therefore \quad l = d\sqrt{\frac{f_c}{a}},$$

which gives the lap of the plate.

A similar method of reasoning would enable us to determine the corresponding quantities in the cases of double-riveted joints, etc.

The above is the ordinary theory of riveted joints: it assumes that the joint should be made equally strong against giving way by any method in which it is possible for it to give way. There are a number of practical considerations which modify more or less the results of this theory, and which can only be determined experimentally. A fuller account of this subject from an experimental point of view will be given later.

§ 178. **Intensity of Stress.** — Whenever the stress over a plane area is uniformly distributed, we obtain its intensity at each point by dividing the total stress by the area over which it acts, thus obtaining the amount per unit of area. When, however, the stress is not uniformly distributed, or when its inten-

sity varies at different points, we must adopt a somewhat different definition of its *intensity at a given point.* In that case, if we assume a small area containing that point, and divide the stress which acts on that area by the area, we shall have, in the quotient, an approximation to the intensity required, which will approach nearer and nearer to the true value of the intensity at that point, the smaller the area is taken.

Hence the intensity of a variable stress at a given point is, —

The limit of the ratio of the stress acting on a small area containing that point, to the area, as the latter grows smaller and smaller.

By dividing the total stress acting on a certain area by the entire area, we obtain the mean intensity of the stress for the entire area.

§ 179. **Graphical Representation of Stress.** — A convenient mode of representing stress graphically is the following: —

Let AB (Fig. 163) be the plane surface upon which the stress acts; let the axes OX and OY be taken in this plane, the axis OZ being at right angles to the plane.

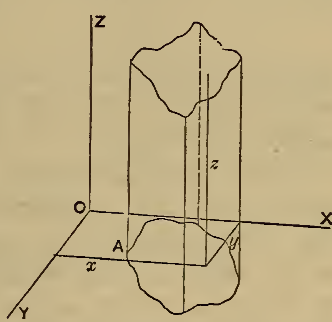


FIG. 163.

Conceive a portion of a cylinder whose elements are all parallel to OZ, bounded at one end by the given plane surface, and at the other by a surface whose ordinate at any point contains as many units of length as there are units of force in the intensity of the stress at that point of the given plane surface where the ordinate cuts it.

The volume of such a figure will evidently be

$$V = \int\int z dx dy = \int\int p dx dy,$$

where $z = p =$ intensity of the stress at the given point.

Hence the volume of the cylindrical figure will contain as many units of volume as the total stress contains units of force; or, in other words, the total stress will be correctly represented by the volume of the body.

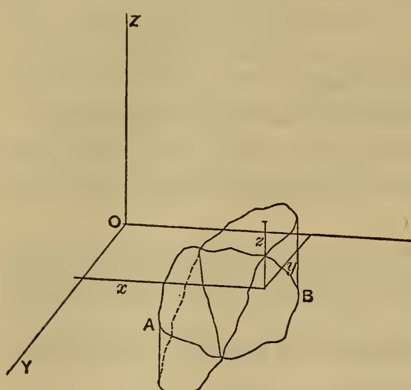


FIG. 164.

If the stress on the plane figure is partly tension and partly compression, the surface whose ordinates represent the intensity of the stress will lie partly on one side of the given plane surface and partly on the other; this surface and the plane surface on which the stress acts, cutting each other in some line, straight or curved, as shown in Fig. 164. In that case, the magnitude of the resultant stress $P = V = \int\int z dx dy$ will be equal to the difference of the wedge-shaped volumes shown in the figure.

It will be observed that the above method of representing stress graphically represents, 1°, the intensity at each point of the surface to which it is applied; and, 2°, the total amount of the stress on the surface. It does not, however, represent its direction, except in the case when the stress is normal to the surface on which it acts.

In this latter case, however, this is a complete representation of the stress.

The two most common uses of stress are, 1°, uniform stress, and, 2°, uniformly varying stress. These two cases are represented respectively in Figs. 165 and 166; the direction also being correctly represented when, as is most frequently the case, the stress is normal to the surface of action. In Fig. 165, AB is supposed to be the surface on which the stress

acts; the stress is supposed to be uniform, and normal to the surface on which it acts; the bounding surface in this case becomes a plane parallel to AB; the intensity of the stress at any point, as P, will be represented by PQ; while the whole cylinder will contain as many units of volume as there are units of force in the whole stress.

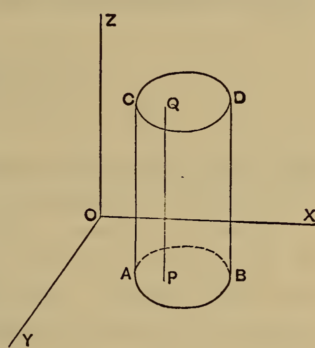


FIG. 165.

Fig. 166 represents a uniformly varying stress. Here, again, AB is the surface of action, and the stress is supposed to vary at a uniform rate from the axis OY. The upper bounding surface of the cylindrical figure which represents the stress becomes a plane inclined to the XOY plane, and containing the axis OY.

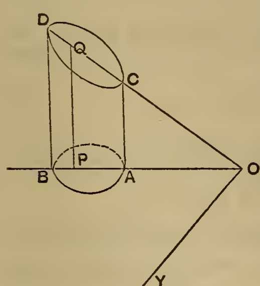


FIG. 166.

In this case, if a represent the intensity of the stress at a unit's distance from OY, the stress at a distance x from OY will be $p = ax$, and the total amount of the stress will be

$$P = \int\int p\,dx\,dy = a\int\int x\,dx\,dy.$$

When a stress is oblique to the surface of action, it may be represented correctly in all particulars, except in direction, in the above-stated way.

§ 180. **Centre of Stress.** — The centre of stress, or the point of the surface at which the resultant of the stress acts, often becomes a matter of practical importance. If, for convenience, we employ a system of rectangular co-ordinate axes, of which the axes OX and OY are taken in the plane of the surface on which the stress acts, and if we let $p = \phi(x, y)$ be

the intensity of the stress at the point (x, y), we shall have, for the co-ordinates of the centre of stress,

$$x_1 = \frac{\int\int xpdxdy}{\int\int pdxdy}, \quad y_1 = \frac{\int\int ypdxdy}{\int\int pdxdy},$$

(see § 42), where the denominator, or $\int\int pdxdy$, represents the total amount of the stress.

When the stress is positive and negative at different parts of the surface, as in Fig. 135, the case may arise when the positive and negative parts balance each other, and hence the stress on the surface constitutes a statical couple. In that case

$$\int\int pdxdy = 0.$$

§ 181. **Uniform Stress.** — In the case of uniform stress, we have —

1°. The intensity of the stress is constant, or $p =$ a constant.

2°. The volume which represents it graphically becomes a cylinder with parallel and equal bases, as in Fig. 165.

3°. The centre of stress is at the centre of gravity of the surface of action; for the formulæ become, when p is constant,

$$x_1 = \frac{p\int\int xdxdy}{p\int\int dxdy} = \frac{\int\int xdxdy}{\int\int dxdy} = x_0,$$

$$y_1 = \frac{p\int\int ydxdy}{p\int\int dxdy} = \frac{\int\int ydxdy}{\int\int dxdy} = y_0,$$

where x_0, y_0, are the co-ordinates of the centre of gravity of the surface.

Examples of uniform stress have already been given in the cases of direct tension, direct compression, and, in the case of riveted joints, for the shearing-force on the rivet.

§ 182. **Uniformly Varying Stress.** — Uniformly varying stress has already been defined as a stress whose intensity varies uniformly from a given line in its own plane; and this line will be called the *Neutral Axis*. Thus, if the plane be taken as the XOY plane (Fig. 166), and the given line be taken as OY, we shall have, if a denotes the intensity of the stress at a unit's distance from OY, and x the distance of any special point from OY, that the intensity of the stress at the point will be

$$p = ax.$$

The total amount of the stress will be

$$P = a\int\int xdxdy.$$

The total moment of the stress about OY will be found by multiplying each elementary stress by its leverage. This leverage is, in the case of normal stress, x; hence in that case the moment of any single elementary force will be

$$(ax\Delta x\Delta y)x = ax^2\Delta x\Delta y,$$

and the total moment of the stress will be

$$M = a\int\int x^2dxdy.$$

In the case of oblique stress, this result has to be modified, as the leverage is no longer x. Confining ourselves to stress normal to the plane of action, we have, for the co-ordinates of the centre of stress,

$$x_1 = \frac{\int\int pxdxdy}{\int\int pdxdy} = \frac{a\int\int x^2dxdy}{P} = \frac{\int\int x^2dxdy}{\int\int xdxdy} = \frac{\int\int x^2dxdy}{x_0A},$$

$$y_1 = \frac{\int\int pydxdy}{\int\int pdxdy} = \frac{a\int\int xydxdy}{P} = \frac{\int\int xydxdy}{\int\int xdxdy} = \frac{\int\int xydxdy}{x_0A},$$

since

$$P = a\int\int xdxdy = ax_0A,$$

where x_0, y_0, are the co-ordinates of the centre of gravity, and A is the area of the surface of action.

§ 183. **Case of a Uniformly Varying Stress which amounts to a Statical Couple.** — Whenever $P = 0$, we have

$$a\int\int xdxdy = 0 \quad \therefore \quad \int\int xdxdy = 0 \quad \therefore \quad x_0 A = 0 \quad \therefore \quad x_0 = 0.$$

In this case, therefore, we have —

1°. There is no resultant stress, and hence the whole stress amounts to a statical couple.

2°. Since $x_0 = 0$, the centre of gravity of the surface of action is on the axis OY, which is the neutral axis.

Hence follows the proposition : —

When a uniformly varying stress amounts to a statical couple, the neutral axis contains (passes through) the centre of gravity of the surface of action.

In this case there is no single resultant of the stress ; but the moment of the couple will be, as has been already shown,

$$M = a\int\int x^2 dxdy.$$

§ 184. **Example of Uniformly Varying Stress.** — One of the most common examples of uniformly varying stress is that of the pressure of water upon the sides of the vessel containing it.

Thus, let Fig. 167 represent the vertical cross-section of a reservoir wall, the water pressing against the vertical face AB. It is a fact established by experiment, that the intensity of the pressure of any body of water at any point is proportional to the depth of the point below the free upper level of the water, and normal to the surface pressed upon. Hence, if we suppose the free upper level of the water to be even with the top of the wall, the intensity of the pressure there will be zero ; and if we represent by CB the intensity of the pressure at the bottom, then, joining A and

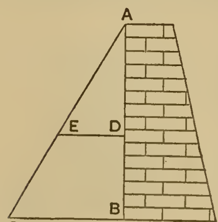


Fig. 167.

C we shall have the intensity of the pressure at any point, as D, represented by ED, where

$$ED : CB = AD : AB.$$

Here, then, we have a case of uniformly varying stress normal to the surface on which it acts.

§ 185. **Fundamental Principles of the Common Theory of the Stresses in Beams under a Transverse Load.** — Fig. 168 shows a beam fixed at one end and loaded at the other, while Fig. 169 shows a beam supported at the ends and loaded at the middle. Let, in each case, the plane of the paper contain a vertical longitudinal section of the beam. In Fig. 168, it is evident that the upper fibres are lengthened, while the lower ones are shortened, and *vice versa* in Fig. 169. In either case, there is, somewhere between the upper and lower fibres, a fibre which is neither elongated nor compressed.

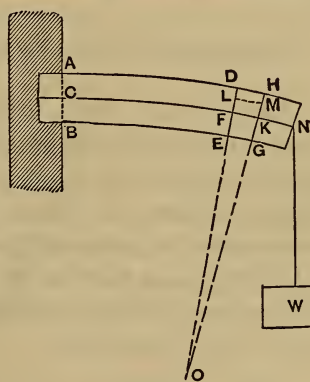


Fig. 168.

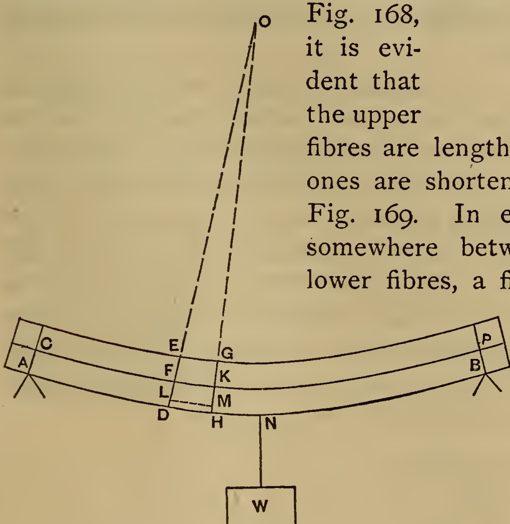


Fig. 169.

Let CN represent that fibre, Fig. 168, and CP, Fig. 169. This line may be called the neutral line of the longitudinal section; and, if a section be made at any point at right

angles to this line, the horizontal line which lies in the cross-section, and cuts the neutral lines of all the longitudinal sections, or, in other words, the locus of the points where the neutral lines of the longitudinal sections cut the cross-section, is called the *Neutral Axis* of the cross-section. In the ordinary theory of the stresses in beams, a number of assumptions are made, which will now be enumerated.

ASSUMPTIONS MADE IN THE COMMON THEORY OF BEAMS.

ASSUMPTION No. 1. — If, when a beam is not loaded, a plane cross-section be made, this cross-section will still be a plane after the load is put on, and bending takes place. From this assumption, we deduce, as a consequence, that, if a certain cross-section be assumed, the elongation or shortening per unit of length of any fibre at the point where it cuts this cross-section, is proportional to the distance of the fibre from the neutral axis of the cross-section.

Proof. — Imagine two originally parallel cross-sections so near to each other that the curve in which that part of the neutral line between them bends may, without appreciable error, be accounted circular. Let ED and GH (Fig. 168 or Fig. 169) be the lines in which these cross-sections cut the plane of the paper, and let O be the point of intersection of the lines ED and GH. Let $OF = r$, $FL = y$, $FK = l$, $LM = l + \alpha l$, in which α is the strain or elongation per unit of length of a fibre at a distance y from the neutral line, y being a variable; then, because FK and LM are concentric arcs subtending the same angle at the centre, we shall have the proportion

$$\frac{r+y}{r} = \frac{l + \alpha l}{l} \quad \text{or} \quad 1 + \alpha = 1 + \frac{y}{r},$$

$$\therefore \alpha = \frac{y}{r} \quad \text{or} \quad \alpha = \left(\frac{1}{r}\right)y;$$

but as y varies for different points in any given cross-section, while r remains the same for the same section, it follows, that, if a certain cross-section be assumed, *the strain of any fibre at the point where it cuts this cross-section is proportional directly to the distance of this fibre from the neutral axis of the cross-section.*

Assumption No. 2. — This assumption is that commonly known as *Hooke's Law.* It is as follows: "*Ut tensio sic vis;*" i.e., The stress is proportional to the strain, or to the elongation or compression per unit of length. As to the evidence in favor of this law, experiment shows, that, as long as the material is not strained beyond safe limits, this law holds. Hence, making these two assumptions, we shall have: *At a given cross-section of a loaded beam, the direct stress on any fibre varies directly as the distance of the fibre from the neutral axis.* Hence it is a uniformly varying stress, and we may represent it graphically as follows: Let $ABCD$, Fig. 170, be the cross-section of a beam, and KL the neutral axis. Assume this for axis OY, and draw the other two axes, as in the figure. If, now, EA be drawn to represent the intensity of the direct (normal) stress at A, then will the pair of wedges $AEFBKL$ and $DCHGKL$ represent the stress graphically, since it is uniformly varying.

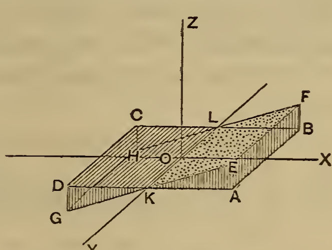


Fig. 170.

POSITION OF NEUTRAL AXIS.

Assumption No. 3. — It will next be shown, that, on the two assumptions made above, and from the further assumption that the only resistances opposed to the bending of the beam

are the direct tensions and compressions of the fibres, it follows that the neutral axis must pass through the centre of gravity of the cross-section.

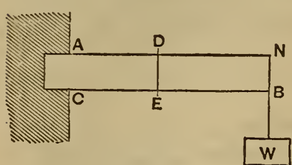


FIG. 171.

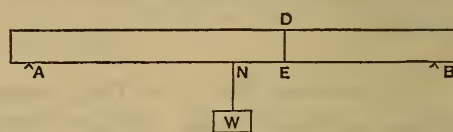


FIG. 172.

Since the curvatures in Figs. 168 and 169 are exaggerated in order to render them visible, Figs. 171 and 172 have been drawn. If, now, we assume a section DE, such that $AD = x$ (Fig. 171) and $NE = x$ (Fig. 172), and consider all the forces acting on that part of the beam which lies to the right of DE (i.e., both the external forces and the stresses which the other parts of the beam exert on this part), we must find them in equilibrium. The external forces are, in Fig. 172, —

1°. The loads acting between B and E; in this case there are none.

2°. The supporting force at B; in this case it is equal to $\frac{W}{2}$, and acts vertically upwards.

In Fig. 143 they are, —

The loads between D and N; in this case there is only the one, W at N.

The internal forces are merely the stresses exerted by the other parts of the beam on this part: they are, —

1°. The resistance to shearing at the section, which is a vertical stress.

2°. The direct stresses, which are horizontal.

Now, since the part of the beam to the right of DE is at rest, the forces acting on it must be in equilibrium; and, since

they are all parallel to the plane of the paper, we must have the three following conditions; viz., —

1°. The algebraic sum of the vertical forces must be zero.

2°. The algebraic sum of the horizontal forces must be zero.

3°. The algebraic sum of the moments of the forces about any axis perpendicular to the plane of the paper must be zero.

But, on the above assumptions, the only horizontal forces are the direct stresses: hence the algebraic sum of these direct stresses must be zero; or, in other words, the direct stresses must be equivalent to a statical couple.

Now, it has already been shown, that, whenever a uniformly varying stress amounts to a statical couple, the neutral axis must pass through the centre of gravity of the surface acted upon. Hence in a loaded beam, if the three preceding assumptions be made, it follows that the neutral axis of any cross-section must contain the centre of gravity of that section.

By way of experimental proof of this conclusion, Barlow has shown by experiment, that, in a cast-iron beam of rectangular section, the neutral axis does pass through the centre of gravity of the section.

RÉSUMÉ.

The conclusions arrived at from the foregoing are as follows: —

1°. That at any section of a loaded beam, if a horizontal line be drawn through the centre of gravity of the section, then the fibres lying along this line will be subjected neither to tension nor to compression; in other words, this line will be the neutral axis of the section.

2°. The fibres on one side of this line will be subjected to tension, those on the other side being subjected to compression; the tension or compression of any one fibre being proportional to its distance from the neutral axis.

§ 186. **Shearing-Force and Bending-Moment.** — In determining the strength of a beam, or the proper dimensions of a beam to bear a certain load, when we assume the neutral axis to pass through the centre of gravity of the cross-section, we have imposed the second of the three last-mentioned conditions of equilibrium. The remaining two conditions may otherwise be stated as follows : —

1°. The total force tending to cause that part of the beam that lies to one side of the section to slide by the other part, must be balanced by the resistance of the beam to shearing at the section.

2°. The resultant moment of the external forces acting on that part of the beam that lies to one side of the section, about a horizontal axis in the plane of the section, must be balanced by the moment of the couple formed by the resisting stresses.

The shearing-force at any section is the force with which the part of the beam on one side of the section tends to slide by the part on the other side. In a beam free at one end, it is equal to the sum of the loads between the section and the free end. In a beam supported at both ends, it is equal in magnitude to the difference between the supporting force at either end, and the sum of the loads between the section and that support.

The bending-moment at any section is the resultant moment of the external forces acting on the part of the beam to one side of the section, these moments being taken about a horizontal axis in the section.

In a beam free at one end, it is equal to the sum of the moments of the loads between the section and the free end, about a horizontal axis in the section.

In a beam supported at both ends, it is the difference between the moment of either supporting force, and the sum of the moments of the loads between the section and that support; all the moments being taken about a horizontal axis in the section.

Hence the two conditions of equilibrium may be more briefly stated as follows : —

1°. The shearing-force at the section must be balanced by the resistance opposed by the beam to shearing at the section.

2°. The bending-moment at the section must be balanced by the moment of the couple formed by the resisting stresses.

It is necessary, therefore, in determining the strength of a beam, to be able to determine the shearing-force and bending-moment at any point, and also the greatest shearing-force and the greatest bending-moment, whatever be the loads.

A table of these values for a number of ordinary cases will now be given ; but I should recommend that the table be merely considered as a set of examples, and that the rules already given for finding them be followed in each individual case.

Let, in each case, the length of the beam be l, and the total load W. When the beam is fixed at one end and free at the other, let the origin be taken at the fixed end ; when it is supported at both ends, let it be taken directly over one support. Let x be the distance of any section from the origin. Then we shall have the results given in the following table : —

Description of Beam.	Distribution of the Load.		Shearing-Force. At Distance x from Origin.	Shearing-Force. Greatest.	Bending-Moment. At Distance x from Origin.	Bending-Moment. Greatest.
Beam fixed at one end, free at the other,	Single load at free end,		W	W	$W(l-x)$	Wl
	Load uniformly distributed,		$\frac{W}{l}(l-x)$	W	$\frac{W}{2l}(l-x)^2$	$\frac{Wl}{2}$
Beam supported at both ends,	Single load at middle,	Between origin and middle,	$\frac{W}{2}$	$\frac{W}{2}$	$\frac{W}{2}x$	$\frac{Wl}{4}$
		Beyond middle,	$-\frac{W}{2}$		$\frac{W}{2}(l-x)$	
	Load uniformly distributed,		$\frac{W}{l}\left(\frac{l}{2}-x\right)$	$\frac{W}{2}$	$\frac{W}{2l}(lx-x^2)$	$\frac{Wl}{8}$
	Single load at distance a from origin,	Between origin and load,	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}x$	$\frac{Wa(l-a)}{l}$
		Beyond load,	$\frac{Wa}{l}$	$\frac{Wa}{l}$	$\frac{Wa}{l}(l-x)$	

In a beam fixed at one end and free at the other, the greatest shearing-force, and also the greatest bending-moment, are at the fixed end. In a beam supported at both ends, and loaded at the middle, or with a uniformly distributed load, the greatest shearing-force is at either support, the greatest bending-moment being at the middle. In the last case (i.e., that of a beam supported at the ends, and having a single load not at the middle), the greatest bending-moment is at the load; the greatest shearing-force being at that support where the supporting force is greatest.

§ 187. **Moments of Inertia of Sections.** — In the usual methods of determining the strength of a beam or column, it is necessary to know, 1°, the distance from the neutral axis of the section to the most strained fibres; 2°, the moment of inertia of the section about the neutral axis. The manner of finding the moments of inertia has been explained in Chap. II.

In the following table are given the areas of a large number of sections, and also their moments of inertia about the neutral axis, which is the axis YY in each case. These results should be deduced by the student.

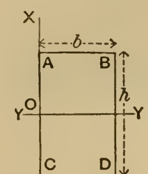
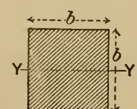
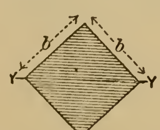
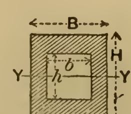
Figure.		Description.	A.	I.	Distance of YY from Most Strained Fibre.
	173	*Rectangle,* — Height $= h$ Breadth $= b$	bh	$\frac{bh^3}{12}$	$\frac{h}{2}$
	174	*Square,* — Side $= b$	b^2	$\frac{b^4}{12}$	$\frac{b}{2}$
	175	*Square,* — Side $= b$	b^2	$\frac{b^4}{12}$	$\frac{1}{2}b\sqrt{2}$
	176	*Rectangular cell,* — Outside dimensions $= B$ and H Inside dimensions $= b$ and h	$BH - bh$	$\frac{1}{12}(BH^3 - bh^3)$	$\frac{H}{2}$

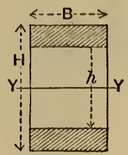
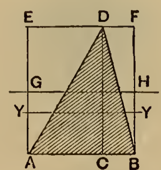
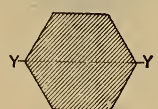
Figure.		Description.	A.	I.	Distance of YY from Most Strained Fibre.
	177	*Hollow rectangle,* — Outside dimensions, B and H Inside dimensions, B and h	$B(H - h)$	$\frac{1}{12}B(H^3 - h^3)$	$\frac{H}{2}$
	178	*Triangle,* — $AB = b,\ DC = h$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{2}{3}h$
		When axis is AB		$\frac{bh^3}{12}$	
		When axis is EF		$\frac{bh^3}{4}$	
	179	*Regular hexagon,* — Side $= a$	$\frac{3a^2\sqrt{3}}{2}$	$\frac{5a^4\sqrt{3}}{16} = 0.541a^4$	$\frac{a\sqrt{3}}{2} = 0.866a$

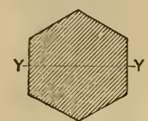
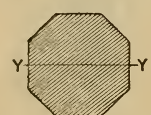
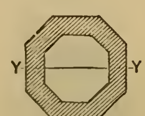
Figure.		Description.	A.	I.	Distance of YY from Most Strained Fibre.
	180	*Regular hexagon*	$\frac{3a^2\sqrt{3}}{2}$	$\frac{5a^4\sqrt{3}}{16} = 0.541a^4$	a
	181	*Regular octagon,* — Radius of circumscribed circle $= a$ Length of one side $= 2a \sin 22\frac{1}{2}°$ $= a\sqrt{2 - \sqrt{2}}$ $= 0.383a$	$2a^2\sqrt{2} = 2.828a^2$	$\frac{a^4}{6}(1 + 2\sqrt{2}) = 0.638a^4$	$\frac{a}{2}\sqrt{2 + \sqrt{2}}$ $= 0.924a$ $= a \cos 22\frac{1}{2}°$
	182	*Octagonal cell,* — Radius of circle circumscribed around outer octagon $= a_1$ Radius of circle circumscribed around inner octagon $= a_2$	$(a_1^2 - a_2^2)2\sqrt{2}$ $= (a_1^2 - a_2^2)(2.828)$	$\frac{1 + 2\sqrt{2}}{6}(a_1^4 - a_2^4)$ $= 0.638(a_1^4 - a_2^4)$	$\frac{a}{2}\sqrt{2 + \sqrt{2}}$

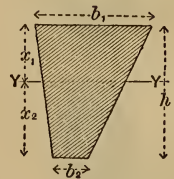
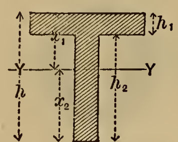
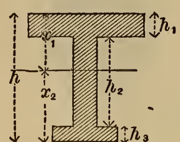
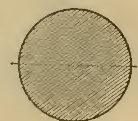
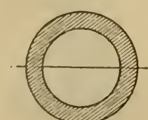
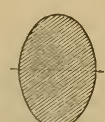
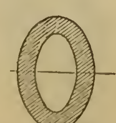
Figure.	Description.	A.	I.	Distance of YY from Most Strained Fibre.
 183	*Trapezoid,* — Parallel sides, b_1 and b_2 respectively. Height h	$(b_1 + b_2)\frac{h}{2}$	$\frac{h^3}{36} \cdot \frac{b_1^2 + 4b_1b_2 + b_2^2}{b_1 + b_2}$	$x_1 = \frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}$ $x_2 = \frac{2b_1 + b_2}{b_1 + b_2} \cdot \frac{h}{3}$
 184	**T**-*section,* — Area of flange $= A_1$ Area of web $= A_2$ Total depth $= h$ $= h_1 + h_2$ $A = A_1 + A_2$	$A_1 + A_2$	$\frac{A_1h_1^2 + A_2h_2^2}{12} + \frac{A_1A_2(h_1 + h_2)^2}{4(A_1 + A_2)}$	$x_1 = \frac{h}{2} - \frac{A_1h_2 - A_2h_1}{2(A_1 + A_2)}$ $x_2 = \frac{h}{2} + \frac{A_1h_2 - A_2h_1}{2(A_1 + A_2)}$
 185	**I**-*section,* — Area of upper flange $= A_1$ Area of web $= A_2$ Area of lower flange $= A_3$	$A_1 + A_2 + A_3$	$\frac{A_1h_1^2 + A_2h_2^2 + A_3h_3^2}{12} + A_1\left(x_1 - \frac{h}{2}\right)^2 + A_2\left(x_1 - h_1 - \frac{h_2}{2}\right)^2 + A_3\left(x_2 - \frac{h_2}{2}\right)^2$	$x_1 = \frac{A_1\frac{h_1}{2} + A_2\left(h_1 + \frac{h_2}{2}\right) + A_3\left(h_1 + h_2 + \frac{h_3}{2}\right)}{A_1 + A_2 + A_3}$

Figure.	Description.	*A.*	*I.*	Distance of *YY* from Most Strained Fibre.
 186	*Circle,* — Radius r	πr^2	$\dfrac{\pi r^4}{4}$	r
 187	*Hollow circle,* — Outer radius $= r$ Inner radius $= r_1$	$\pi(r^2 - r_1^2)$	$\dfrac{\pi(r^4 - r_1^4)}{4}$	r
 188	*Ellipse,* — Vertical axis $= h = 2a$ Horizontal axis $= b_1 = 2b$	$\pi ab = \dfrac{\pi b_1 h}{4}$	$\dfrac{\pi a^3 b}{4} = \dfrac{\pi b_1 h^3}{64}$	$a = \dfrac{h}{2}$
 189	*Hollow ellipse,* — Outer semi-axes a and b Inner semi-axes a_1 and b_1	$\pi(ab - a_1 b_1)$	$\dfrac{\pi a^3 b}{4} - \dfrac{\pi a_1^3 b_1}{4}$	a

In the following table, the cross-sections are considered as composed of their central lines; the area of any given portion being found by multiplying the thickness of the iron by the corresponding length of line, just as was done in the corresponding cases under "Centre of Gravity."

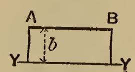
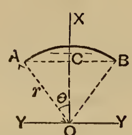
Figure.		Description.	A.	I.	Distance of Axis YY from Most Strained Fibre.
	190	Straight line AB about an axis YY through one end	A	$\frac{1}{3}Ab^2$	
	191	Straight line AB about an axis YY through the middle	A	$\frac{1}{3}Ab^2$	b
	192	Straight line AB about about an axis parallel to it	A	Ab^2	
	193	Circular arc AB about an axis YY through the centre of the circle, and parallel to the chord AB; $t =$ thickness of iron	$A = 2r\theta t$	$tr^3(\theta + \sin\theta\cos\theta) = \frac{Ar^2}{2}\left[1 + \frac{\sin\theta\cos\theta}{\theta}\right]$	

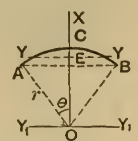
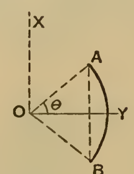
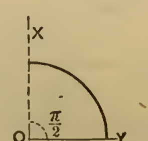
Figure.		Description.	A.	I.	Distance of Axis YY from Most Strained Fibre.
	194	Circular arc AB about an axis YY through the centre of gravity, and parallel to the chord	$A = 2r\theta t$	$tr^3\left[\theta + \sin\theta\cos\theta - \frac{2\sin^2\theta}{\theta}\right]$ $= \frac{Ar^2}{2}\left[1 + \frac{\sin\theta\cos\theta}{\theta} - \frac{\sin^2\theta}{\theta^2}\right]$	$CE = r\left(1 - \frac{\sin\theta}{\theta}\right)$
	195	Circular arc AB about an axis OY at right angles to the chord, and passing through the centre of the circle	$A = 2r\theta t$	$tr^3(\theta - \sin\theta\cos\theta)$ $= \frac{Ar^2}{2}\left[1 - \frac{\sin\theta\cos\theta}{\theta}\right]$	$r\sin\theta$
	196	Quarter-arc of circle about an axis through one end and the centre of the circle	$A = \frac{\pi r}{2}$	$\frac{Ar^2}{2}$	

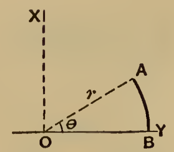
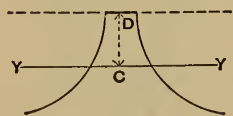
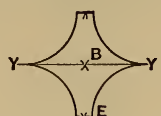
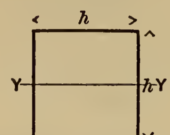
Figure.		Description.	A.	I.	Distance of Axis YY from Most Strained Fibre.
	197	Circular arc AB	$A = tr\theta$	$\frac{tr^3}{2}(\theta - \sin\theta\cos\theta)$ $= \frac{Ar^2}{2}\left[1 - \frac{\sin\theta\cos\theta}{2}\right]$	
	198	*Barlow rail,*— Two quadrants of circular arcs and a web. If area of arcs $= A$, then web $= \frac{3}{11}A$. YY goes through the centre of gravity of the rail	$\frac{14}{11}A$	$\frac{2}{11}Ar^2$, calling $\pi = \frac{22}{7}$	$\frac{r}{2}$
	199	*Double Barlow rail,*— Joint area of arcs $= 2A$, and of webs $\frac{6}{11}A$	$\frac{28}{11}A$	Ar^2	r
	200	Square cell	$A = 4ht$	$\frac{2}{3}h^3t = \frac{1}{6}Ah^2$	$\frac{h}{2}$

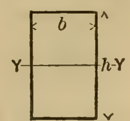
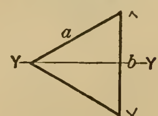
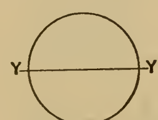
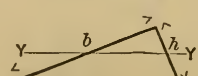
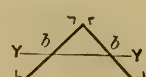
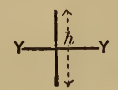
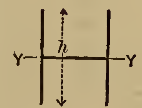
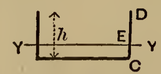
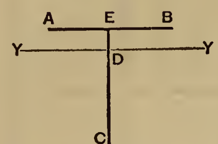
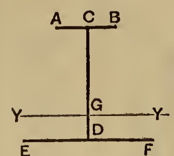
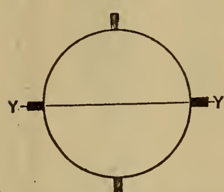
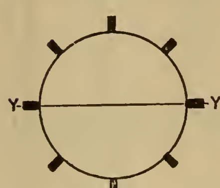
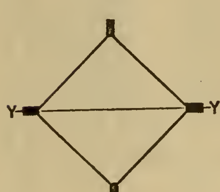
Figure.		Description.	$A.$	$I.$	Distance of Axis YY from Most Strained Fibre.
	201	Rectangular cell	$A = 2t(b+h)$	$\frac{th^2}{6}(3b + h)$	$\frac{h}{2}$
	202	Triangular cell	$A = t(b+2a)$	$\frac{b^2t}{12}(2a + b) = \frac{Ab^2}{12}$	$\frac{b}{2}$
	203	*Circular cell,* — Radius r	$2\pi rt$	$\pi r^3 t = \frac{Ar^2}{2}$	r
	204	*Angle iron,* — Unequal arms, Thickness t, Length of one arm $= b$, and of other $= h$	$t(b + h)$	$\frac{b^2h^2t\,(b + h)}{12\,(b^2 + h^2)} = \frac{Ab^2h^2}{12\,(b^2 + h^2)}$	$\frac{1}{2}\frac{bh}{\sqrt{b^2 + h^2}}$
	205	Angle of equal arms	$2bt$	$\frac{b^3t}{12} = \frac{Ab^2}{24}$	$\frac{1}{2}b\sqrt{2}$

Figure.		Description.	A.	I.	Distance of Axis YY from Most Strained Fibre.
	206	Cross of equal arms	$2ht$	$\frac{h^3t}{12} = \frac{Ah^2}{24}$	$\frac{h}{2}$
	207	H-*section*, — Area of web $= B$ Combined area of flanges $= A$	$A + B$	$\frac{Ah^2}{12}$	$\frac{h}{2}$
	208	*Channel-section*, — Area of web $= B$ Combined area of flanges $= A$	$A + B$	$h^2\left[\frac{A}{12} + \frac{AB}{4(A + B)}\right]$	$DE = \frac{h}{2}\frac{A + 2B}{A + B}$ $CE = \frac{h}{2}\frac{A}{A + B}$
	209	T-*section*, — Area of flange $= A$ Area of web $= B$	$A + B$	$\frac{Bh^2}{12}\left(\frac{4A + B}{A + B}\right)$	$DE = \frac{h}{2}\frac{B}{A + B}$ $DC = \frac{h}{2}\frac{2A + B}{A + B}$
	210	I-*iron*, — Area upper flange $= A$ Area lower flange $= A_1$ Area of web $= B$	$A + A_1 + B$	$\frac{h^2}{12}\left[\frac{12AA_1 + 4B(A + A_1) + B^2}{A + A_1 + B}\right]$	$CG = \frac{h}{2}\frac{2A_1 + B}{A + A_1 + B}$ $GD = \frac{h}{2}\frac{2A + B}{A + A_1 + B}$

§ 188. **Cross-Sections of Phœnix Columns considered as made of Lines.** — It is to be observed that the moments of inertia are the same for all axes passing through the centre. Thickness $= t$, radius of round ones $= r$, area of each flange $= a$, length of each flange $= l$.

Figure.	Description.	*A.*	*I.*
 211	Four flanges	$2\pi rt + 4a$	$\pi r^3 t + 2a\left(r + \frac{l}{2}\right)^2$
 212	Eight flanges	$2\pi rt + 8a$	$\pi r^3 t + 4a\left(r + \frac{l}{2}\right)^2$
 213	Square, four flanges, $r =$ radius of circumscribed circle	$4rt\sqrt{2} + 4a$	$\frac{4r^3t\sqrt{2}}{3} + 2a\left(r + \frac{l}{2}\right)^2$
 214	Six flanges	$2\pi rt + 6a$	$\pi r^3 t + 3a\left(r + \frac{l}{2}\right)^2$

§ 189. **Graphical Representation of Bending-Moments.** — The bending-moment at each point of a loaded beam may be represented graphically by lines laid off to scale, as will be shown by examples.

I. Suppose we have the cantilever shown in Fig. 215, loaded at D with a load W: then will the bending-moment at any section, as at F, be obtained by multiplying W by FD; that at AC being $W \times (AB)$. If, now, we lay off CE to scale to represent this, i.e., having as many units of length as there are units of moment in the product $W \times (AB)$, and join E with D, then will the ordinate FG of any point, as G, represent (to the same scale) the bending-moment at a section through F.

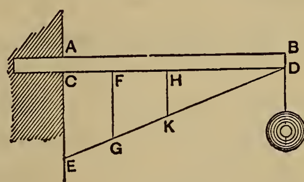


FIG. 215.

II. If we have a uniformly distributed load, we should have, for the line corresponding to CE in Fig. 215, a curve. This is shown in Fig. 216, where we have the uniformly distributed load $EIGF$. If we take the origin at D, as before, we have, for the bending-moment, at a distance x from the origin, as has been shown, $\frac{W}{2l}(l - x)^2$; and by giving x different values, and laying off the corresponding value of the bending-moment, we obtain the curve CA, any ordinate of which will represent the bending-moment at the corresponding point of the beam.

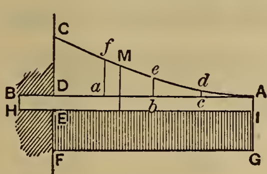


FIG. 216.

When we have more than one load on a beam, we must draw the curve of bending-moments for each load separately, and then find the actual bending-moment at any point of the beam

by taking the sum of the ordinates (drawn from that point) of each of these separate curves or straight lines. If we then draw a new curve, whose ordinates are these sums, we shall have the actual curve of bending-moments for the beam as loaded. Some examples will now be given, which will explain themselves.

III. Fig. 217 shows a cantilever with three concentrated loads. The line of bending-moments for the load at C is CE, that for the load at O is OF, and for the load at P is PG. They are combined above the beam by laying off $AH = DE$, $HK = DF$, and $KL = DG$, and thus obtaining the broken line $LMNB$, which is the line of bending-moments of the beam loaded with all three loads.

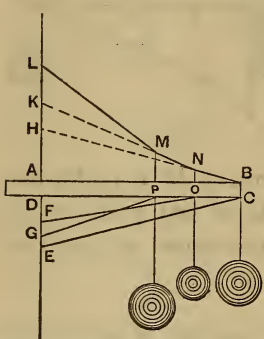


FIG. 217.

IV. Fig. 218 shows the case of a beam supported at both ends, and loaded at a single point D; ALB is the line of bending-moments when the weight of the beam is disregarded, so that $xy =$ bending-moment at x.

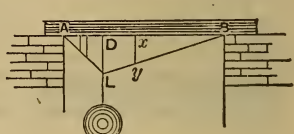


FIG. 218.

V. Fig. 219 shows the case of a beam supported at the ends, and loaded with three concentrated loads at the points B, C, and D respectively; the lines of bending-moments for each individual load being respectively AFE, AGE, and AHE, and the actual line of bending-moments being $AKLME$.

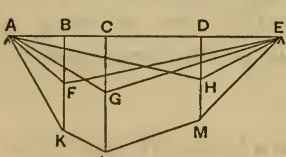


FIG. 219.

VI. Fig. 220 shows the case of a beam supported at the ends, and loaded with a uniformly distributed load; the line of bending-moments being a curve, *ACDB*, as shown in the figure.

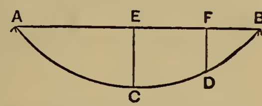


FIG. 220.

VII. In Fig. 221 we have the case of a beam, over a part of which, viz., *EF*, there is a distributed load; the rest of the beam being unloaded. The line of bending-moments is curvilinear between *E* and *F*, and straight outside of these limits. It is *AGSHB;* and, when the curve is plotted, we can find the greatest bending-moment graphically by finding its greatest ordinate. We can also determine it analytically by first determining the bending-moment at a distance x from the origin, and on the side towards the resultant of the load, and then differentiating. This process is shown in the following:—

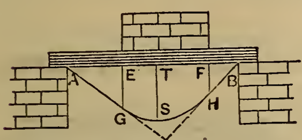


FIG. 221.

Let A (Fig. 222) be the point where the resultant of the load acts, and O the middle of the beam, and let w be the load per unit of length; let $OA = a$, and $AB = AC = b$, so that the whole load $= 2wb$: therefore supporting force at $D = 2wb\dfrac{a+c}{2c} = \dfrac{wb(a+c)}{c}$.

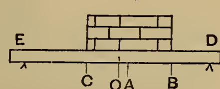


FIG. 222.

If we take a section at a distance x from O to the right, we shall have, for the bending-moment at that section,

$$\frac{wb(a+c)}{c}(c-x) - \frac{w}{2}(a+b-x)^2 = \text{a maximum.}$$

Differentiate, and we have

$$\frac{-wb(a+c)}{c} + w(a+b-x) = 0 \quad \therefore \quad x = \frac{a(c-b)}{c};$$

hence the greatest bending-moment will be

$$\frac{wb(a+c)}{c}\left(c - \frac{a(c-b)}{c}\right) - \frac{w}{2}\left(a + b - a + \frac{ab}{c}\right)^2$$

$$= \frac{wb}{c^2}(a+c)(c^2 - ac + ab) - \frac{wb^2}{2c^2}(a^2 + 2ac + c^2)$$

$$= \frac{wb}{2c^2}(a^2b - 2a^2c + 2c^3 - bc^2).$$

VIII. In Figs. 223 and 224 we have the case of a beam supported at the ends, and loaded with a uniformly distributed load, and also with a concentrated load. In the first figure, the greatest bending-moment is at D, and in the second at C.

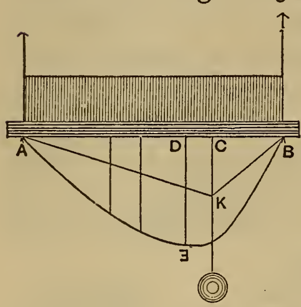


Fig. 223.

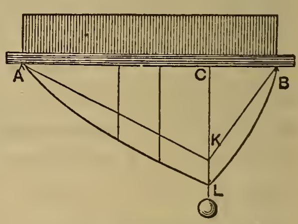


Fig. 224.

IX. In Fig. 225 we have a beam supported at A and B, and loaded at C and D with equal weights; the lengths of AC and BD being equal. We have, consequently, between A and B, a uniform bending-moment; while on the left of A and on the right of B we have a varying bending-moment. The line of bending-moments is, in this case, $CabD$.

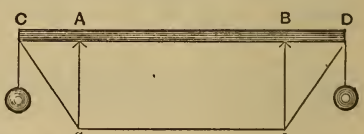


Fig. 225.

We may, in a similar way, derive curves of bending-moment for all cases of loading and supporting beams.

§ 190. Mode of Procedure for Ascertaining the Stresses at Different Parts of a Beam when the Loads and the Dimensions are given. — When the dimensions of a beam, the load and its distribution, and the manner of supporting are given, and it is desired to find the actual intensity of the stress on any particular fibre at any given cross-section, we must proceed as follows: —

1°. Find the actual bending-moment (M) at that cross-section.

2°. Find the moment of inertia (I) of the section about its neutral axis.

3°. Observe, that, from what has already been shown, the moment of the couple formed by the tensions and compressions is aI, where $a =$ intensity of stress of a fibre whose distance from the neutral axis is unity, and that this moment must equal the bending-moment at the section in order to secure equilibrium. Hence we must have

$$aI = M.$$

Moreover, if p denote the (unknown) intensity of the stress of the fibre where the stress is desired, and if y denote the distance of this fibre from the neutral axis, we shall have

$$a = \frac{p}{y},$$

$$\therefore \quad \frac{p}{y}I = M, \qquad\qquad \therefore \quad p = \frac{My}{I},$$

from which equation we can determine p.

EXAMPLES.

1. Given a beam 18 feet span, supported at both ends, and loaded uniformly (its own weight included) with 1000 lbs. per foot of length. The cross-section is a T, where area of flange = 3 square inches, area of web = 4 square inches, height = 10 inches. Find (a) the

bending at 3 feet from one end; (*b*) the greatest bending-moment; (*c*) the greatest intensity of the tension at each of the above sections; (*d*) the greatest intensity of the compression at each of these sections.

2. Given an I-beam with equal flanges, area of each flange = 3 square inches, area of web = 3 square inches, height = 10 inches; the beam is 12 feet long, supported at the ends, and loaded uniformly (its own weight included) with a load of 2000 lbs. per foot of length. Find (*a*) the bending-moment at a section one foot from the end; (*b*) the greatest bending-moment; (*c*) the greatest intensity of the stress at each of the above bross-sections.

§ 191. **Mode of Procedure for Ascertaining the Dimensions of a Beam to bear a Certain Load, or the Load that a Beam of Given Dimensions and Material is Capable of Bearing.** — If we wish to determine the proper dimensions of the beam when the load and its distribution, as well as the manner of supporting, are given, so that it shall nowhere be strained beyond safe limits, or if we wish to determine the greatest load consistent with safety when the other quantities are given, we must impose the condition that the greatest intensity of the tension to which any fibre is subjected shall not exceed the safe working-strength for tension of the material of which the beam is made, and the greatest intensity of the compression to which any fibre is subjected shall not exceed the safe working-strength of the material for compression.

Thus, we must in this case first determine where is the section of greatest bending-moment (this determination sometimes involves the use of the Differential Calculus).

Next we must determine the magnitude of the greatest bending-moment, absolutely if the load and length of the beam are given (if not, in terms of these quantities), and then equate this to the moment of the resisting couple.

Thus, if M_0 is the greatest bending-moment, I_0 the moment of inertia of that section where this greatest bending-moment

acts, and if f_t = working-strength per square inch for tension, f_c = working-strength per square inch for compression, y_t = distance of most stretched fibre from the neutral axis, and y_c = distance of most compressed fibre from the neutral axis, then will $\frac{f_t}{y_t}$ be the greatest tension per square inch, at a unit's distance from the neutral axis, consistent with safety against tearing, and $\frac{f_c}{y_c}$ the greatest compression per square inch, at a unit's distance from the neutral axis, consistent with safety against crushing.

Of course the least of these must not be exceeded in the actual beam. Hence we must put

$$M_o = \frac{f}{y}I,$$

where $\frac{f}{y}$ is taken as the lesser of the two quantities $\frac{f_t}{y_t}$ and $\frac{f_c}{y_c}$.

MODULUS OF RUPTURE.

The modulus of rupture is the greatest tension or compression per square inch to which the most strained fibre of the beam is subjected when the beam is just on the point of breaking.

WORKING-STRENGTH.

The working-strength per square inch of a material for transverse strength is the greatest stress per square inch to which it is safe to subject the most strained fibre of the beam. It is usually obtained by dividing the modulus of rupture by some factor of safety, as 3 or 4.

§ 192. EXAMPLES.

1. Given, as the modulus of rupture of a spruce beam, 4000 lbs. per square inch: find its breaking-strength, assuming it to be 4 inches wide and 12 inches deep, the span being 18 feet, the load being uniformly distributed over its entire length.

2. Suppose such a beam to break with a load at the middle of 5000 lbs.: find its modulus of rupture.

3. Given a T-beam fixed at one end, and loaded uniformly. The area of the flange is 3 square inches, that of the web also 3 square inches, height = 10 inches. The beam is 4 feet long. Find the greatest load it will bear with safety, the working-strength per square inch of the material being, for tension, 10000 lbs., and for compression 8000 lbs.

4. Given an I-beam, area of each flange being 3 square inches, and area of web 3 square inches, height = 12 inches, span 8 feet, supported at the ends, and loaded uniformly: what load will it bear with safety, the working-strength of the iron for tension, and also for compression, being 12000 lbs. per square inch.

5. Given a beam (Fig. 226) supported at both ends, and loaded, 1°, with w pounds per unit of length uniformly, and 2°, with a single load W at a distance a from the left-hand support: find the position of the section of greatest bending-moment, and the value of the greatest bending-moment.

Solution.

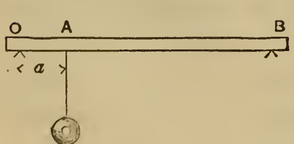


FIG. 226.

1. Left-hand supporting-force

$$= \frac{wl}{2} + \frac{W(l - a)}{l}.$$

Right-hand supporting-force

$$= \frac{wl}{2} + \frac{Wa}{l}.$$

2. Assume a section at a distance x from the left-hand support (this support being the origin), and the bending-moment at that section is, —

when $x < a$,

$$\left\{\frac{wl}{2} + \frac{W(l-a)}{l}\right\}x - \frac{wx^2}{2};$$

and when $x > a$,

$$\left\{\frac{wl}{2} + \frac{W(l-a)}{l}\right\}x - \frac{wx^2}{2} - W(x-a).$$

To find the value of x for the section of greatest bending-moment, differentiate each, and put the first differential co-efficient = zero.

We shall thus have, in the first case,

$$\frac{wl}{2} + \frac{W(l-a)}{l} - wx = 0, \text{ or } x = \frac{l}{2} + \frac{W(l-a)}{wl};$$

and in the second case,

$$\frac{wl}{2} + \frac{W(l-a)}{l} - wx - W = 0, \text{ or } x = \frac{l}{2} + \frac{W(l-a)}{wl} - \frac{W}{w}.$$

Now, whenever the first is $< a$, or the second is $> a$, we shall have in that one the value of x corresponding to the section of greatest bending-moment. But if the first is $> a$, *and* the second $< a$, then the greatest bending-moment is at the concentrated load.

These conclusions will be evident on drawing a diagram representing the bending-moments graphically, as in Figs. 223 and 224; and the greatest bending-moment may then be found by substituting, in the corresponding expression for the bending-moment, the deduced value of x.

6. Given an I-beam 10 feet long, supported at both ends, and loaded, at a distance 2 feet to the left of the middle, with 20000 pounds. Find the bending-moment at the middle, the greatest bending-moment, also the greatest intensity of the tension, and that of the compression at each of these sections.

Given Area of upper flange = 8 sq. in.
Area of lower flange = 5 sq. in.
Area of web = 7 sq. in.
Total depth = 14 in.

7. Given a beam (Fig. 227) 18 feet long, loaded at A with 1000 lbs., and at B with 2000 lbs.; the beam weighing 200 lbs. ($OA = 3$ feet, $OB = 10$ feet). Find the section of greatest bending-moment, and the bending-moment at that section.

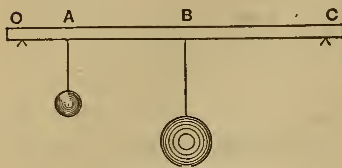


FIG. 227.

§ 193. **Beams of Uniform Strength.** — A beam of uniform strength (technically so called) is one in which the dimensions of the cross-section are varied in such a manner, that, at each cross-section, the greatest intensity of the tension shall be the same, and so also the greatest intensity of the compression.

Such beams are very rarely used; and, as the cross-section varies at different points, it would be decidedly bad engineering to make them of wood, for it would be necessary to cut the wood across the grain, and this would develop a tendency to split.

In making them of iron, also, the saving of iron would generally be more than offset by the extra cost of rolling such a beam. Nevertheless, we will discuss the form of such beams in the case when the section is rectangular.

In all cases we have the general equation

$$M = \frac{p}{y} I$$

applying at each cross-section, where M = bending-moment (section at distance x from origin), I = moment of inertia of same section, y = distance from neutral axis to most strained fibre, and p = intensity of stress on most strained fibre; the condition for this case being that p is a constant for all values of x (i.e., for all positions of the section), while M, I, and y are functions of x.

As we are limiting ourselves to rectangular sections, if we let b = breadth and h = depth of rectangle (one or both varying with x), we shall have

$$M = \frac{p}{6} bh^2$$

as the condition for such a beam, with p a constant for all values of x, when the same load remains on the beam.

We must, therefore, have bh^2 proportional to M. Hence, assuming the origin as before,

1°. Fixed at one end, load at the other, $bh^2 = \left(\frac{6}{p}\right) W(l - x)$.

2°. Fixed at one end, uniformly loaded, $bh^2 = \left(\frac{6}{p}\frac{W}{2l}\right)(l - x)^2$.

3°. Supported at ends, loaded at middle,
$$\begin{cases} \text{for } x < \frac{l}{2}, & bh^2 = \left(\frac{6}{p}\frac{W}{2}\right)x; \\ \text{for } x > \frac{l}{2}, & bh^2 = \left(\frac{6}{p}\frac{W}{2}\right)(l - x). \end{cases}$$

4°. Supported at ends, uniformly loaded, $bh^2 = \left(\frac{6}{p}\frac{W}{2l}\right)(lx - x^2)$.

Now, this variation of section may be accomplished in one of two ways: 1st, by making h constant, and letting b vary; and 2d, by making b constant, and letting h vary. Thus, in the first case above mentioned, if h is constant, we have, for the plan of the beam,

$$b = \left(\frac{6W}{ph^2}\right)(l - x):$$

and if one side be taken parallel to the axis of the beam, this will be the equation of the other side; and, as this is the equation of a straight line, the plan will be a triangle.

If, on the other hand, b be constant, and h vary, we shall have, for the vertical longitudinal section of the beam,

$$h^2 = \left(\frac{6W}{pb}\right)(l - x);$$

and, if one side be taken as a straight line in the direction of the axis, the other will be a parabola.

A similar reasoning will give the plan or elevation respectively in each case; and these can be readily plotted from their equations.

CROSS-SECTION OF EQUAL STRENGTH.

A cross-section of equal strength (technically so called) is one so proportioned that the greatest intensity of the tension shall bear the same ratio to the breaking tensile strength of the material as the greatest intensity of the compression bears to the breaking compressive strength of the material. This is accomplished, as will be shown directly, by so arranging the form and dimensions of the section that the distance of the neutral axis from the most stretched fibre shall bear to its distance from the most compressed fibre the same ratio that the tensile bears to the compressive strength of the material.

Let f_c = breaking-strength per square inch for compression,
f_t = breaking-strength per square inch for tension,
y_c = distance of neutral axis from most compressed fibre,
y_t = distance of neutral axis from most stretched fibre.

If p_c = actual greatest intensity of compression, and p_t = actual greatest intensity of tension, then, for a cross-section of equal strength, we must have, according to the definition, $\frac{p_c}{p_t} = \frac{f_c}{f_t}$; but we have $\frac{p_c}{y_c} = \frac{p_t}{y_t}$ = intensity of stress at a unit's

distance from the neutral axis. Hence, combining these two, we obtain

$$\frac{y_c}{y_t} = \frac{f_c}{f_t}.$$

EXAMPLE.

Suppose we have $f_c = 80000$ lbs. per square inch, and $f_t = 20000$ lbs. per square inch. : find the proper proportion between the flange A_1 and the web A_2 of a **T**-section whose depth is h.

§ 194. **Deflection of Beams.** — We have already seen (§ 185), that, in the case of a beam which is bent by a transverse load, we have

$$\alpha = \frac{1}{r}(y),$$

where (having assumed a certain cross-section whose distance from the origin is x) $\alpha =$ the *strain* of a fibre whose distance from the neutral axis is y, and $r =$ radius of curvature of the neutral lamina at the section in question. Hence follows the equation

$$\frac{1}{r} = \frac{\alpha}{y};$$

but from the definition of E, the modulus of elasticity, we shall have

$$\alpha = \frac{p}{E},$$

where $p =$ intensity of the stress at a distance y from the neutral axis.

Hence it follows, assuming Hooke's law, that

$$\frac{1}{r} = \frac{p}{Ey} = \frac{1}{E}\frac{p}{y}$$

We have already seen, that, disregarding signs, $M = \dfrac{p}{y} I$

(making, of course, the two assumptions already spoken of when this formula was deduced), where $M =$ bending-moment at, and $I =$ moment of inertia of, the section in question; i.e., of that section whose distance from the origin is x. This gives $\frac{p}{y} = -\frac{M}{I}$, if, denoting tension by the $+$ sign, and taking y positive upwards, we call M positive when it tends to cause tension on the lower, and compression on the upper, side; these being the conventions in regard to signs which we shall adopt in future. Hence, by substitution, we have

$$\frac{1}{r} = \frac{p}{Ey} = -\frac{M}{EI}. \qquad (1)$$

Now, if we assume the axis of x coincident with the neutral line of the central longitudinal section of the beam, and the axis of v at right angles to this, and v positive upwards, no matter where the origin is taken, we shall always have, as is shown in the Differential Calculus,

$$\frac{1}{r} = \frac{-\frac{d^2v}{dx^2}}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}}.$$

Hence equation (1) becomes

$$\frac{\frac{d^2v}{dx^2}}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}} = \frac{M}{EI}. \qquad (2)$$

M and I being functions of x: and, when we can integrate this equation, we can obtain v in terms of x, thus having the equation of the elastic curve of the neutral line; and, by computing the value of v corresponding to any assumed value of x, we can obtain the deflection at that point of the beam.

The above equation (2) is, as a rule, too complicated to be integrated, except by approximation; and the approximation usually made is the following: —

Since in a beam not too heavily loaded, the slope, and consequently the tangent of the slope (or angle the neutral line makes with the horizontal at any point), is necessarily small, it follows that $\frac{dv}{dx}$ is very small, and hence $\left(\frac{dv}{dx}\right)^2$ is also very small, and $1 + \left(\frac{dv}{dx}\right)^2$ is nearly equal to unity. Making this substitution, we obtain, in place of equation (2),

$$\frac{d^2v}{dx^2} = \frac{M}{EI}; \qquad (3)$$

and this is the equation with which we always start in computing the slope and deflection of a loaded beam, or in finding the equation of the elastic line.

By one integration (suitably determining the arbitrary constant) we obtain the slope whose tangent is $\frac{dv}{dx}$, and by a second integration we obtain the deflection v at a distance x from the origin; and thus, by substituting any desired value for x, we can obtain the deflection at any point.

§ 195. **Ordinary Formulæ for Slope and Deflection.** — We may therefore write, if i is the circular measure of the slope at a distance x from the origin, since $i = \tan i = \frac{dv}{dx}$ nearly,

$$\frac{d^2v}{dx^2} = \frac{M}{EI},$$

$$i = \frac{dv}{dx} = \int \frac{M}{EI}\,dx,$$

$$v = \int\int \frac{M}{EI}\,dx^2.$$

In these equations, of course, E is taken as a constant, M must ALWAYS be expressed in terms of x, and so also must I whenever the section varies at different points. When, however, the section is uniform, I is constant, and the formulæ reduce to

$$i = \frac{1}{EI}\int M dx, \qquad v = \frac{1}{EI}\int\int M dx^2.$$

§ 196. **Special Cases.** — 1°. Let us take a cantilever loaded with a single load at the free end. Assume the origin, as before, at the fixed end, and let the beam be one of uniform section. We then have $M = -W(l - x)$,

$$\therefore \quad i = -\frac{W}{EI}\int (l - x) dx = -\frac{W}{EI}\left(lx - \frac{x^2}{2}\right) + c.$$

To determine c, observe that when $x = 0$, $i = 0$;

$$\therefore \quad c = 0 \qquad \therefore \quad i = -\frac{W}{EI}\left(lx - \frac{x^2}{2}\right) \qquad (1)$$

is the slope at a distance x from the origin.

The deflection at the same point will be

$$v = \int i dx = -\frac{W}{EI}\int\left(lx - \frac{x^2}{2}\right)dx = -\frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + c;$$

but when $x = 0$, $v = 0$ $\quad\therefore\quad c = 0 \quad\therefore\quad$ the deflection at a distance x from the origin will be

$$v = -\frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right). \qquad (2)$$

The equations (1) and (2) give us the means of finding the slope and deflection at any point of the beam.

To find the greatest slope and deflection, we have that both expressions are greatest when $x = l$. Hence, if i_0 and v_0 represent the greatest slope and deflection respectively,

$$i_0 = -\frac{Wl^2}{2EI}, \qquad v_0 = -\frac{Wl^3}{3EI}.$$

2°. Next take the case of a beam supported at both ends and loaded uniformly, the load per unit of length being w.

Assume the origin at the left-hand end; then

$$M = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2) \qquad \text{and} \qquad W = wl$$

$$\therefore \quad i = \frac{w}{2EI}\int (lx - x^2)\,dx = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + c.$$

To determine c, we have that when $x = \frac{l}{2}$, then $i = 0$;

$$\therefore \quad 0 = \frac{w}{2EI}\left(\frac{l^3}{8} - \frac{l^3}{24}\right) + c \qquad \therefore \quad c = -\frac{wl^3}{24EI}$$

$$\therefore \quad i = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) - \frac{wl^3}{24EI} = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3) \quad (1)$$

$$\therefore \quad v = \int i\,dx = \frac{w}{24EI}\int (6lx^2 - 4x^3 - l^3)\,dx$$

$$= \frac{w}{24EI}(2lx^3 - x^4 - l^3x) + c.$$

But when $x = 0$, $v = 0$;

$$\therefore \quad c = 0$$

$$\therefore \quad v = \frac{w}{24EI}(2lx^3 - x^4 - l^3x). \qquad (2)$$

For the greatest slope, we have $x = 0$, or $x = l$;

$$\therefore \quad i_0 = \frac{wl^3}{24EI} = \frac{Wl^2}{24EI}.$$

For the greatest deflection, $x = \frac{l}{2}$;

$$\therefore \quad v_0 = \frac{w}{24EI}\frac{5l^4}{16} = \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}.$$

3°. Take the case of a beam supported at both ends, and loaded at the middle with a load W.

Assume, as before, the origin at the left-hand support. Then we shall have

$$M = \frac{W}{2}x, \quad x < \frac{l}{2}, \quad \text{and} \quad M = \frac{W}{2}(l - x) \text{ when } x > \frac{l}{2}.$$

Therefore, for the slope up to the middle, we have

$$i = \frac{W}{2EI}\int xdx = \frac{W}{2EI}\frac{x^2}{2} + c.$$

When $x = \frac{l}{2}$, then $i = 0$;

$$\therefore \quad c = -\frac{Wl^2}{16EI}$$

$$\therefore \quad i = \frac{W}{4EI}\left(x^2 - \frac{l^2}{4}\right), \tag{1}$$

and

$$v = \frac{W}{4EI}\int\left(x^2 - \frac{l^2}{4}\right)dx = \frac{W}{4EI}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right) + c.$$

But when $x = 0$, $v = 0$;

$$\therefore \quad c = 0$$

$$\therefore \quad v = \frac{W}{4EI}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right). \tag{2}$$

The slope is greatest when $x = 0$;

$$\therefore \quad i_0 = \frac{Wl^2}{16EI}.$$

The deflection is greatest when $x = \frac{l}{2}$;

$$\therefore \quad v_0 = \frac{Wl^3}{48EI}.$$

In this case the symmetry of the beam and load makes it

unnecessary to examine the part where $x > \frac{l}{2}$; but, if this were to be done, we should have, for that part,

$$i = \frac{W}{2EI}\int (l - x)dx \quad \text{and} \quad v = \frac{W}{2EI}\int\int (l - x)dx^2.$$

4°. Following will be found a table of some deflections, which may be regarded as examples simply.

Uniform Cross-Section.	Greatest Slope.	Greatest Deflection.
Fixed at one end, loaded at the other .	$\frac{1}{2}\frac{Wl^2}{EI}$	$\frac{1}{3}\frac{Wl^3}{EI}$
Fixed at one end, loaded uniformly . .	$\frac{1}{6}\frac{Wl^2}{EI}$	$\frac{1}{8}\frac{Wl^3}{EI}$
Supported at ends, load at middle . . .	$\frac{1}{16}\frac{Wl^2}{EI}$	$\frac{1}{48}\frac{Wl^3}{EI}$
Supported at ends, uniformly loaded . .	$\frac{2}{24}\frac{Wl^2}{EI}$	$\frac{5}{384}\frac{Wl^3}{EI}$
Uniform Strength and Uniform Depth.		
Fixed at one end, load at the other . .	$\frac{Wl^2}{EI}$	$\frac{1}{2}\frac{Wl^3}{EI}$
Fixed at one end, uniformly loaded . .	$\frac{1}{2}\frac{Wl^2}{EI}$	$\frac{1}{4}\frac{Wl^3}{EI}$
Supported at both ends, load at middle .	$\frac{1}{8}\frac{Wl^2}{EI}$	$\frac{1}{32}\frac{Wl^3}{EI}$
Supported at both ends, uniformly loaded,	$\frac{1}{16}\frac{Wl^2}{EI}$	$\frac{1}{64}\frac{Wl^3}{EI}$

§ 197. **Deflection with Uniform Bending-Moment.** — If the bending-moment is uniform, then M is constant; and, if I is also constant, we have

$$i = \frac{M}{EI}\int dx = \frac{Mx}{EI} + c;$$

but when $x = \frac{l}{2}$, then $i \doteq 0$;

$$\therefore \quad c = -\frac{Ml}{2EI}$$

$$\therefore \quad i = \frac{M}{EI}\left(x - \frac{l}{2}\right) = \frac{dv}{dx}$$

$$\therefore \quad v = \frac{M}{EI}\left(\frac{x^2}{2} - \frac{lx}{2}\right),$$

the constant disappearing because $v = 0$ when $x = 0$.

Hence, for a beam where the bending-moment is uniform, we have

$$i = \frac{M}{EI}\left(x - \frac{l}{2}\right), \quad v = \frac{M}{EI}\left(\frac{x^2}{2} - \frac{lx}{2}\right);$$

and for greatest slope and deflection, we have

$$i_0 = \frac{-Ml}{2EI}, \quad v_0 = \frac{M}{EI}\left(\frac{l^2}{8} - \frac{l^2}{2}\right) = -\frac{3}{8}\frac{Ml^2}{EI}.$$

§ 198. **Resilience of a Beam.** — *The resilience of a beam is the mechanical work performed in deflecting it to the amount it would deflect under its greatest allowable gradually applied load.* In the case of a concentrated load, if W is the greatest allowable gradually applied load, and v_1 the corresponding deflection at the point of application of the load, then will the mean value of the load that produces this deflection be $\frac{W}{2}$, and the resilience of the beam will be $\frac{W}{2}v_1$.

§ 199. **Slope and Deflection of a Beam with a Concentrated Load not at the Middle.** — Take, as the next case, a beam (Fig. 228). Let the load at A be W, and distance $OA = a$, and let $a > \frac{l}{2}$.

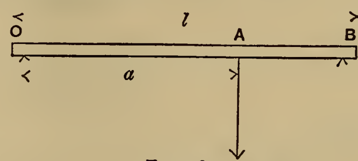


FIG. 228.

$$x < a \quad M = \frac{W(l - a)}{l} x,$$

$$x < a \quad M = \frac{Wa}{l}(l - x),$$

$$\therefore \quad x < a \quad i = \frac{W(l - a)}{lEI} \int x dx = \frac{W(l - a)}{2lEI} x^2 + c.$$

When $x = 0$, $i = i_0 =$ undetermined slope at O;

$$\therefore \quad c = i_0, \qquad \therefore \quad i = \frac{W(l - a)}{2lEI} x^2 + i_0, \tag{1}$$

and

$$\therefore \quad v = \frac{W(l - a)}{2lEI} \int x^2 dx + i_0 \int dx = \frac{W(l - a)}{6lEI} x^3 + i_0 x + c.$$

When $x = 0$, $v = 0$;

$$\therefore \quad v = \frac{W(l - a)}{6lEI} x^3 + i_0 x, \tag{2}$$

$$x > a \quad i = \frac{Wa}{lEI} \int (l - x) dx = \frac{Wa}{lEI}\left(lx - \frac{x^2}{2}\right) + c.$$

To determine c, observe that when $x = a$, this value of i and that deduced from (1) must be identical.

$$\frac{Wa}{lEI}\left(la - \frac{a^2}{2}\right) + c = \frac{W(l - a)a^2}{2lEI} + i_0 \qquad \therefore \quad c = -\frac{Wa^2}{2EI} + i_0$$

$$\therefore \quad i = \frac{Wa}{lEI}\left(lx - \frac{x^2}{2}\right) - \frac{Wa^2}{2EI} + i_0,$$

or

$$i = \frac{Wa}{2lEI}(2lx - x^2 - la) + i_0, \qquad (3)$$

and

$$v = \frac{Wa}{2lEI}\int(2lx - x^2 - la)dx + i_0\int dx$$

$$= \frac{Wa}{2lEI}\left(lx^2 - \frac{x^3}{3} - lax\right) + i_0 x + c.$$

To determine c, observe that when $x = a$, this value of v and (2) must be identical;

$$\therefore \quad \frac{Wa}{2lEI}\left(-\frac{a^3}{3}\right) + i_0 a + c = \frac{W(l-a)}{6lEI}a^3 + i_0 a$$

$$\therefore \quad c = \frac{W}{6lEI}(la^3 - a^4 + a^4) = \frac{Wla^3}{6lEI} = \frac{Wa^3}{6EI}$$

$$\therefore \quad v = \frac{Wa}{6lEI}(3lx^2 - x^3 - 3lax + la^2) + i_0 x. \qquad (4)$$

To determine i_0, we have that when $x = l$, $v = 0$;

$$\therefore \quad 0 = \frac{Wa}{6lEI}(2l^3 - 3al^2 + la^2) + i_0 l$$

$$\therefore \quad i_0 = \frac{Wa}{6l^2EI}(3al^2 - 2l^3 - la^2) = \frac{Wa}{6lEI}(3al - 2l^2 - a^2).$$

Substituting this value of i_0 in the equations (1), (2), (3), and (4), we obtain for

$$(1) \quad i = \frac{W(l-a)}{2lEI}x^2 + \frac{Wa}{6lEI}(3al - 2l^2 - a^2),$$

$$(2) \quad v = \frac{W(l-a)}{6lEI}x^3 + \frac{Wa}{6lEI}(3al - 2l^2 - a^2)x,$$

$$(3)\ i = \frac{Wa}{2lEI}(2lx - x^2 - al) + \frac{Wa}{6lEI}(3al - 2l^2 - a^2),$$

$$(4)\ v = \frac{Wa}{6lEI}(3lx^2 - x^3 - 3lax + la^2) + \frac{Wa}{6lEI}(3al - 2l^2 - a^2)x.$$

To find the greatest deflection, differentiate (2), and place the first differential co-efficient equal to zero: or, which is the same thing, place $i = 0$ in (1), and find the value of x; then substitute this value in (2), and we shall have the greatest deflection.

We thus obtain

$$(l - a)x^2 = \frac{-a}{3}(3al - 2l^2 - a^2) \qquad \therefore\ x^2 = \frac{a}{3}\left(\frac{2l^2 - 3al + a^2}{l - a}\right),$$

or

$$x^2 = \frac{a}{3}(2l - a) \qquad \therefore\ x = \frac{\sqrt{2al - a^2}}{\sqrt{3}};$$

and the greatest deflection becomes

$$v_0 = -\frac{Wa(l - a)(2l - a)}{9lEI}\,\frac{\sqrt{2al - a^2}}{\sqrt{3}}.$$

§ 200. *EXAMPLES.*

1. In example 1, p. 284, find the greatest deflection of the beam when it is loaded with $\frac{1}{4}$ of its breaking-load, assuming $E = 1200000$.

2. In the same case, find what load will cause it to deflect $\frac{1}{400}$ of its span.

3. What will be the stress at the most strained fibre when this occurs.

4. In example 3, p. 284, find the load the beam will bear without deflecting more than $\frac{1}{400}$ of its span, assuming $E = 24000000$.

5. Find the stress at the most strained fibre when this occurs.

6. In example 6, p. 285, find the greatest deflection under a load $\frac{1}{4}$ the breaking-load.

§ 201. **Deflection and Slope under Working-Load.** — If we take the four cases of deflection given in the first part of the table on p. 295, and calling f the modulus of rupture of the material, and y the distance of the most strained fibre from the neutral axis, and if we make the applied load the working-load, we shall have respectively —

$$1°.\quad Wl = \frac{fI}{y} \qquad \therefore \quad W = \frac{fI}{ly}.$$

$$2°.\quad \frac{Wl}{2} = \frac{fI}{y} \qquad \therefore \quad W = \frac{2fI}{ly}.$$

$$3°.\quad \frac{Wl}{4} = \frac{fI}{y} \qquad \therefore \quad W = \frac{4fI}{ly}.$$

$$4°.\quad \frac{Wl}{8} = \frac{fI}{y} \qquad \therefore \quad W = \frac{8fI}{ly}.$$

And the values of slope and deflection will become respectively,

	Slope.	Deflection.		Slope.	Deflection.
1°.	$\frac{1}{2}f\frac{l}{Ey}$	$\frac{1}{3}f\frac{l^2}{Ey}$	3°.	$\frac{1}{4}f\frac{l}{Ey}$	$\frac{1}{12}\frac{fl^2}{Ey}$
2°.	$\frac{1}{3}f\frac{l}{Ey}$	$\frac{1}{4}f\frac{l^2}{Ey}$	4°.	$\frac{1}{3}f\frac{l}{Ey}$	$\frac{5}{48}f\frac{l^2}{Ey}$

From these values, and those given on p. 295, we derive the following two propositions: —

1°. If we have a series of beams differing only in length, and we apply the same load in the same manner to each, their greatest slopes will vary as the squares of their lengths, and their greatest deflections as the cubes of their lengths.

2°. If, however, we load the same beams, not with the same load, but each one with its working-load, as determined by allowing a given greatest fibre stress, then will their greatest slopes vary as the lengths, and their greatest deflections as the squares of their lengths.

§ 202. **Slope and Deflection of Rectangular Beams.** — If the beams are rectangular, so that $I = \frac{bh^3}{12}$ and $y = \frac{h}{2}$, the values of slope and deflection above referred to become further simplified, and we have the following tables: —

	Given Load W.		Working-Load. Greatest Fibre Stress $= f$.	
	Slope.	Deflection.	Slope.	Deflection.
1°.	$\frac{6Wl^2}{Ebh^3}$	$\frac{4Wl^3}{Ebh^3}$	$\frac{fl}{Eh}$	$\frac{2}{3}\frac{fl^2}{Eh}$
2°.	$\frac{2Wl^2}{Ebh^3}$	$\frac{3}{2}\frac{Wl^3}{Ebh^3}$	$\frac{2}{3}\frac{fl}{Eh}$	$\frac{1}{2}\frac{fl^2}{Eh}$
3°.	$\frac{3}{4}\frac{Wl^2}{Ebh^3}$	$\frac{1}{4}\frac{Wl^3}{Ebh^3}$	$\frac{1}{2}\frac{fl}{Eh}$	$\frac{1}{6}\frac{fl^2}{Eh}$
4°.	$\frac{1}{2}\frac{Wl^2}{Ebh^3}$	$\frac{5}{32}\frac{Wl^3}{Ebh^3}$	$\frac{2}{3}\frac{fl}{Eh}$	$\frac{5}{24}\frac{fl^2}{Eh}$

So that, in the case of rectangular beams similarly loaded and supported, we may say that —

Under a given load W, the slopes vary as the squares of the lengths, and inversely as the breadths and the cubes of the depths; while the deflections vary as the cubes of the lengths, and inversely as the breadths and the cubes of the depths.

On the other hand, under their working-loads, the slopes vary directly as the lengths, and inversely as the depths; while the deflections vary as the squares of the lengths, and inversely as the depths.

§ 203. **Beams Fixed at the Ends.** — The only cases which we shall discuss here are the two following; viz., —

1°. Uniform section loaded at the middle.

2°. Uniform section, load uniformly distributed.

CASE I. — *Uniform Section loaded at the Middle.* — The fixing at the ends may be effected by building the beam for some distance into the wall, as shown in Fig. 229. The same result, as far as the effect on the beam is concerned, might be effected as follows: Having merely supported it, and placed upon it the loads it has to bear, load the ends outside of the supports just enough to make the tangents at the supports horizontal.

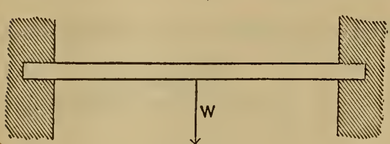


FIG. 229.

These loads on the ends would, if the other load was removed, cause the beam to be convex upwards: and, moreover, the bending-moment due to this load would be of the same amount at all points between the supports; i.e., a uniform bending-moment. Moreover, since the effect of the central load and the loads on the ends is to make the tangents over the supports horizontal, it follows that the upward slope at the support due to the uniform bending-moment above described must be just equal in amount to the downward slope due to the load at the middle, which occurs when the beam is only supported.

Hence the proper method of proceeding is as follows: —

1°. Calculate the slope at the support as though the beam were supported, and not fixed, at the ends; and we shall have, if we represent this slope by i_1, the equation

$$i_1 = -\frac{Wl^2}{16EI}. \qquad (1)$$

2°. Determine the uniform bending-moment which would produce this slope.

To do this, we have, if we represent this uniform bending-moment by M_1, that the slope which it would produce would be

$$-\frac{M_1 l}{2EI}; \qquad (2)$$

and, since this is equal to i_1, we shall have the equation

$$-\frac{M_1 l}{2EI} - \frac{Wl^2}{16EI} = 0 \qquad (3)$$

$$\therefore \quad M_1 = -\frac{Wl}{8}. \qquad (4)$$

This is the actual bending-moment at either fixed end; and the bending-moment at any special section at a distance x from the origin will be

$$M + M_1,$$

where M is the bending-moment we should have at that section if the beam were merely supported, and not fixed. Hence, when it is fixed at the ends, we shall have, for the bending-moment at a distance x from O, where O is at the left-hand support,

$$M = \frac{W}{2}x - \frac{W}{8}l. \qquad (5)$$

When $x = \frac{l}{2}$, we obtain, as bending-moment at the middle,

$$M_0 = \frac{Wl}{8}; \qquad (6)$$

and, since $M_1 = -M_0$, it follows that the greatest bending-moment is

$$\frac{Wl}{8},$$

this being the magnitude of the bending-moment at the middle and also at the support.

POINTS OF INFLECTION.

The value of M becomes zero when

$$x = \frac{l}{4} \text{ and when } x = \frac{3l}{4};$$

hence it follows that at these points the beam is not bent, and that we thus have two points of inflection half-way between the middle and the supports.

SLOPE AND DEFLECTION UNDER A GIVEN LOAD.

We shall have, as before,

$$i = \int \frac{M}{EI} dx = \frac{Wx^2}{4EI} - \frac{Wlx}{8EI} + c;$$

and since, when $x = 0$, $i = 0$,

$$\therefore \quad c = 0$$

$$\therefore \quad i = \frac{dv}{dx} = \frac{W}{8EI}(2x^2 - lx) \tag{7}$$

$$\therefore \quad v = \frac{W}{8EI}\left(\frac{2x^3}{3} - \frac{lx^2}{2}\right), \tag{8}$$

the constant vanishing because $v = 0$ when $x = 0$. The slope becomes greatest when $x = \frac{l}{4}$, and the deflection when $x = \frac{l}{2}$. Hence for greatest slope and deflection, we have

$$i_0 = -\frac{Wl^2}{64EI}, \tag{9}$$

$$v_0 = -\frac{Wl^3}{192EI}. \tag{10}$$

SLOPE AND DEFLECTION UNDER THE WORKING-LOAD.

If f represent the working-strength of the material per square inch, and if W represent the centre working-load, we shall have

$$\frac{Wl}{8} = \frac{fI}{y}$$

$$\therefore \quad W = \frac{8fI}{ly} \tag{11}$$

$$\therefore \quad i_0 = -\frac{1}{8}\frac{fl^2}{Ey} \quad (12) \qquad v_0 = -\frac{1}{24}\frac{fl^3}{Ey}. \quad (13)$$

CASE II. — *Uniform Section, Load uniformly Distributed.* — Pursuing a method entirely similar to that adopted in the former case, we have —

1°. Slope at end, on the supposition of supported ends, is

$$i_1 = -\frac{Wl^2}{24EI}. \tag{1}$$

2°. Slope at end under uniform bending-moment M_1 is

$$-\frac{M_1 l}{2EI}. \tag{2}$$

Hence, since their sum equals zero,

$$M_1 = -\frac{Wl}{12}, \tag{3}$$

which is the bending-moment over either support.

The bending-moment at distance x from one end is

$$M = \frac{W}{2l}(lx - x^2) - \frac{Wl}{12}. \tag{4}$$

This is greatest when $x = 0$, and is then $-\frac{Wl}{12}$. Hence greatest bending-moment is, in magnitude,

$$\frac{Wl}{12}. \tag{5}$$

POINTS OF INFLECTION.

$$M \text{ becomes zero when } x = \frac{l}{2} \pm \frac{l}{2}\frac{1}{\sqrt{3}}. \qquad (6)$$

Hence the two points of inflection are situated at a distance $\frac{l}{2\sqrt{3}}$ on either side of the middle.

SLOPE AND DEFLECTION.

$$i = \int \frac{M}{EI} dx = \frac{W}{12lEI}(3lx^2 - 2x^3 - l^2x), \qquad (7)$$

the constant vanishing because $i = 0$ when $x = 0$.

$$v = \frac{W}{12lEI}\left\{lx^3 - \frac{x^4}{2} - \frac{l^2x^2}{2}\right\}, \qquad (8)$$

the constant vanishing because $v = 0$ when $x = 0$. Hence for greatest slope and deflection we have, i is greatest when $x = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$, and v is greatest when $x = \frac{l}{2}$;

$$\therefore \quad i_0 = -\frac{Wl^2}{72\sqrt{3}EI}, \qquad (9)$$

$$v_0 = -\frac{Wl^3}{384EI}. \qquad (10)$$

SLOPE AND DEFLECTION UNDER WORKING-LOAD.

For working-load we have

$$\frac{Wl}{12} = \frac{fI}{y} \qquad (11)$$

$$\therefore \quad W = \frac{12fI}{ly} \qquad (12)$$

$$\therefore \quad i_0 = -\frac{fl}{6\sqrt{3}Ey}, \qquad (13)$$

$$v_0 = -\frac{fl^2}{32Ey}. \qquad (14)$$

EXAMPLES.

1. Given a 4-inch by 12-inch yellow-pine beam, span 20 feet, fixed at the ends; find its safe centre load, its safe uniformly distributed load, and its deflection under each load. Assume a modulus of rupture 5000 lbs. per square inch, and factor of safety 4. Modulus of elasticity, 1200000.

2. Find the depth necessary that a 4-inch wide yellow-pine beam, 20 feet span, fixed at the ends, may not deflect more than one four-hundredth of the span under a load of 5000 lbs. centre load.

§ 204. **Variation of Bending-Moment with Shearing-Force.** — *If, in any loaded beam whatever,* M *represent the bending-moment, and* F *the shearing-force at a distance* x *from the origin, then will*

$$F = \frac{dM}{dx}. \qquad (1)$$

Proof (*a*). — In the case of a cantilever (Fig. 230), assume the origin at the fixed end; then, if M represent the bending-moment at a distance x from the origin, and $M + \Delta M$ that at a distance $x + \Delta x$ from the origin, we shall have the following equations: —

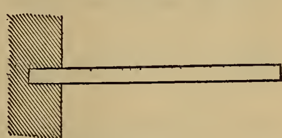


Fig. 230.

$$M = -\sum_{x=x}^{x=l} W(l - x),$$

$$M + \Delta M = -\sum_{x=x}^{x=l} W(l - x - \Delta x) \text{ nearly.}$$

Hence, by subtraction,

$$\Delta M = \Delta x \sum_{x=x}^{x=l} W \text{ nearly}$$

$$\therefore \quad \frac{\Delta M}{\Delta x} = \sum_{x=x}^{x=l} W;$$

and, if we pass to the limit, and observe that

$$F = \sum_{x=x}^{x=l} W,$$

we shall obtain

$$\frac{dM}{dx} = F. \qquad (2)$$

(*b*) In the case of a beam supported at the ends (Fig. 231), assume the origin at the left-hand end, and let the left-hand supporting-force be S; then, if a represent the distance from the origin to the point of application of W, we shall have the equations

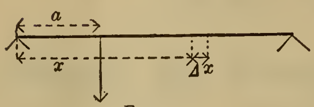


FIG. 231.

$$M = Sx - \sum_{x=0}^{x=x} W(x - a),$$

$$M + \Delta M = S(x + \Delta x) - \sum_{x=0}^{x=x} W(x - a + \Delta x) \text{ nearly.}$$

Hence, by subtraction,

$$\Delta M = S \,.\, \Delta x - \sum_{x=0}^{x=x} W\Delta x \text{ nearly}$$

$$\therefore \quad \frac{\Delta M}{\Delta x} = S - \sum_{x=0}^{x=x} W \text{ nearly};$$

and, if we pass to the limit, and observe that

$$F = S - \sum_{x=0}^{x=x} W,$$

we shall obtain

$$\frac{dM}{dx} = F, \qquad (3)$$

as before.

§ 205. **Longitudinal Shearing of Beams.** — The resistance of a beam to longitudinal shearing sometimes becomes a matter of importance, especially in timber, where the resistance to shearing along the grain is very small. We will therefore proceed to ascertain how to compute the intensity of the longitudinal shear at any point of the beam, under any given load; as this should not be allowed to exceed a certain safe limit, to be determined experimentally. Assume a section AC (Fig. 232) at a distance x from the origin, and let the bending-moment at that section be M. Let the section BD be at a distance $x + \Delta x$ from the origin, and let the bending-moment at that section be $M + \Delta M$.

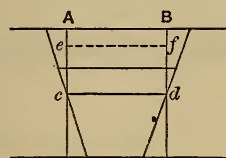


Fig. 232.

Let y_0 be the distance of the outside fibre from the neutral axis; and let $ca = y_1$ be the distance of a, the point at which the shearing-force is required, from the neutral axis.

Consider the forces acting on the portion $ABba$, and we shall have —

1°. Intensity of direct stress at $A = \frac{My_0}{I}$.

2°. Intensity of direct stress at a unit's distance from neutral axis $= \frac{M}{I}$.

3°. Intensity of direct stress at e, where $ce = y$, is $\frac{My}{I}$.

So, likewise, intensity of direct stress at f is $\frac{(M + \Delta M)y}{I}$.

Therefore, if z represent the width of the beam at the point e, we shall have —

$$\text{Total stress on face } Aa = \frac{M}{I}\int_{y_1}^{y_0} yzdy,$$

$$\text{Total stress on face } Bb = \frac{M + \Delta M}{I}\int_{y_1}^{y_0} yzdy;$$

$$\therefore \quad \text{Difference} = \frac{\Delta M}{I}\int_{y_1}^{y_0} yzdy:$$

and this is the total horizontal force tending to slide the piece $AabB$ on the face ab.

Area of face ab, if z_1 is its width, is

$$z_1 \Delta x;$$

therefore intensity of shear at a is approximately

$$\frac{\dfrac{\Delta M}{I}\displaystyle\int_{y_1}^{y_0} yzdy}{z_1 \Delta x},$$

or exactly (by passing to the limit)

$$\frac{\left(\dfrac{dM}{dx}\right)}{z_1 I}\int_{y_1}^{y_0} yzdy.$$

And, observing that $F = \dfrac{dM}{dx}$, this intensity reduces to

$$\frac{F}{z_1 I}\int_{y_1}^{y_0} yzdy. \qquad (1)$$

We may reduce this expression to another form by observing, that, if y_2 represent the distance from c to the centre of gravity of area Aa, and A represent its area, we have

$$\int_{y_1}^{y_0} yzdy = y_2 A;$$

therefore intensity of shear (at distance y_1 from neutral axis) at point $a =$

$$\frac{F}{z_1 I}(y_2 A). \qquad (2)$$

This may be expressed as follows :—

Divide the shearing-force at the section of the beam under consideration, by the product of the moment of inertia of the section and its width at the point where the intensity of the shearing-force is desired, and multiply the quotient by the statical moment of the portion of the cross-section between the point in question and the outer fibre; this moment being taken about the neutral axis. The result is the required intensity of shear.

The last factor is evidently greatest at the neutral axis; hence the intensity of the shearing-force is greatest at the neutral axis.

LONGITUDINAL SHEARING OF RECTANGULAR BEAMS.

For rectangular beams, we have

$$I = \frac{bh^3}{12}, \qquad z_1 = b.$$

Hence formula (2) becomes

$$\frac{12F}{b^2h^3}(y_2A). \tag{3}$$

For the intensity at the neutral axis, we shall have, therefore,

$$\frac{12F}{b^2h^3}\left(\frac{h}{4}\frac{bh}{2}\right) = \frac{3}{2}\frac{F}{bh}, \tag{4}$$

since for the neutral axis we have

$$y_2 = \frac{h}{4} \quad \text{and} \quad A = \frac{bh}{2}.$$

EXAMPLES.

1. What is the intensity of the tendency to shear at the neutral axis of a rectangular 4-inch by 12-inch beam, of 14 feet span, loaded at the middle with 5000 lbs.

2. What is that of the same beam at the neutral axis of the cross-section at the support, when the beam has a uniformly distributed load of 12000 lbs.

3. What is that of a 9-inch by 14-inch beam, 20 feet span, loaded with 15000 lbs. at the middle.

§ 206. **Strength of Hooks.** — The following is the method to be pursued in determining the stresses in a hook due to a given load; or, *vice versa*, the proper dimensions to use for a given load.

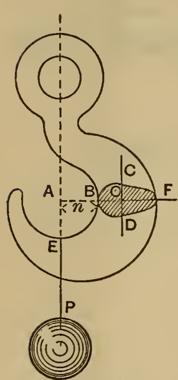


Fig. 233.

Suppose (Fig. 233) a load hung at E; the load being P, and the distance from EA to the inside of the most strained section being

$$AB = n.$$

Let O be the centre of gravity of this section, and let $OB = y$. Conceive two equal and opposite forces, each equal and parallel to P, acting at O.

Let A = area of section, and let I = its moment of inertia about CD ($BCDF$ represents the section revolved into the plane of the paper); then —

1°. The downward force at O causes a uniformly distributed stress over the section, whose intensity is

$$p_1 = \frac{P}{A}.$$

2°. The downward force at E and the upward force at O constitute a couple, whose moment is

$$P(n + y);$$

and this is resisted, just as the bending-moment in a beam, by a uniformly varying stress, producing tension on the left, and compression on the right, of CD.

If we call p_2 the greatest intensity of the tension due to this bending-moment, viz., that at B, we have

$$p_2 = \frac{P(n+y)y}{I};$$

therefore the actual greatest intensity of the tension is

$$p = p_1 + p_2 = \frac{P}{A} + \frac{P(n+y)y}{I}, \qquad (1)$$

and this must be kept within the working-strength if the load is to be a safe one.

EXAMPLES.

1. Suppose the hook to be made of 1-inch diameter iron, and $n = 1$ inch: what is the working-load, modulus of rupture = 50000 lbs. per square inch, factor of safety 6.

2. A tension-rod hanging vertically bears a load at a horizontal distance of three inches from its centre of gravity: find the necessary diameter, supposing it to be of wrought-iron.

§ 207. **Short Struts.**—The case of a short strut, with the load applied at some point other than the centre of gravity of the section, is similar to that of the hook. Thus, let O' (Fig. 234) be the centre of gravity of the lower section, and let $A'O' = x_0$.

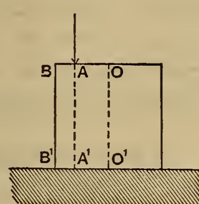


FIG. 234.

Conceive two equal and opposite forces at O', each equal and parallel to P, and we have—

1°. Downward force along line OO' causes uniform stress of intensity,

$$p_1 = \frac{P}{A}.$$

2°. The other two form a couple, whose moment is

$$Px_0;$$

therefore the greatest intensity of the compression due to this will be that at B, or

$$p_2 = \frac{(Px_0)a}{I},$$

where $a = O'B'$. Hence total greatest intensity is

$$p = \frac{P}{A} + \frac{Px_0a}{I};$$

or, if we write

$$I = A\rho^2,$$

where ρ = radius of gyration of lower section about the axis through O' perpendicular to the plane of the paper, we have

$$p = \frac{P}{A}\left(1 + \frac{x_0a}{\rho^2}\right);$$

and this should be kept within the limits of the working-strength of the material.

EXAMPLES.

1. Given a cylindrical column of 8 inches diameter, and let $x_0 = 2$ inches: find greatest stress per square inch under a load of 100000 lbs.

2. Given $P = 200000$ lbs., $x_0 = 2$ inches: find diameter of a yellow-pine strut suitable to bear the load, with factor of safety 4. Compressive strength of yellow pine = 4400 lbs. per square inch.

§ 208. **Strength of Columns.** — The formulæ in common use for the strength of columns are of three kinds; viz, —

1°. Euler's formulæ, where it is assumed, that, for any given material, there is a certain definite ratio of length to diameter, below which a column will give way by direct crushing, while one whose ratio of length to diameter is greater will give way wholly by transverse bending.

2°. Hodgkinson's empirical formulæ, based upon his experiments upon small columns of a variety of ratios of length to diameter.

3°. Gordon's formulæ, where it is assumed that all columns give way by a combination of crushing and bending.

It is very much to be regretted that none of these sets of formulæ are borne out by experiment upon the large scale, and that thus far we have no formulæ for columns that are borne out generally. Euler's are evidently faulty in the fundamental assumption; Hodgkinson's experiments were made on small columns, and do not agree well with those on large ones; Gordon's, or, as they are otherwise called, Rankine's, are probably correct in their fundamental assumption, but there is a serious lapse in the reasoning by which they are deduced.

The formulæ most frequently used in American practice are those of Gordon. Hence we will take those first.

§ 209. **Gordon's Formulæ for Columns.** (*a*) *Column fixed in Direction at Both Ends.* — Let CAD be the central axis of the column, P the breaking-load, and v the greatest deflection, AB. Conceive at A two equal and opposite forces, each equal to P; then —

1°. The downward force at A causes a uniformly distributed stress over the section, of intensity,

$$p_1 = \frac{P}{A}.$$

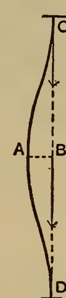


FIG. 235.

2°. The downward force at C and the upward force at A constitute a couple, whose moment is

$$M = Pv;$$

and this is resisted, just as the bending-moment in a beam, by a uniformly varying stress, producing compression on the right, and tension on the left, of A.

If we call p_2 the greatest intensity of the compression due to this bending, we have

$$p_2 = \frac{(Pv)y}{I},$$

where y = distance from the neutral axis to the most strained fibre of the section at A. Then will the greatest intensity of the stress of compression at section A be

$$p = p_1 + p_2 = \frac{P}{A} + \frac{Pvy}{I};$$

and, since P is the breaking-load, p must be equal to the breaking-strength for compression per square inch $= f$.

Hence

$$f = \frac{P}{A}\left(1 + \frac{vy}{\rho^2}\right), \qquad (1)$$

where ρ = smallest radius of gyration of section at A.

Thus far the reasoning appears sound; but in the next step it is assumed, that because, in a loaded beam, the greatest deflection under the breaking-load varies as the square of the length, and inversely as the distance from the neutral axis to the most strained fibre, therefore in this case it is assumed that we must have also

$$v \propto \frac{l^2}{y},$$

or

$$v = \frac{1}{c}\frac{l^2}{y},$$

where c is a constant to be determined by experiment. Hence

$$vy = \frac{1}{c}l^2;$$

therefore, substituting this in (1),

$$\therefore \quad f = \frac{P}{A}\left\{1 + \frac{l^2}{c\rho^2}\right\}$$

$$\therefore \quad P = \frac{fA}{1 + \dfrac{l^2}{c\rho^2}}, \qquad (2)$$

which is the required formula for a column fixed in direction at both ends.

(*b*) *Column hinged at the Ends.* — It is assumed in the previous case that the points of inflection are halfway between the middle and the ends, and hence that, by taking the middle half, we have the case of bending of a column hinged at the ends (Fig. 236). Hence, to obtain the formula suitable for this case, substitute, in (2), $2l$ for l, and we obtain

FIG. 236.

$$P = \frac{fA}{1 + \frac{4l^2}{cp^2}}. \qquad (3)$$

(*c*) *Column fixed at One End and hinged at the Other* (Fig. 237). — In this case we should, in accordance with these assumptions, take $\frac{3}{4}$ of the column fixed in direction at both ends; hence, to obtain the formula for this case, substitute, in (2), $\frac{4}{3}l$ for l, and we thus obtain

FIG. 237.

$$P = \frac{fA}{1 + \frac{16l^2}{9cp^2}}. \qquad (4)$$

Rankine gives, for values of f and c, the following, based upon Hodgkinson's experiments: —

	f (lbs. per sq. in.).	c.
Wrought-iron	36000	36000
Cast-iron	80000	6400
Dry timber	7200	3000

§ 210. **Euler's Rules for the Strength of Columns.** — The following are the rules for determining the strength of a column of uniform cross-section, according to Euler: —

(*a*) *Column fixed in Direction at One End only, which bends, as shown in the Figure.*

1°. Calculate the breaking-load on the assumption that the column will give way by direct compression. This will be

$$P_1 = fA, \tag{1}$$

where $f =$ crushing-strength per square inch, and $A =$ area of cross-section in square inches.

2°. Calculate the load that would break the column if it were to give way by bending, by means of the following formula: —

$$P_2 = \left(\frac{\pi}{2l}\right)^2 EI, \tag{2}$$

where $E =$ modulus of elasticity of the material, $I =$ smallest moment of inertia of the cross-section, and $l =$ length of column.

Then will the actual breaking-strength, according to Euler, be the smaller of these two results.

To deduce the latter formula, assume the origin at the hinged end, and take x vertical and y horizontal.

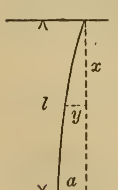


FIG. 238.

Let $\rho =$ radius of curvature at point (x, y), and let $M =$ bending-moment at the same point.

Then we shall have, just as was shown in the case of the deflection of beams,

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{Py}{EI}. \tag{3}$$

But as was there shown,

$$\frac{1}{\rho} = -\frac{d^2y}{dx^2},$$

$$\therefore \quad -\frac{d^2y}{dx^2} = \frac{P}{EI}y,$$

$$\therefore \quad -\int \frac{dy}{dx} \cdot \frac{d^2y}{d^2x}dx = \frac{P}{EI}\int y\frac{dy}{dx}dx$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = -\frac{P}{EI}y^2 + c;$$

and, since for

$$y = a, \qquad \frac{dy}{dx} = 0, \qquad \therefore \quad c = \frac{P}{EI}a^2$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{P}{EI}(a^2 - y^2) \tag{4}$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{P}{EI}}\sqrt{a^2 - y^2}$$

$$\therefore \quad \frac{dy}{\sqrt{a^2 - y^2}} = \sqrt{\frac{P}{EI}}dx$$

$$\therefore \quad \sin^{-1}\frac{y}{a} = \sqrt{\frac{P}{EI}}x + c.$$

And since, when

$$x = 0, \qquad y = 0, \qquad \therefore \quad c = 0,$$

and we have

$$\sin^{-1}\left(\frac{y}{a}\right) = \sqrt{\frac{P}{EI}}x. \tag{5}$$

When $y = a$, we know that $x = l$; hence, substituting in (5), we have

$$\frac{\pi}{2} = \sqrt{\frac{P}{EI}}l,$$

or

$$P = \left(\frac{\pi}{2l}\right)^2 EI. \tag{6}$$

(*b*) *Column hinged at Both Ends* (Fig. 236).

1°. Calculate the crushing-load, as before, from the formula

$$P_1 = fA.$$

2°. Calculate the load that would break it, if it were to give way wholly by transverse bending, from the formula

$$P_2 = \left(\frac{\pi}{l}\right)^2 EI, \qquad (7)$$

this being derived from (2) or (6) by substituting $\frac{l}{2}$ for l; the reasoning being the same for this substitution as was adopted with Gordon's formula.

(*c*) *Column fixed in Direction at Both Ends* (Fig. 235). — We have for the crushing-load the same formula as before; viz., —

$$P_1 = fA;$$

and for the bending we have

$$P_2 = \left(\frac{2\pi}{l}\right)^2 EI, \qquad (8)$$

this being obtained from (2) or (6) by substituting $\frac{l}{4}$ for l.

(*d*) These rules may be summed up as follows: —

1°. Calculate the crushing-load by the formula

$$P_1 = fA.$$

2°. Calculate the load that would break the column by bending, from the following formulæ: —

(*a*) $$P_2 = \left(\frac{\pi}{2l}\right)^2 EI$$

if fixed in direction at one end only;

$$(\beta) \qquad P_2 = \left(\frac{\pi}{l}\right)^2 EI$$

if hinged or rounded at both ends;

$$(\gamma) \qquad P_2 = \left(\frac{2\pi}{l}\right)^2 EI$$

if fixed in direction at both ends.

Then will the actual breaking-strength be the least of the two results.

(*e*) In order to ascertain the length where incipient flexure occurs, according to this theory we should place the two results equal to each other, and from the resulting equation determine l. We should thus obtain, for the three cases respectively, —

$$(\alpha) \qquad l = \frac{\pi}{2}\sqrt{\frac{EI}{fA}}, \qquad (9)$$

$$(\beta) \qquad l = \pi\sqrt{\frac{EI}{fA}}, \qquad (10)$$

$$(\gamma) \qquad l = 2\pi\sqrt{\frac{EI}{fA}}. \qquad (11)$$

Hence all columns whose length is less than that given in these formulæ will, according to Euler, give way by direct crushing; and those of greater length, by bending only.

§ 211. **Hodgkinson's Rules for the Strength of Columns.** — Eaton Hodgkinson made a very extensive series of tests of columns, especially of cast-iron, and deduced from these tests certain empirical formulæ. These tests form, even at the present time, the basis of the most used formulæ for the strength of columns. The strength of pillars of the ordinary sizes used in practice has been computed by means of Hodgkinson's for-

mulæ, and tabulated by Mr. James B. Francis: and we find in his book the following rules for the strength of solid cylindrical pillars of cast-iron, with the ends flat; i.e., "finished in planes perpendicular to the axis, the weight being uniformly distributed on these planes."

For pillars whose length exceeds thirty times their diameter, he gives the formula,

$$W = 99318 \frac{D^{3.55}}{l^{1.7}}, \tag{1}$$

where $D =$ diameter of column in inches, $l =$ length in feet, $W =$ breaking-weight in pounds.

If, on the other hand, the length does not exceed thirty times the diameter, he gives, for the breaking-weight, the following formula: —

$$W' = \frac{Wc}{W + \frac{3}{4}c}, \tag{2}$$

where $W =$ breaking-weight that would be derived from the preceding formula, $W' =$ actual breaking-weight, $c =$ weight which would crush the pillar, or

$$c = 109801\left(\frac{\pi D^2}{4}\right). \tag{3}$$

For hollow cast-iron pillars, if $D =$ external diameter in inches, $d =$ internal diameter in inches, we should have, in place of (1),

$$W = 99318 \frac{D^{3.55} - d^{3.55}}{l^{1.7}}, \tag{4}$$

and in place of (3),

$$c = 109801 \frac{\pi(D^2 - d^2)}{4}. \tag{5}$$

For very long wrought-iron pillars, Hodgkinson found the strength to be 1.745 times that of a cast-iron pillar of the same dimensions; but, for very short pillars, he found the strength of

the wrought-iron pillar very much less than that of the cast-iron one of the same dimensions. With a length of 30 diameters and flat ends, the wrought-iron exceeded the cast-iron by about ten per cent.

§ 212. **Strength of Shafting.** — The usual criterion for the strength of shafting is, that it shall be sufficiently strong to resist the twisting to which it is exposed in the transmission of power.

Proceeding in this way, let EF (Fig. 239) be a shaft, AB the driving, and CD the following, pulley. Then, if two cross-sections be taken between these two pulleys, the portion of the shaft between these two cross-sections will, during the transmission of power, be in a twisted condition; and if, when the shaft is at rest, a pair of vertical parallel diameters be drawn in these sections, they will, after it is set in motion, no longer be parallel, but will be inclined to each other at an angle depending upon the power applied. Let GH be a section at a distance x from O, and let KI be another section at a distance $x + dx$ from O. Then, if di represent the angle at which the originally parallel diameters of these sections diverge from each other, and if $r =$ the radius of the shaft, we shall have, for the length of an arc passed over by a point on the outside,

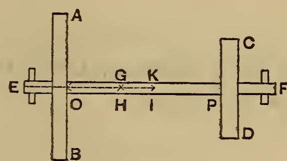


FIG. 239.

$$rdi;$$

and for the length of an arc that would be passed over if the sections were a unit's distance apart, instead of dx apart,

$$\frac{rdi}{dx} = r\frac{di}{dx}.$$

This is called the *strain* of the outer fibres of the shaft, as it is the distortion per unit of length of the shaft.

In all cases where the shaft is homogeneous and symmetrical, if i is the angle of divergence of two originally parallel diameters whose distance apart is x, we shall have the strain,

$$v = r\frac{di}{dx} = r\frac{i}{x}.$$

This also is the tangent of the angle of twist.

A fibre whose distance from the axis of the shaft is unity, will have, for its strain,

$$\frac{di}{dx} = \frac{i}{x}.$$

A fibre whose distance from the axis of the shaft is ρ, will have, for its strain,

$$v = \rho\frac{di}{dx} = \rho\frac{i}{x}.$$

Fixing, now, our attention upon one cross-section, GH, we have that the strain of a fibre at a distance ρ from the axis (ρ varying, and being the radius of any point whatever) is

$$\rho\left(\frac{i}{x}\right),$$

where $\frac{i}{x}$ is a constant for all points of this cross-section.

Hence, assuming Hooke's law, "*Ut tensio sic vis*," we shall have, if C represent the shearing modulus of elasticity, that the *stress* of a fibre whose distance from the axis is ρ, is

$$p = Cv = C\rho\left(\frac{di}{dx}\right) = C\rho\left(\frac{i}{x}\right),$$

which quantity is proportional to ρ, or varies uniformly from the centre of the shaft.

The intensity at a unit's distance from the axis is

$$C\left(\frac{i}{x}\right);$$

and if we represent this by a, we shall have for that at a distance ρ from the axis,

$$p = a\rho.$$

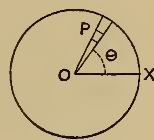


FIG. 240.

Hence we shall have (Fig, 240), that, on a small area,

$$dA = d\rho(\rho d\theta) = \rho d\rho d\theta,$$

the stress will be

$$pdA = a\rho dA = a\rho^2 d\rho d\theta.$$

The moment of this stress about the axis of the shaft is

$$\rho p dA = a\rho^2 dA = a\rho^3 d\rho d\theta,$$

and the entire moment of the stress at a cross-section is

$$a\int\rho^2 dA = a\int\int\rho^3 d\rho d\theta = aI,$$

where $I = \int\rho^2 dA$ is the moment of inertia of the section about the axis of the shaft.

This moment of the stress is evidently caused by, and hence must be balanced by, the twisting-moment due to the pull of the belt. Hence, if M represent the greatest allowable twisting-moment, and a the greatest allowable intensity of the stress at a unit's distance from the axis, we shall have

$$M = aI = \frac{p}{\rho} I.$$

If f is the safe working shearing-strength of the material per square inch, we shall have f as the greatest safe stress per square inch at the outside fibre, and hence

$$M = \frac{f}{r} I$$

will be the greatest allowable twisting-moment.

For a circle, radius r,

$$I = \frac{\pi r^4}{2} \qquad \therefore \quad M = f\frac{\pi r^3}{2} = f\frac{\pi d^3}{16}.$$

For a hollow circle, outside radius r_1, inside radius r_2,

$$I = \frac{\pi(r_1^4 - r_2^4)}{2} \qquad \therefore \quad M = f\frac{\pi}{2r_1}(r_1^4 - r_2^4).$$

Moreover, if the dimensions of a shaft are given, and the actual twisting-moment to which it is subjected, the stress at a fibre at a distance ρ from the axis will be found by means of the formula

$$p = \frac{M\rho}{I}.$$

The more usual data are the horse-power transmitted and the speed, rather than the twisting-moment.

If we let $P =$ force applied, and $R =$ its leverage, as, for instance, when $P =$ difference of tensions of belt, and $R =$ radius of pulley, we have

$$M = P.R;$$

and if $HP =$ number of horses-power transmitted, and $N =$ number of turns per minute, then

$$HP = \frac{P(2\pi RN)}{33000}$$

$$\therefore \quad PR = \frac{33000 HP}{2\pi N} = M.$$

EXAMPLE.

Given working-strength for shearing of wrought-iron as 10000 lbs. per square inch; find proper diameter of shaft to transmit 20-horse power, making 100 turns per minute.

§ 213. **Angle of Torsion.** — From the formula, § 212, $p = \frac{M\rho}{I}$, combined with

$$p = a\rho = C\rho\frac{i}{x},$$

we have

$$C\rho\frac{i}{x} = \frac{M\rho}{I}$$

$$\therefore \quad i = \frac{Mx}{CI},$$

which gives the circular measure of the angle of divergence of two originally parallel diameters whose distance apart is x; the twisting-moment being M, and the modulus of shearing elasticity of the material, C.

EXAMPLES.

1. Find the angle of twist of the shaft given in example 1, § 212, when the length is 10 feet, and $C = 8500000$.

2. What must be the diameter of a shaft to carry 80 horses-power, with a speed of 300 revolutions per minute, and factor of safety 6, breaking shearing-strength of the iron per square inch being 50000 lbs.

§ 214. **Transverse Deflection of Shafts.** — In determining the proper diameter of shaft to be used in any given case, we ought not merely to consider the resistance to twisting, but also the deflection under the transverse load of the belt-pulls, weights of pulleys, etc. This deflection should not be allowed to exceed $\frac{1}{100}$ of an inch per foot of length. Hence the deflection should be determined in each case.

The formulæ for computing this deflection will not be given here, as the methods to be pursued are just the same as in the case of a beam, and can be obtained from the discussions on that subject.

§ 215. **Combined Twisting and Bending.** — The most common case of a shaft is for it to be subjected to combined twisting and bending. The discussion of this case involves the theory of elasticity, and will not be treated here; but the formulæ commonly given will be stated, without attempt to prove them until a later period. These formulæ are as follows: —

Let M_1 = greatest bending-moment,

M_2 = greatest twisting-moment,

r = external radius of shaft,

I = moment of inertia of section about a diameter, for a solid shaft $I = \dfrac{\pi r^4}{4}$,

f = working-strength of the material = greatest allowable stress at outside fibre;

then

1°. According to Grashof,

$$f = \frac{r}{I}\left\{\tfrac{3}{8}M_1 + \tfrac{5}{8}\sqrt{M_1^2 + M_2^2}\right\}. \qquad (1)$$

2°. According to Rankine,

$$f = \frac{r}{I}\left\{M_1 + \sqrt{M_1^2 + M_2^2}\right\}. \qquad (2)$$

CHAPTER VII.

STRENGTH OF MATERIALS AS DETERMINED BY EXPERIMENT.

§ 216. **General Remarks.** — Whatever computations are made to determine the form and dimensions of pieces that are to resist stress and strain, must, if they are to have any practical value, be based upon experiments made upon the materials themselves.

The most valuable experiments in any given case, whenever the results of such experiments are available, are those made upon pieces of the same quality, size, and form as those to which the results are to be applied, and under conditions entirely similar to those to which the pieces are subjected in actual practice.

It is very seldom that the results of such experiments are available; and hence we must, in general, make use of such tests as have been or can be made, and from them determine the strength of the pieces in actual use by computation, making good use of our judgment.

As time goes on, and experimental science advances, a greater number of the conditions that exist in actual practice are introduced into the experiments; and hence the reliability of the experimental results, and their applicability to practical cases, are increased. Nevertheless, it is necessary to use the utmost caution when applying the results of the experiments to cases where the conditions are different from those under which the experiments were made. An attempt will be made in this

chapter to give an account of the most important results of experiment on the strength of materials, and to explain the modes of using the results that are now employed. As to the way in which the experiments have generally been carried on, we may observe: —

1°. In by far the greater number of cases, the test pieces have been very much smaller than the pieces to be used. Indeed, it is only of late years that the importance of testing full-size pieces has been recognized; and hence most of the experiments upon such pieces are of very recent date.

2°. In the greater part of the experiments that have been made, the phenomena observed have been those that occur during the application of the load for a short time only; very little having been done by way of determining the behavior of the pieces under a long-continued action of the load, or under repeated applications of the load, such as occur in practice.

3°. Very few experiments have been made on the effect of applying two kinds of stress simultaneously, as tension and bending, or twisting and bending, or on applying stresses of opposite kinds, as tension and compression, successively.

4°. The tests thus far made have had for their object more frequently to determine the breaking-strength of the piece. Next to this, the subject most frequently experimented upon has been the limit of elasticity; and less has been done by way of determining the modulus of elasticity, and other matters.

5°. The fact that the breaking-strength alone is not a sufficient criterion by which to determine the suitability of a material for use in construction, has been recognized only by the later experimenters.

6°. In order to understand what is meant by "the limit of elasticity," we must observe, that, if a small load be applied to the piece under test, and then removed, the deformation or distortion caused by the application of the load apparently vanishes, and the piece resumes its original form and dimensions

on the removal of the load; in other words, no permanent set takes place. When the load, however, is increased beyond a certain point, the piece under test does not return entirely to its original dimensions on the removal of the load, but retains a certain permanent set.

The load upon the application of which permanent set *apparently* begins, is called the *limit of elasticity*, and is found by experiment to be at about one-third the breaking-weight in iron, and from one-third upwards in steel, sometimes reaching nearly three-fourths.

Experiments show, however, that even a very small load will produce a permanent set, and that the apparent return of the piece to its original dimensions upon the removal of the load is only due to the want of delicacy in the measuring-instruments that have been used in the tests. A better definition of the limit of elasticity would therefore be, that load upon the application of which the permanent set begins to be noticeable, with such rough means of measuring as a pair of dividers.

7°. It has often been assumed, that, if the load applied to the piece in practice exceeded the elastic limit, the piece would be permanently injured in its properties for resisting stress, and that the deformation and injury would continue increasing, until eventually fracture would occur. It has been proved experimentally, however, that it is sometimes advantageous to apply once a load to a piece somewhat greater than the elastic limit, and that by this means the elastic limit is increased. This process of using up a part of the elasticity of the piece cannot continue indefinitely, and the data to show how far it can be advantageously carried are but few.

8°. The determination of the modulus of elasticity, which has been defined (§ 167) as the ratio of the stress to the strain, is a very important matter; as it gives us the means of computing the deformation under any given load, and thus determining the safe load by prescribing the greatest deformation to

be allowed, rather than by prescribing that the safe load shall be a certain fraction of the breaking-load.

9°. Other important matters which guide us in judging of the suitability of a piece for the purpose to which it is to be applied, are, the appearance of its fracture, its density, its homogeneity, its composition, and the care taken in its manufacture, or the circumstances of its growth and seasoning if it is wood, also its brittleness, hardness, malleability, ductility, the amount of warning it gives before giving way, etc.

10°. From such data as we have furnished to us by experiment, we decide, to the best of our ability, as to the suitability of the piece for the use for which it is intended, and also as to the amount it will safely bear, when we know its dimensions, or the proper dimensions to bear safely the required stress.

11°. Tests have been made on tension, compression, shearing, transverse and torsional strength, of the different materials used in construction, especially cast-iron, wrought-iron, steel, and wood, also copper and other metals; but the tests on tensile strength are by far the most numerous in the case of iron.

§ 217. **Cast-Iron.** — Cast-iron is a combination of iron with 2 per cent to 6 per cent of carbon. The large amount of carbon which it contains is its distinguishing feature, and determines its behavior in most respects.

Pig-Iron is the result of the first smelting, being obtained directly from the smelting-furnace. The ore and fuel are put into the furnace, together with a flux, which is of a calcareous nature when the ore is argillaceous, or which contains clay when the ore is calcareous. The mass is brought to a high heat, a strong blast of air being introduced. The mass is thus melted; the fluid iron settling to the bottom, while slag, which is the result of the combination of the flux with the impurities of the ore, rises to the top. The iron is drawn off in the liquid state, and run into moulds, the result being pig-iron.

The result of this first melting is very rarely used for any

casting; but the pig-iron is usually re-melted in a cupola furnace before being used, the result of this re-melting being the ordinary cast-iron of commerce.

The pig-iron is divided into classes, according to the purpose for which it is intended, and the amount of carbon it contains.

Those pigs that have a considerable amount of carbon in mechanical mixture, and show a gray color on being fractured, are used by the founder to melt over, and make cast-iron. This is called "foundry iron," and is divided into foundry iron Nos. 1, 2, and 3, from which are subsequently made gray cast-iron Nos. 1, 2, and 3. When there is less carbon, it is sometimes called "foundry No. 4," etc.; but it is only used to make wrought-iron of an inferior quality.

Those pigs which are to be used in making wrought-iron and steel, and which have been fused at a low heat and with little fuel, are called "forge-iron."

Cast-iron is of two kinds, *white cast-iron* and *gray cast-iron.* The first is a chemical compound of iron with 2 per cent to 6 per cent of carbon, almost all of the carbon being chemically combined with the iron. The second, or gray cast-iron, contains part of the carbon in chemical combination, and the remainder in the state of graphite mechanically mixed with the iron.

Gray Cast-Iron is divided into three classes, known respectively as Nos. 1, 2, and 3.

No. 1 contains the largest amount of carbon in mechanical mixture, the effect of which is to render it soft and fusible, though not as strong as Nos. 2 and 3. It is, therefore, very suitable for making castings where precision in form is a desideratum, as its fusibility causes it to fill the mould well. It is not as suitable, however, where strength is required.

No. 2 is that which is most suitable for use in construction, as it is stronger than No. 1, and not so soft.

No. 3, on the other hand, contains the smallest amount of carbon in the graphitic form, and is, hence, harder and more

brittle. It is suited, therefore, only for the massive and heavy parts of machinery.

White Cast-Iron contains hardly any free carbon. It is of two kinds, granular and crystalline. The crystalline variety is of no use in construction: it is hard and very brittle. The granular variety is also unsuitable for use in construction, but forms the hard skin on the surface of a piece that has been chilled.

As to the adaptability of cast-iron to construction, it presents certain advantages and certain disadvantages. It is the cheapest form of iron. It is easy to give it any desired form. It resists oxidation better than either wrought-iron or steel. It has a very high compressive strength. On the other hand, its tensile strength is comparatively small, averaging, in common varieties, 15000 pounds per square inch, or thereabouts. It cannot be riveted or welded when broken. It is brittle, breaking off without giving much warning, and stretching but little before giving way. It is liable to hidden and small surface defects and air-bubbles, which render its strength somewhat doubtful. It is also liable to absorb impurities from the fuel or flux in the furnace, the most injurious being sulphur and phosphorus; the effect of the former being to produce red-shortness, or brittleness when hot, and that of the latter to produce cold shortness, or brittleness when cold.

Another very serious drawback in the use of cast-iron in construction is its liability to initial strains from the inequality in cooling. Thus, if one part of the casting is very thin and another very thick, the thin part cools first, and, in cooling, contracts; and the thick part, cooling afterwards, causes stresses in the thin part, which may be sufficient to break it, or, if not, there may be so much stress established, that but little more will break it. Thus, the change of temperature from summer to winter is sometimes sufficient to break the arms of a pulley from off the rim. Its quality depends largely upon its composition and its density.

The fracture should be of a bluish-gray color, and close-grained texture, with considerable metallic lustre if the iron is of good quality. If the fracture is mottled, with patches of darker or lighter iron, or crystalline spots, it is an indication of unsoundness, especially so if there are air-bubbles.

It is not well adapted to bear tension, on account of its low tensile strength, and also on account of its brittleness and treacherousness.

In former times it was extensively used for iron beams to bear a transverse load, but has now been almost entirely superseded by wrought-iron in this regard.

It is still used for columns and posts in buildings, on account of its high compressive strength ; but, in cases where the length of the column is so great as to cause it to give way by bending, as in bridge columns, its use has been almost wholly abandoned, and wrought-iron and steel are taking its place. In the case of bridge columns, it is also necessary to use a metal which can easily be riveted, and to which the other members can be readily attached; and wrought iron is more suitable than cast for this purpose: also another reason is, that wrought-iron and steel are much better suited to resist shocks than cast-iron.

In machinery, it is used in all those parts where weight, mass, or form is of more importance than strength, as in the frames and bed-plates of machines, also for hangers, pulleys, and gear-wheels.

Cast-iron is also used for water-mains where great pressure is to be resisted, also in hydraulic presses, and in heavy ordnance. For shafting, wrought-iron has taken its place, and so also for the shells of steam-boilers, with the exception of some sectional boilers, partly on account of its low tensile strength and general untrustworthiness, but especially because of its liability to give way without warning when subjected to the sudden expansions and contractions which it would have to undergo if used in a steam-boiler.

Malleable Cast-Iron. — When a casting is to be made in a rather intricate form, it is frequently the custom to malleableize the cast-iron. This is done by heating it to a bright-red heat in an annealing oven, in powdered hematite ore, with a suitable flux. By this process a part of the carbon is removed, and the result is — provided the casting is not large — a product that can be hammered into any desired shape when cold, but is very brittle when hot. It is used in cases where toughness is required, together with the possession of an intricate form: thus gun-locks, pokers, tongs, etc., are sometimes made by this means, and sometimes also screw propellers.

§ 218. **Tensile and Compressive Strength of Cast-Iron.** — A list of the principal experimenters on the strength and elasticity of cast-iron will be given, and references to the accounts of their work, which the student who wishes to pursue the subject further will do well to consult: —

1°. Eaton Hodgkinson: (*a*) Report of the Commissioners on the Application of Iron to Railway Structures.
(*b*) London Philosophical Transactions. 1840.
(*c*) Experimental Researches on the Strength and other Properties of Cast-Iron. 1846.

2°. W. H. Barlow: Barlow's Strength of Materials.

3°. Sir William Fairbairn: On the Application of Cast and Wrought Iron to Building Purposes.

4°. Major Wade (U.S.A.): Report of the Ordnance Department on the Experiments on Metals for Cannon. 1856.

5°. Capt. T. J. Rodman: Experiments on Metals for Cannon.

6°. Col. Rosset: Resistenza dei Principali Metalli da Bocchi di Fuoco.

7°. John Anderson: Strength of Materials.

Taking up first the experiments of Hodgkinson and the other members of the commission, we find that they consist of a series of tests made to compare the strength of iron from different parts of the kingdom. These tests show the average strength of English cast-iron at that time.

The following table gives the tensile and also the crushing strength per square inch of the different kinds tested : —

Description of the Iron.	Tensile Strength per Square Inch, in lbs.	Height of Specimen, in inches.	Crushing-Strength per Square Inch, in lbs.
Lowmoor, No. 1	12694	$\frac{3}{4}$ $1\frac{1}{2}$	64534 56445
Lowmoor, No. 2	15458	$\frac{3}{4}$ $1\frac{1}{2}$	99525 92332
Clyde, No. 1	16125	$\frac{3}{4}$ $1\frac{1}{2}$	92896 88741
Clyde, No. 2	17807	$\frac{3}{4}$ $1\frac{1}{2}$	109992 102030
Clyde, No. 3	23468	$\frac{3}{4}$ $1\frac{1}{2}$	107197 104881
Blaenavon, No. 1	13938	$\frac{3}{4}$ $1\frac{1}{2}$	90860 80561
Blaenavon, No. 2 (first sample) .	16724	$\frac{3}{4}$ $1\frac{1}{2}$	117605 102408
Blaenavon, No. 2 (second sample),	14291	$\frac{3}{4}$ $1\frac{1}{2}$	68559 68532
Calder, No. 1	13735	$\frac{3}{4}$ $1\frac{1}{2}$	72193 75983
Coltness, No. 3	15278	$\frac{3}{4}$ $1\frac{1}{2}$	100180 101831
Brymbo, No. 1	14426	$\frac{3}{4}$ $1\frac{1}{2}$	74815 75678
Brymbo, No. 3	15508	$\frac{3}{4}$ $1\frac{1}{2}$	76133 76958
Bowling, No. 2	13511	$\frac{3}{4}$ $1\frac{1}{2}$	76132 73984
Ystalyfera, No. 2 (anthracite) . .	14511	$\frac{3}{4}$ $1\frac{1}{2}$	99926 95559
Yniscedwyn, No. 1 (anthracite) .	13952	$\frac{3}{4}$ $1\frac{1}{2}$	83509 78659
Yniscedwyn, No. 2 (anthracite) .	13348	$\frac{3}{4}$ $1\frac{1}{2}$	77124 75369
Morries Stirling's (second quality),	25764	$\frac{3}{4}$ $1\frac{1}{2}$	125333 119457
Morries Stirling's (third quality) .	23461	$\frac{3}{4}$ $1\frac{1}{2}$	158653 129876

The specimens used for determining the tensile strength were all cruciform in section; those for determining the crushing-strength were cylinders, with a diameter of $\frac{3}{4}$ inch. The machine used was simply a single lever, with a scale-pan at the extreme end, in which weights were placed. There was no provision for taking up the stretch of the specimen, and therefore the specimens were all short.

The average tensile strength of these specimens, omitting the Stirling metal, was 15298 lbs. per square inch; and the average compressive strength was 82296 lbs. per square inch.

It will be observed, that, omitting the Stirling iron, which is a mixture of cast and wrought iron, the tensile strength ranged from 12694 to 23468 lbs. per square inch, and the compressive strength, from 56445 to 117605 lbs. per square inch. The reader will doubtless observe, that, as a rule, the crushing-strengths obtained from the longer specimens were less than those obtained with the shorter ones; but this, it seems to the writer, is probably due to the nature of the testing-machine, and not to any bending in the specimen due to its length.

Mr. Hodgkinson next proceeded to compare the tensile strength of specimens cruciform in section, with that of specimens circular in section and of about the same area. He found but little difference; and this could readily be accounted for by the fact, that, the perimeter of the cruciform one being greater, the proportion of hard skin would be greater in the cruciform than in the circular, the effect of this being, perhaps, partially counteracted by some little initial stress, on account of unequal cooling of the different parts.

Hodgkinson also made a few experiments to determine the laws of extension of cast-iron, and for this purpose used rods 10 feet long and 1 square inch in section. The table of average results is the following: —

RESULTS OF NINE TENSILE TESTS.

Weights laid on, in lbs.	Extensions, in inches.	Sets, in inches.	Strains, in fractions of the length.	Sets, in fractions of the length.	Ratio of Weight to Extensions	Modulus of Elasticity.
1053.77	0.0090	–	0.00007	–	117086	14050320
1580.65	0.0137	0.00022	0.00011	0.0000018	115131	13815720
2107.54	0.0186	0.00055	0.00016	0.0000046	113309	13597080
3161.31	0.0287	0.00107	0.00024	0.0000089	110150	13218000
4215.08	0.0391	0.00175	0.00033	0.0000146	107803	12936360
5268.85	0.0500	0.00265	0.00042	0.0000221	105377	12645240
6322.62	0.0613	0.00372	0.00051	0.0000310	103142	12377040
7376.39	0.0734	0.00517	0.00061	0.0000431	100496	12059520
8430.16	0.0859	0.00664	0.00072	0.0000553	98139	11776680
9483.94	0.0995	0.00844	0.00083	0.0000703	95316	11437920
10537.71	0.1136	0.01062	0.00095	0.0000885	92762	11131440
11591.48	0.1283	0.01306	0.00107	0.0001088	90347	10841640
12645.25	0.1448	0.01609	0.00121	0.0001341	87329	10479480
13699.83	0.1668	0.02097	0.00139	0.0001748	82133	9855960
14793.10	0.1859	0.02410	0.00155	0.0002008	79576	9549120

RESULTS OF EIGHT COMPRESSIVE TESTS.

Weights laid on, in lbs.	Compressions, in inches.	Sets, in inches.	Strains, in fractions of the length.	Sets, in fractions of the length.	Ratio of Weight to Compressions.	Modulus of Elasticity.
2064.75	0.01875	0.00047	0.00016	0.0000039	110120	13214400
4129.49	0.03878	0.00226	0.00032	0.0000188	106485	12778200
6194.24	0.05978	0.00400	0.00050	0.0000333	103617	12434040
8258.98	0.07879	0.00645	0.00066	0.0000538	104823	12578760
10323.73	0.09944	0.00847	0.00083	0.0000706	103819	12458280
12388.48	0.12030	0.01088	0.00100	0.0000907	102980	12357600
14453.22	0.14163	0.01405	0.00118	0.0001171	102049	12245880
16517.97	0.16338	0.01712	0.00136	0.0001427	101102	12132240
18582.71	0.18505	0.02051	0.00154	0.0001709	100420	12050400
20647.46	0.20624	0.02484	0.00172	0.0002070	100114	12013680
24776.95	0.24961	0.03220	0.00208	0.0002683	99263	11911560
28906.45	0.29699	0.04300	0.00247	0.0003583	97331	11679720
33030.80	0.35341	0.06096	0.00295	0.0005080	93463	11215560

These tables show that the modulus of elasticity of cast-iron varies with the load, growing gradually smaller as the load increases.

The following table enables us to compare the modulus of elasticity for tension with that for compression under nearly the same load.

Tension.		Compression.	
Load per Square Inch.	Modulus of Elasticity.	Load per Square Inch.	Modulus of Elasticity.
2107	13597080	2064	13214400
4215	12936360	4129	12778200
6322	12377040	6194	12434000
8430	11776680	8258	12578760
10537	11131440	10323	12458280
12645	10479480	12388	12357600
14793	9549120	14453	12245880

This table shows, that, with moderate loads, the modulus of elasticity for tension of cast-iron does not differ materially from that for compression, and that the difference increases as the load becomes greater.

This fact is one of very considerable importance, inasmuch as it is one of the fundamental assumptions in the common theory of beams; and we thus see that experiment justifies our making this assumption for cast-iron whenever the load is not excessive. The justification is even greater with wrought-iron and steel.

The gradual decrease of the modulus of elasticity with the increase of load shows that Hooke's law, "*Ut tensio sic vis*" (the stress is proportional to the strain), does not hold true in

cast-iron ; nevertheless, it is more nearly true for moderate loads than for larger ones.

The experiments of Major Wade of the United-States army, most of which were made at the South-Boston Iron Foundry, are recorded in the Report of the Ordnance Department on metals for cannon, printed in 1856. The Report contains experiments in regard to the effect of keeping the iron in a state of fusion for a long time, and also on the effect of successive re-meltings upon the quality of the iron ; and he says, —

"It is found that the same iron is greatly improved in quality by retaining it in the furnace after it is melted for considerable periods of time, and that it is further improved by casting it into pigs, and again re-melting it. But it is known also that a continuation of this process will ultimately impair the tenacity of the iron, and render it wholly unfit for use. It is found also that the different kinds of iron require a different kind of treatment to produce the best effect. The breaking-instrument enables one to ascertain the effect produced by these processes in all their several stages of progress, and to decide on that which is found most suitable for making guns of the best quality."

On p. 279 of the Report is given the following experiment in this regard : —

"The first sample was a rough, crude pig of grade No. 1, Greenwood iron.

	Fusion.	Tensile Strength.
Pig .	1st	14000
Bars .	2d	20900
	3d	30229
	4th	35786
Head,	4th	33724

"The second, third, and fourth are the same iron, cast and cooled in like manner, and differ only in the number of times melted ; and they exhibit the changes effected in the strength by the repeated meltings only. The fifth sample is from the same melting as the fourth, from which it differs only in cooling, being cast in a large mass, and cooled slowly."

The compressive strength of cast-iron tested by Major Wade varied from 84500 lbs. per square inch to 175000 lbs. per square inch. The following table of results of a number of tests of metal taken from different cannon, gives the average tensile strength, specific gravity, and proportion of carbon in the lot of specimens examined: —

	Specific Gravity.	Tensile Strength, lbs., per Square Inch.	Total Carbon.	Combined Carbon.	Allotropic Carbon.
1st-class guns,	7.204	28805	0.0384	0.0178	0.0206
2d-class guns,	7.154	24767	0.0376	0.0146	0.0230
3d-class guns,	7.087	20148	0.0365	0.0082	0.0283

It will be noticed that an increase in tensile strength is generally accompanied by an increase in specific gravity.

Thus another lot of iron gave the following results: —

	Tensile Strength, lbs., per Square Inch.	Specific Gravity.
Mean .	27232	7.302
Least .	22402	7.163
Greatest,	31027	7.402

He also experimented on the difference between hot and cold blast iron, and recommends decidedly the cold blast.

With the hot blast, unless the materials — i.e., the ore, fuel, and flux — are very pure, more of the impurities are fused, and combine with the iron; whereas the iron can be made much more rapidly by its use. Indeed, very little cold-blast charcoal iron is made at the present time; i-e., iron where the blast is cold, and where charcoal is the only fuel used.

The rules of the Ordnance Department of the United States require that all cast-iron which is used for cannon shall have at least a tensile strength of 30000 lbs. per square inch.

The specimens used for testing the tensile strength of cast-iron have generally been made with shoulders: and the smallest part has had, in most cases, no length; and the specimen has had, therefore, very little opportunity to stretch.

Colonel Rosset, of the Arsenal at Turin, made a series of experiments upon the influence of the shape of the specimen upon the tensile strength. For this purpose he used specimens with shoulders; and, among other tests, he compared the strength of the same iron by using specimens the lengths of whose smallest parts were respectively 1 metre, 30 millimetres, and 0 millimetres, with the following results: —

Length of Specimen.	Tensile Strength, in lbs., per Square Inch.		
	1st Cannon.	2d Cannon.	3d Cannon.
1 metre . .	31291	25601	28019
30 millimetres .	32571	34562	30011
0 millimetres .	33993	36411	30011

It will thus be seen, that, before we can decide upon the quality of cast-iron as affected by the tensile strength, it is necessary to know the length of that part of the specimen

which has the smallest area. Colonel Rosset's tests of cast-iron were almost entirely confined to high-grade irons, suitable to use in cannons.

He deduced, for mean value of the modulus of elasticity of the specimens 1 metre in length, 20419658 lbs. per square inch: this, of course, is a modulus only adapted to these high grades, and is not applicable to common cast-iron.

Before proceeding to iron columns, the values given for the tensile and compressive strength, and modulus of elasticity, of cast-iron, in lbs., per square inch, given by Rankine and Weisbach, will be stated; which values are copied by many handbooks: —

		Tensile Strength.	Compressive Strength.	Modulus of Elasticity.
Professor Rankine .	Various qualities,	13400 to 29000	82000 to 145000	14000000 to 22900000
	Average	16500	112000	17000000
Professor Weisbach,	Average	18500	–	14220000

§ 219. **Cast-Iron Columns.** *Hodgkinson's Experiments.* — The high compressive strength of cast-iron would seem to render it a very suitable material for all cases of columns or struts.

Nevertheless, its use for the compression members of bridge and roof trusses has now been almost entirely abandoned; and wrought-iron has taken its place for these purposes, — partly because these struts are very long, and, if of cast-iron, might, on account of the bending, bring into play its tensile strength; and partly because wrought-iron and steel can be easily riveted, joined to other pieces, and repaired, whereas cast-iron cannot so easily.

Cast-iron is still in use very extensively for the columns of

buildings, as these are generally shorter, and not liable to such varying loads, as is the case with bridges.

Almost the only experiments on the strength of cast-iron columns are those made by Eaton Hodgkinson, and recorded in the London Philosophical Transactions for 1840.

These experiments were made by him at the works of Sir William Fairbairn, who, as he says, put every means for a full investigation into his hands, and expressed the wish that he should extend the inquiry to pillars of various kinds, ancient as well as modern, and leave no part of the subject in uncertainty for the want of experiments sufficiently varied and extensive.

The pillars with which the experiments were made were mostly of cast-iron, as being the material in most general use at that time for that purpose; but some were of wrought-iron, and a few of wood.

Until the time when Hodgkinson made his experiments, the prevailing theory of the strength of columns was that of Euler, which has been already explained in § 210.

According to this theory, a column of any given dimensions and material requires a certain weight to bend it, even in the slightest degree; and, with less than this weight, it would not be bent at all.

This weight producing incipient flexure is to be found, as we have already seen, by the formulæ,

$P = \left(\frac{\pi}{2l}\right)^2 EI$ for one end fixed in direction and the other rounded.

$P = \left(\frac{\pi}{l}\right)^2 EI$ for both ends rounded.

$P = \left(\frac{2\pi}{l}\right)^2 EI$ for both ends fixed in direction, where $l =$ length of column, $I =$ moment of inertia of section, $E =$ modulus of elasticity of the material.

The load P is, then, according to Euler, the breaking-load, unless the column is so short that the breaking-load by direct crushing is less than the value of P, in which case the column will fail by direct crushing. Starting from this point of view, Hodgkinson says, in his report, "My first object was to supply the deficiencies of Euler's theory of the strength of pillars if it should appear capable of being rendered practically useful, and, if not, to endeavor to adapt the experiments so as to lead to useful results." He also says, in regard to Euler's theory, "I have many times sought experimentally, with great care, for the weight producing incipient flexure according to the theory of Euler, but have hitherto been unsuccessful. So far as I can see, flexure commences with weights far below those with which pillars are usually loaded in practice. It seems to be produced by weights much smaller than are sufficient to render it capable of being measured."

"With respect to the conclusions of some writers, that flexure does not take place with less than about half the breaking-weight, this, I conceive, could only mean large and palpable flexure; and it is not improbable that the writers were in some degree deceived, from their having generally used specimens thicker, compared with their length, than have been usually employed in the present effort."

Another matter to which Hodgkinson devoted considerable attention in this investigation, was a comparison of the strength of pillars with flat and with rounded ends respectively. By flat ends is meant having the ends finished in planes perpendicular to the axis, and having the resultant of the weight act along the axis: the ends are then fixed in direction. By rounded ends Hodgkinson actually meant rounded ends; and he used such in his experiments, making them generally hemispherical in form. The results have been assumed generally to apply to pin ends, or to any irregularity of fixing the ends which does not absolutely fix them in direction.

As a result of his experiments, he states, that, in all pillars whose length is thirty times the diameter or upward, the strength of those with flat ends seems to be about three times as great as the strength of those with rounded ends; the mean ratio being 3.167.

In pillars shorter than thirty diameters, the ratio decreased when the ratio of the length to the diameter decreased.

The experiments were all made on circular or hollow circular pillars, the lengths varying from $7\frac{1}{2}$ diameters to 121 diameters; a case of the first being one $15\frac{1}{8}$ inches long and 0.51 inch diameter, and of the last, $30\frac{1}{4}$ inches long and 1.01 inch diameter; the largest diameters used being about two inches.

The empirical formulæ given by Hodgkinson for the strength of cast-iron columns more than thirty diameters long, are as follows; viz., —

FOR SOLID CYLINDRICAL COLUMNS.

(*a*) With rounded ends,

$$P = 33379 \frac{d^{3.6}}{l^{1.7}}; \tag{1}$$

(*b*) With flat ends,

$$P = 98982 \frac{d^{3.6}}{l^{1.7}}: \tag{2}$$

where $d =$ diameter in inches, $l =$ length in feet, $P =$ breaking-weight in pounds.

FOR HOLLOW CYLINDRICAL COLUMNS.

(*a*) With rounded ends,

$$P = 29120 \frac{D^{3.6} - d^{3.6}}{l^{1.7}}; \tag{3}$$

(*b*) With flat ends,

$$P = 99318 \frac{D^{3.6} - d^{3.6}}{l^{1.7}}: \qquad (4)$$

where $D =$ outside diameter, $d =$ inside diameter in inches, and $l =$ length in feet.

For columns less than thirty diameters long, Hodgkinson gives the following formula for the breaking-strength :—

$$y = \frac{bc}{b + \frac{3}{4}c}, \qquad (5)$$

where $c =$ force which would crush the pillar without bending it, $b =$ breaking-weight that would be obtained by using the formula for columns more than thirty diameters long, $y =$ actual breaking-strength.

For determining the value of c, we have as the crushing-strength of cast-iron, from some experiments of Hodgkinson made at the time on cast-iron, the value 109801 lbs. per square inch ; so that we should have,

For solid circular columns,

$$c = 109801 \frac{\pi d^2}{4}; \qquad (6)$$

For hollow circular columns,

$$c = 109801 \frac{\pi(D^2 - d^2)}{4}. \qquad (7)$$

The number 33379 of equation (1) is the mean result from eighteen pillars, varying in length from 121 times the diameter down to 15 times the diameter; the 98982 of equation (2) is similarly obtained from eleven pillars, varying in length from 78 to 25 times the diameter.

Mr. James B. Francis has computed and published a series

of tables of the strength of cast-iron pillars of the ordinary sizes used in practice, the computations being made by means of Hodgkinson's formulæ.

We have also Gordon's formulæ, which have been already given in § 209, the constants of which are based upon Hodgkinson's experiments. While we have, in the case of wrought-iron and wood, the results of experiments made upon full-size columns, which furnish us more reliable data upon which to base our conclusions in designing such columns, we have, in the case of cast-iron, no way of obtaining more reliable data than those furnished by Hodgkinson's experiments; and hence we can use these results only with a large factor of safety. Mr. Francis recommends five.

§ 220. **Transverse Strength of Cast-Iron.** — At one time cast-iron was very largely used for beams and girders to support a transverse load. Its use for this purpose has now been almost entirely abandoned, as it has been superseded by wrought-iron.

A great many experiments have been made on the transverse strength of cast-iron; the specimens used in some cases being small, and in others large. The records of a great many experiments of this kind are to be found in the first four books of the list already enumerated in § 218. The details of these tests will not be considered here, but an outline will be given of some of the main difficulties that arise in applying the results and in using the beams.

That cast-iron is treacherous and liable to hidden flaws; and that it is brittle, is well known by all who use it. It is also a fact, that in casting any piece where the thickness varies in different parts, the unequal cooling is liable to establish initial strains in the metal, and that, therefore, those parts where such strains have been established have their breaking-strength diminished in proportion to the amount of these strains. These defects are enough to exclude cast-iron from use for beams when we have a much better material in wrought-iron.

Another element of uncertainty, which has vexed experimenters, and which has not yet received a complete solution, is the fact, that whereas we can, with some approximation to correct results, predict the strength of a wrought-iron beam, from simply knowing the tensile and compressive strength of the iron per square inch, by applying our ordinary theory of beams, it is impossible to do so in the case of a cast-iron beam. Thus, in the case of a cast-iron which would have a tensile strength of 16000 to 18000 lbs. per square inch, and a compressive strength of about 80000 lbs. per square inch, the modulus of rupture would be variable, but very seldom either of these two numbers.

In Rankine's tables, the modulus of rupture for rectangular beams of cast-iron is given on the average as 40000 lbs. per square inch, and that for openwork beams as 17000.

What the meaning of this discrepancy was, and how to get a uniform modulus of rupture, engaged the attention of many experimenters. One explanation offered was, that the neutral axis was not at the centre of gravity of the section, but that its position was in some way affected by the relation between the tensile and the compressive strength of the iron. A moment's thought, and a consideration of the assumptions made in the common theory of beams, will show the absurdity of any such conclusion. As well might we conclude that a loaded beam, resting on two supports at the ends, would have its pressure on these supports regulated by the strength of the supports, instead of by the principle of the lever.

Besides this, the younger Barlow tried an experiment to determine the position of the neutral axis in a rectangular cast-iron beam, and found it at the middle of the depth.

The elder Barlow proposed a very elaborate explanation, deducing some much more complicated formulæ for the strength of beams than the ordinary ones; but there is not good evidence of the truth of this theory.

The most plausible explanation seems to be that offered by Rankine; viz., that, besides unequal cooling and the consequent establishing of initial strains, the variation is caused by the proportion of the hardened, tough skin of the metal. This skin theory is borne out, to a great extent, by a series of tests made by Edwin Clark, and detailed in D. K. Clark's "Rules and Tables," p. 562.

§ 221. **Wrought-Iron.** — Wrought-iron is the product obtained by removing the carbon from cast-iron. It is produced by melting the iron, and passing an oxidizing flame over it. When the carbon is burned out, the mass of iron is left in a pasty condition, full of holes. It is then taken out, and hammered or rolled in order to unite it into one mass. Wrought-iron is thus, from the commencement of its manufacture, a series of welds; and the perfection or imperfection of these welds affects very seriously the quality of the iron.

The result of this first process is not suitable to use in any construction of importance; but it requires to be re-heated and re-rolled a number of times, in order to make it more homogeneous, and to remove flaws from within the iron.

At best, however, wrought-iron is a series of welds; and, if a piece be broken, the separate layers of which it is composed can be seen plainly. It is also subject to the impurities of the cast-iron from which it is made. Thus, the presence of sulphur makes it red-short, or brittle when hot; and the presence of phosphorus makes it cold-short, or brittle when cold.

It cannot, like cast-iron, be melted and run into moulds: but it can be welded; that is, two masses of wrought-iron can be united by being brought to a proper temperature, and then hammered together.

Wrought-iron is much more capable of bearing a tensile or transverse stress than cast-iron: it is tougher, it stretches more, and gives more warning before fracture. At one time cast-iron was almost the only form in which iron was used in

construction; but now wrought-iron and steel are superseding it in by far the majority of cases where strength and toughness, and the ability to resist varied stresses, are demanded.

Wrought-iron is also expected to withstand a great many trials that would seriously injure cast-iron: thus, two pieces of wrought-iron are generally united together by riveting; the holes for the rivets have to be punched or drilled, and then the rivets have to be hammered; the entire process tending to injure the iron. Wrought-iron has to withstand flanging, and is liable to severe shocks when in use; as, for instance, those that occur from the difference of temperature, and the changes of temperature in the different parts of a steam-boiler.

§ 222. **Tensile and Compressive Strength of Wrought-Iron.** — As to the experimenters on the tensile strength and elasticity of wrought-iron, those who preceded Hodgkinson are of little more than historic interest. The following list includes a number of the most known experimenters: —

1°. Eaton Hodgkinson: (*a*) Report of Commissioners on the Application of Iron to Railway Structures.
(*b*) London Philosophical Transactions. 1840.
2°. William H. Barlow: Barlow's Strength of Materials.
3°. Sir William Fairbairn: On the Application of Cast and Wrought Iron to Building Purposes.
4°. Franklin Institute Committee: Report of the Committee of the Franklin Institute. In the Franklin Institute Journal of 1837.
5°. L. A. Beardslee, Commander, U.S.N.: Experiments on the Strength of Wrought-Iron and of Chain Cables. Revised and enlarged by William Kent, M.E., or Executive Document 98, 45th Congress, as stated below.
6°. David Kirkaldy: Experiments on Wrought-Iron and Steel.
7°. Professor Bauschinger: Mittheilungen aus dem Mech.-Tech. Laboratorium der K. Pol. Schule in München.
8°. G. Bouscaren: Report on the Progress of Work on the Cincinnati Southern Railway, by Thomas D. Lovett. Nov. 1, 1875.

9°. Government Testing Machine at Watertown, and Government Commission: (*a*) Executive Document 98, 45th Congress U.S.A., 2d session.
(*b*) Executive Document 23, 46th Congress U.S.A., 2d session.
(*c*) Executive Document 12, 47th Congress U.S.A., 1st session.
(*d*) Executive Document 1, 47th Congress U.S.A., 2d session.
(*e*) Executive Document 5, 48th Congress U.S.A., 1st session.

10°. A. Wöhler: (*a*) Die Festigkeits versuche mit Eisen und Stahl.
(*b*) Strength and Determination of the Dimensions of Structures of Iron and Steel, by Dr. Phil. Jacob J. Weyrauch. Translated by Professor Dubois.

11°. Alexander Holley: Executive Document 23, 46th Congress U.S.A., 2d session.

12°. Professor R. H. Thurston: Materials of Engineering.

A few tests were made on the tensile strength of wrought-iron by Eaton Hodgkinson: two of these were on the tensile strength and elasticity of rods about fifty feet long; each rod being made in three parts, these parts being united by couplings.

Below is given the table of results of the first of these tests, as recorded in the commissioners' report, to which table is added here the column of modulus of elasticity:—

Weights applied, in lbs.	Extension, in inches.	Sets, in inches.	Weights per Square Inch of Section, in lbs.	Strains, in Fractions of the Length.	Sets, in Fractions of the Length.	Ratio of Weight to Extension.	Modulus of Elasticity.
560	0.0485	Perceptible	2668	0.000082	–	270544	32465280
1120	0.1095	Perceptible after one hour	5335	0.000185	–	239565	28297800
1680	0.1675	0.0015	8003	0.000284	0.0000025	234889	28186680
2240	0.2240	0.0020	10670	0.000379	0.0000034	234201	28104120
2800	0.2805	0.0027	13338	0.000475	0.0000042	233791	28054920
3360	0 3370	0.0030	16005	0.000571	0.0000047	233518	27017160
3920	0.3930	0.0040	18673	0.000666	0.0000068	233616	28033920
4480	0.4520	0.0075	21340	0.000766	0.0000127	232138	27856560
5040	0.5155	0.0195	24008	0.000874	0.0000330	228975	27477000

Weights applied, in lbs.	Extension, in inches.	Sets, in inches.	Weights per Square Inch of Section, in lbs.	Strains, in Fractions of the Length.	Sets, in Fractions of the Length.	Ratio of Weight to Extension.	Modulus of Elasticity.
5600	0.5980	0.0490	26676	0.001014	0.0000830	219317	26318040
6160	0.7600	0.1545	29343	0.001288	0.0002619	189825	22779000
6720	1.3100	–	32011	0.002228	–	–	–
In 10 min.	1.3780	0.6670	–	0.002356	0.0011305	113228	13587360
In 12 min.	1.3900	–	–	–	–	–	–
7280	2.5310	1.8125	34678	0.004290	0.0030720	67363	8083560
7840	5.4060	5.0000	37346	0.009163	0.0084746	33966	4075920
Repeated	5.8750	5.0625	–	0.009957	0.0085805	–	–
8400	6.2190	5.3750	40013	0.010166	0.0091083	32798	3935760
In 1 hour	6.9370	–	–	0.011758	–	–	–
In 2 hours	7.0000	–	–	0.011866	–	–	–
In 3 hours	7.0470	–	–	0.011775	–	–	–
In 4 hours	7.0500	–	–	0.011950	–	–	–
In 5 hours	7.0620	–	–	0.011966	–	–	–
In 6 hours	7.0640	–	–	0.011975	–	–	–
In 7 hours	7.0940	–	–	0.012025	–	–	–
In 10 hours	7.0940	–	–	0.012025	–	–	–
Left over night; next morning	7.0000	–	–	–	–	–	–
8960	10.5620	9.7500	42681	0.017900	0.0165250	19870	2384400
In 5 min.	11.5000	–	–	0.019492	–	–	–
In 10 min.	11.7190	–	–	0.019858	–	–	–
In 1 hour	–	10.875	–	–	0.0184333	–	–
In 46 hours	11.9370	11.0000	–	0.020233	0.0186417	–	–
9520	12.6870	11.6880	45348	0.021500	0.0198083	17577	2109240
In 1 hour	12.8100	–	–	0.021708	–	–	–
In 2 hours	12.8150	–	–	0.021716	–	–	–
In 19 hours	12.8150	11.8150	–	0.021716	0.0200250	–	–
10080	14.6250	–	48016	0.024792	–	–	–
In 10 min.	–	13.4370	–	–	0.0227750	16139	1936680
In 1 hour	14.8440	–	–	0.025158	–	–	–
In 11 hours	14.8910	–	–	0.025242	–	–	–
Next morning	14.8750	–	–	–	–	–	–
10640	–	–	–	–	–	–	–
In 10 min.	20.6250	19.3750	50684	0.034958	0.0328417	12082	1449840
Repeated	20.7780	–	–	0.035217	–	–	–
In 7 hours	20.7810	–	–	0.035225	–	–	–
In 12 hours	20.7810	–	–	0.035225	–	–	–
Next morning	20.7500	–	–	–	–	–	–
11200	–	–	53351	–	–	–	–

With this last weight the rod broke at one of the weldings, where there was a slight defect; perhaps a rather smaller weight would have broken it.

In connection with this table of results, the following observations will be made: —

1°. The breaking-strength per square inch in this case was 53351 lbs. per square inch.

2°. Permanent set began with very small loads.

3°. The limit of elasticity was about 29000 lbs. per square inch, this being about the load where permanent set began to increase rapidly.

4°. The modulus of elasticity of the rod in this case was about 28000000 lbs. per square inch, this being the ratio of the stress to the strain under loads less than the limit of elasticity.

5°. This experiment shows, that, under moderate loads, the modulus of elasticity, and hence the ratio of the stress to the strain, remains much more nearly constant with wrought than with cast iron.

6°. The other experiment, with a rod 50 feet long, gave as modulus of elasticity 27691200 lbs. per square inch.

The remaining experiments of Hodgkinson on wrought-iron may be found in the report of the commissioners already referred to.

Barlow's experiments on wrought-iron will not be detailed here, except only to say that he tried seven experiments to determine the modulus of elasticity of wrought-iron, and that he obtained, for mean extension per English ton per square inch, maximum, 0.0001082; minimum, 0.0000841; mean, 0.0000956 of the length. These correspond to moduli of elasticity respectively equal to 20702400, 26634900, and 23430900 lbs. per square inch.

The results obtained by experimenters before Hodgkinson's time are very discordant and very uncertain, many of them attributing to wrought-iron a strength far greater than

can be attained at the present time. There is no satisfactory evidence, however, to show that the wrought-iron of that time was any better than (if as good as) that made at the present time; and it is more probable either that there were errors in making the tests, or else that the supposed iron was really steel, and possibly brittle. The paucity of the records, and the impossibility of obtaining the details of the tests, render any search for the reason in any special cases futile, and throw doubt upon the greater part of these tests.

Among the later English experimenters, we have Sir William Fairbairn. An account of his tests on tensile strength will be found in the book already referred to.

One of the most prominent English experimenters, and one who has done a great deal towards rendering our knowledge of this subject more accurate, is David Kirkaldy. His results up to 1866 are detailed in his book entitled, "Experiments on Wrought-Iron and Steel," published in that year.

In the early part of his book will be found a summary of what had been done in this line by the early experimenters.

Kirkaldy tested a large number of English irons; and a summary of his results will be given here, together with his sixty-six concluding observations.

Names of Makers or Works.	Shape.	Breaking-Weight per Square Inch of Area.			Contraction of Area.		Extreme Elongation, per cent.	
		Original, lbs.	Stretched, lbs.	Fractured, lbs.	Stretched, per cent.	Fractured, per cent.		
Bradley & Co., L . . .	Round,	57216	72700	146521	21.2	60.9	30.2	Bright gray, very fine and soft.
Lowmoor	"	61798	76290	131676	19.0	53.1	26.5	Bright lightish gray, close and very fine.
Farnley	"	62886	77200	127425	18.5	50.6	25.6	
Bradley & Co., B.B. Scrap,	"	59370	74378	123805	19.9	52.0	26.6	
Lowmoor	Square,	60364	70131	117147	13.9	48.5	24.9	
Bowling	Round,	62404	74487	114220	16.2	45.3	24.4	
Govan	"	58199	64675	116549	10.0	50.1	21.4	Light gray, uniformly fine.
"	"	58746	67708	113700	13.2	48.3	25.2	
"	"	57598	64630	114866	10.9	49.8	24.8	
Govan Ex. B. Best . . .	"	58169	68446	101863	15.0	42.9	19.2	Brightish gray, close and uniform.
Glasgow B. Best	"	59300	68160	99612	13.0	40.4	20.0	
Govan Ex. B. Best . . .	Square,	56655	64797	99000	12.6	42.7	19.1	
" " . . .	Round,	59109	68140	98527	13.3	40.0	22.3	
" " . . .	"	58358	70645	97821	17.4	40.3	23.8	
Bradley & Co. S.C. . .	"	62231	75133	97575	17.2	36.2	22.2	Bright gray, fine but irregular.
Glasgow B. Best	"	58910	67404	97559	12.6	40.3	21.3	Brightest gray, close and generally uniform.
" "	"	58885	70260	97548	16.2	39.6	23.2	
Govan B. Best	"	64795	74246	97245	12.5	33.4	17.3	
" "	"	59548	67333	95706	11.5	37.7	16.9	

Names of Makers or Works.	Shape.	Breaking-Weight per Square Inch of Area.			Contraction of Area.		Extreme Elongation, per cent.	
		Original, lbs.	Stretched, lbs.	Fractured, lbs.	Stretched, per cent.	Fractured, per cent.		
Govan B. Best	Round,	61341	73527	96442	16.6	36.4	20.0	Brightest gray, close and generally uniform.
Govan	"	61887	73869	95319	16.1	35.1	18.8	
Govan Ex. B. Best . . .	"	57591	66242	95248	13.0	39.5	17.3	
" " . . .	"	57400	68018	92880	15.6	38.0	17.6	
Blochairn B. Best . . .	"	56141	66272	90313	15.3	37.8	21.3	
Malinslee Best . . .	Flat,	56289	65596	88300	14.2	37.0	21.4	Generally close and fine, but irregular.
Port Dundas Ex. B. Best .	Round,	54594	62645	85563	12.7	36.1	20.6	
Glasgow B. Best	"	54579	61496	85012	11.2	35.8	20.3	
Govan B. Best	"	60879	69161	84770	12.0	28.2	17.0	
" "	"	62849	73111	88550	13.7	28.9	19.1	
Govan	"	63956	74535	88512	14.2	27.7	15.8	Rather coarse and irregular.
"	"	59424	68106	79373	12.6	25.1	16.4	
"	"	58326	67046	78139	13.0	25.3	16.7	
Bagnall J.B.	"	55000	61700	75351	10.9	27.0	17.3	
Dundyvan	"	53352	56889	58304	6.2	8.5	6.8	Coarse and very hard.
"	"	51327	53436	54100	4.0	5.1	6.3	
Ystalyfera Puddled . . .	Flat,	38526	38996	39470	1.2	2.4	2.0	Coarse and open.

ROLLED BARS FOR RIVETS.

Names of Makers or Works.	Shape.	Breaking-Weight per Square Inch of Area.			Contraction of Area:		Extreme Elongation, per cent.	
		Original, lbs.	Stretched, lbs.	Fractured, lbs.	Stretched, per cent.	Fractured, per cent.		
Lowmoor	Round,	60075	72685	125775	17.3	52.2	20.5	Uniformly very fine and soft.
Bradley & Co. S.C. .	"	56715	70503	112336	19.5	49.5	22.5	
Ulverston Rivet Best . .	"	53775	67324	104680	20.1	48.6	21.6	
Thorneycroft T.N.S. . .	"	59278	71540	99595	17.1	40.4	22 4	Generally fine and uniform.
R. Solloch E. Best . . .	"	57425	65590	96959	12.4	40.7	17.7	
Coatbridge Best Rivet . .	"	61723	70893	96267	13.2	35.9	21.6	
Glasgow Best Rivet. . .	"	57092	67103	96205	14.9	40.7	23.7	
Lord Ward L W.R.O.,	"	59753	71621	95724	16.4	37.6	18.6	
Blochairn Best Rivet . .	"	59219	67505	89279	12.2	34.3	19.4	
St. Rollox Best Rivet . .	"	56981	63325	77383	9.6	26.3	16.6	Extremely irregular and hard.

SUMMARY OF KIRKALDY'S RESULTS.—IRON PLATES.

Names of Makers or Works.	Lengthwise or Crosswise.	Breaking-Weight per Square Inch of Area.		Contraction of Area at Fracture, per cent.	Extreme Elongation, per cent.
		Original, lbs.	Fractured, lbs.		
Farnley	L.	58437	83112	29.6	17.0
	C.	55033	68961	20.1	11.3
Farnley	L.	58487	70538	20.0	10.9
	C.	54098	59698	11.4	5.9
Lowmoor	L.	52000	64746	19.7	13.2
	C.	50515	57383	12.1	9.3
Farnley	L.	56005	68763	17.8	14.1
	C.	46221	53293	13.2	7.6
Bowling	L.	52235	61716	15.3	11.6
	C.	46441	50009	6.9	5.9
◇ Govan ◇	L.	54644	66728	18.1	11.6
	C.	49399	54020	8.5	6.5
Bradley & Co. S. C.	L.	55831	67406	17.2	12.5
	C.	50550	55206	9.0	5.5
Bradley & Co., L. F.	L.	56996	66858	15.0	13.0
	C.	51251	56070	8.6	5.9
Bradley & Co.	L.	55708	65652	14.9	10.7
	C.	49425	54002	8.1	5.1
Malinslee Best	L.	52572	62131	15.4	8.6
	C.	50627	55746	9.3	5.8
Consett Best Best	L.	51245	59183	13.1	8.9
	C.	46712	52050	10.2	6.4
Consett Best Best	L.	53559	62346	14.4	11.5
	C.	45677	48358	5.6	4.0
Consett Best Best	L.	49120	55472	11.3	8.0
	C.	46755	50000	6.3	5.2
Thorneycroft Best Best	L.	54847	62747	12.5	11.2
	C.	45585	47712	4.6	4.6
Snedshill ♛ Best	L.	52362	61581	15.0	9.6
	C.	43036	45300	5.3	2.8
Wells Best ♛ Best	L.	45997	51140	10.4	6.7
	C.	49311	54842	10.5	7.0
Glasgow Best Boiler	L.	53849	60522	11.0	9.3
	C.	48848	52252	6.6	4.6

SUMMARY OF KIRKALDY'S RESULTS.—IRON PLATES.—*Concluded.*

Names of Makers or Works.	Lengthwise or Crosswise.	Breaking-Weight per Square Inch of Area. Original, lbs.	Breaking-Weight per Square Inch of Area. Fractured, lbs.	Contraction of Area at Fracture, per cent.	Extreme Elongation, per cent.
Glasgow Best Best	L.	53399	59557	10.6	9.0
	C.	41791	43614	3.7	2.6
Wells Best ♛ Best	L.	47410	51521	6.7	4.0
	C.	46630	48348	4.9	3.4
Maker's stamp uncertain	L.	47598	53182	9.8	5.9
	C.	40682	43426	5.0	2.5
K. B. M.	L.	46404	51896	10.2	6.1
	C.	44764	47891	6.4	4.3
Lloyds, Foster, & Co., Best . . .	L.	44967	49162	8.7	5.3
	C.	44732	48344	6.9	4.6
Glasgow Best Best	L.	45626	48208	5.3	4.3
	C.	41340	42430	2.2	2.4
Glasgow Ship	L.	47773	49816	4.8	3.6
	C.	44355	45343	3.4	2.1
Mossend Best Best	L.	43433	46038	5.7	3.3
	C.	41456	43622	4.9	2.9
Govan Best	L.	43942	45886	6.0	3.4
	C.	39544	40624	2.6	1.4
Glasgow Best Scrap	L.	50844	58412	13.0	10.5

Fracture 1–12.—Light gray of various shades, close and very fine.
Fracture 13–30.—Dullish gray of various shades, close and generally fine.
Fracture 31–44.—Dull gray, generally rather coarse and irregular.
Fracture 45–52.—Irregular, generally coarse and open.
Fracture 53.—Close and fine.

KIRKALDY'S SIXTY-SIX CONCLUSIONS.

1°. The breaking-strain does not indicate the quality, as hitherto assumed.

2°. A high breaking-strain may be due to the iron being of superior quality, dense, fine, and moderately soft, or simply to its being very hard and unyielding.

3°. A low breaking-strain may be due to looseness and coarseness in the texture, or to extreme softness, although very close and fine in quality.

4°. The contraction of area at fracture, previously overlooked, forms an essential element in estimating the quality of specimens.

5°. The respective merits of various specimens can be correctly ascertained by comparing the breaking-strain jointly with the contraction of area.

6°. Inferior qualities show a much greater variation in the breaking-strain than superior.

7°. Greater differences exist between small and large bars in coarse than in fine varieties.

8°. The prevailing opinion of a rough bar being stronger than a turned one, is erroneous.

9°. Rolled bars are slightly hardened by being forged down.

10°. The breaking-strain, and contraction of area, of iron plates are greater in the direction in which they are rolled than in a transverse direction.

11°. A very slight difference exists between specimens from the centre and specimens from the outside of crank-shafts.

12°. The breaking-strain, and contraction of area, are greater in those specimens cut lengthways out of crank-shafts than in those cut cross-ways.

13°. The breaking-strain of steel, when taken alone, gives no clew to the real qualities of various kinds of that material.

14°. The contraction of area at fracture of specimens of steel must be ascertained, as well as in those of iron.

15°. The breaking-strain, jointly with the contraction of area, affords the means of comparing the peculiarities in various lots of specimens.

16°. Some descriptions of steel are found to be very hard, and consequently suitable for some purposes; whilst others are extremely soft, and equally suitable for other uses.

17°. The breaking-strain, and contraction of area, of puddled steel plates, as in iron plates, are greater in the direction in which they are rolled; whereas in cast-steel they are less.

18°. Iron, when fractured suddenly, presents invariably a crystalline appearance; when fractured slowly, its appearance is invariably fibrous.

19°. The appearance may be changed from fibrous to crystalline by merely altering the shape of the specimen so as to render it more liable to snap.

20°. The appearance may be changed by varying the treatment so as to render the iron harder and more liable to snap.

21°. The appearance may be changed by applying the strain so suddenly as to render the specimen more liable to snap from having less time to stretch.

22°. Iron is less liable to snap the more it is worked and rolled.

23°. The "skin," or outer part of the iron, is somewhat harder than the inner part, as shown by appearance of fracture in rough and turned bars.

24°. The mixed character of the scrap-iron used in large forgings is proved by the singularly varied appearance of the fractures of specimens cut out of crank-shafts.

25°. The texture of various kinds of wrought-iron is beautifully developed by immersion in dilute hydrochloric acid, which, acting on the surrounding impurities, exposes the metallic portion alone for examination.

26°. In the fibrous fractures the threads are drawn out, and are viewed externally; whilst in the crystalline fractures the threads are snapped across in clusters, and are viewed internally or sectionally. In the latter cases, the fracture of the specimen is always at right angles to the length; in the former, it is more or less irregular.

27°. Steel invariably presents, when fractured slowly, a silky, fibrous appearance; when fractured suddenly, the appearance is invariably granular, in which case also the fracture is always at right angles to the length; when the fracture is fibrous, the angle diverges always more or less from 90°.

28°. The granular appearance presented by steel suddenly fractured is nearly free from lustre, and unlike the brilliant crystalline appearance of iron suddenly fractured: the two combined in the same specimen are shown in iron bolts partly converted into steel.

29°. Steel which previously broke with a silky, fibrous appearance is changed into granular by being hardened.

30°. The little additional time required in testing those specimens

whose rate of elongation was noted, had no injurious effect in lessening the amount of breaking-strain, as imagined by some.

31°. The rate of elongation varies not only extremely in different qualities, but also to a considerable extent in specimens of the same brand.

32°. The specimens were generally found to stretch equally throughout their length until close upon rupture, when they more or less suddenly draw out; usually at one part only, sometimes at two, and in a few exceptional cases at three, different places.

33°. The ratio of ultimate elongation may be greater in short than in long bars in some descriptions of iron, whilst in others the ratio is not affected by difference in the length.

34°. The lateral dimensions of specimens form an important element in comparing either the rate of or the ultimate elongations, — a circumstance which has been hitherto overlooked.

35°. Steel is reduced in strength by being hardened in water, while the strength is vastly increased by being hardened in oil.

36°. The higher steel is heated (without, of course, running the risk of being burned), the greater is the increase of strength by being plunged into oil.

37°. In a highly converted or hard steel the increase in strength and in hardness is greater than in a less converted or soft steel.

38°. Heated steel, by being plunged into oil instead of water, is not only considerably *hardened*, but *toughened*, by the treatment.

39°. Steel plates hardened in oil and joined together with rivets are fully equal in strength to an unjointed soft plate, or, the loss of strength by riveting is more than counterbalanced by the increase in strength by hardening in oil.

40°. Steel rivets fully larger in diameter than those used in riveting iron plates of the same thickness, being found to be greatly too small for riveting steel plates, the probability is suggested that the proper proportion for iron rivets is not, as generally assumed, a diameter equal to the thickness of the two plates to be joined.

41°. The shearing-strain of steel rivets is found to be about a fourth less than the tensile strain.

42°. Iron bolts case-hardened bore a less breaking-strain than when

wholly iron, owing to the superior tenacity of the small proportion of steel being more than counterbalanced by the greater ductility of the remaining portion of iron.

43°. Iron highly heated, and suddenly cooled in water, is hardened, and the breaking-strain, when gradually applied, increased; but at the same time it is rendered more liable to snap.

44°. Iron, like steel, is softened, and the breaking-strain reduced, by being heated, and allowed to cool slowly.

45°. Iron subjected to the cold-rolling process has its breaking-strain greatly increased by being made extremely hard, and not by being "consolidated," as previously supposed.

46°. Specimens cut out of crank-shaft are improved by additional hammering.

47°. The galvanizing or tinning of iron plates produces no sensible effects on plates of the thickness experimented on. The results, however, may be different should the plates be extremely thin.

48°. The breaking-strain is materially affected by the shape of the specimen. Thus, the amount borne was much less when the diameter was uniform for some inches of the length than when confined to a small portion, — a peculiarity previously unascertained and not even suspected.

49°. It is necessary to know correctly the exact conditions under which any tests are made, before we can equitably compare results obtained from different quarters.

50°. The startling discrepancy between experiments made at the Royal Arsenal and by the writer is due to the difference in the shape of the respective specimens, and not to the difference in the two testing-machines.

51°. In screwed bolts, the breaking-strain is found to be greater when old dies are used in their formation than when the dies are new, owing to the iron becoming harder by the greater pressure required in forming the screw-thread when the dies are old and blunt than when new and sharp.

52°. The strength of screw-bolts is found to be in proportion to their relative areas; there being only a slight difference in favor of the smaller compared with the larger sizes, instead of the very material difference previously imagined.

53°. Screwed bolts are not necessarily injured, although strained nearly to their breaking-point.

54°. A great variation exists in the strength of iron bars which have been cut and welded: whilst some bear almost as much as the uncut bar, the strength of others is reduced fully a third.

55°. The welding of steel bars, owing to their being so easily burned by slightly over-heating, is a difficult and uncertain operation.

56°. Iron is injured by being brought to a white or welding heat if not at the same time hammered or rolled.

57°. The breaking-strain is considerably less when the strain is applied suddenly, instead of gradually, though some have imagined that the reverse is the case.

58°. The contraction of area is also less when the strain is suddenly applied.

59°. The breaking-strain is reduced when the iron is frozen. With the strain gradually applied, the difference between a frozen and unfrozen bolt is lessened as the iron is warmed by the drawing-out of the specimen.

60°. The amount of heat developed is considerable when the specimen is suddenly stretched, as shown in the formation of vapor from the melting of the layer of ice on one of the specimens, and also by the surface of others assuming tints of various shades of blue and orange, not only in steel, but also, although in a less marked degree, in iron.

61°. The specific gravity is found generally to indicate pretty correctly the quality of specimens.

62°. The density of iron is *decreased* by the process of wire-drawing, and by the similar process of cold-rolling, instead of *increased*, as previously imagined.

63°. The density in some descriptions of iron is also decreased by additional hot-rolling in the ordinary way: in others the density is very slightly increased.

64°. The density of iron is decreased by being drawn out under a tensile strain, instead of increased, as believed by some.

65°. The most highly converted steel does not, as some may suppose, possess the greatest density.

66°. In cast-steel the density is much greater than in puddled steel, which is even less than in some of the superior descriptions of wrought-iron.

TESTS OF COMMANDER BEARDSLEE.

One of the most valuable sets of tests of wrought-iron is that obtained by committees D, H, and M of the Board appointed by the United-States Government to test iron and steel; the special duties of these committees being to test such iron as would be used in chain-cable, and the chain-cable itself. The chairman of these three committees, which were consolidated into one, was Commander L. A. Beardslee of the United-States Navy. The full account of the tests is to be found in Executive Document 98, 45th Congress, second session; and an abridged account of them was published by William Kent, as has been already mentioned.

The samples of bar-iron tested were round, and varied from one inch to four inches in diameter. The mills from which the samples tested came were as follows:—

Bentoni	Pennsylvania.
Burden and Sons	New York.
Burgess	Ohio.
Catasauqua	Pennsylvania.
New-Jersey Iron and Steel Company	New Jersey.
Niles Iron Company	Ohio.
Pembroke	Massachusetts.
Pencoyd	Pennsylvania.
Phœnix	Pennsylvania.
Sligo	Pennsylvania.
Tamaqua	Pennsylvania.
Tredegar	Virginia.
Trego and Thompson	Maryland.
Wyeth Brothers	Maryland.

Certain conclusions which they reached refer to all kinds of wrought-iron, and will be given here before giving a table of the results of the tests.

1°. Kirkaldy considers the breaking-strength per square inch of fractured area as the main criterion by which to determine the merits of a piece of iron or steel. Commander Beardslee, on the other hand, thinks that a better criterion is what he calls the "tensile limit;" i.e., the maximum load the piece sustains divided by the area of the smallest section when that load is on, i.e., just before the load ceases to increase in the testing-machine.

2°. Kirkaldy had already called attention to the fact that the tensile strength of a specimen is very much affected by its shape, and that, in a specimen where the shape is such that the length of that part which has the smallest cross-section is practically zero (as is the case when a groove is cut around the specimen), the breaking-strength is greater than it is when this portion is long; the excess being in some cases as much as 33 per cent.

Commander Beardslee undertook, by actually testing specimens whose smallest areas varied in length, to determine what must be the least length of that part of the specimen whose cross-section area is smallest, in order that the tensile strength may not be greater than with a long specimen. The conclusion reached was, that no test-piece should be less than one-half inch in diameter, and that the length should never be less than four diameters; while a length of five or six diameters is necessary with soft and ductile metal in order to insure correct results. The following results of testing steel are given in Mr. Kent's book, as confirming the same rule in the case of steel. The tests were make upon Bessemer steel by Col. Wilmot at the Woolwich arsenal.

	Tensile Strength.	Pounds per Square Inch.
By groove form	Highest	162974
	Lowest	136490
	Average	153677
By cylinder	Highest	123165
	Lowest	103255
	Average	114460

3°. Commander Beardslee also noticed that rods of certain diameters of the same kind of iron bore less in proportion than rods of other diameters; and, after searching carefully for the reason, he found it to lie in the proportion between the diameter of the rod and the size of the pile from which it is rolled. The following examples are given: —

$1\frac{1}{8}$-in.	diameter,	6.62%	of pile,	56543	lbs. per	sq. in.	tensile	strength.
$1\frac{1}{4}$	"	8.18%	"	56478	"	"	"	"
$1\frac{3}{8}$	"	9.90%	"	54277	"	"	"	"
$1\frac{1}{2}$	"	11.78%	"	53550	"	"	"	"
$1\frac{5}{8}$	"	7.68%	"	56344	"	"	"	"
$1\frac{3}{4}$	"	8.90%	"	55018	"	"	"	"
$1\frac{7}{8}$	"	10.22%	"	54034	"	"	"	"
2	"	11.63%	"	51848	"	"	"	"

He therefore claims, that, in any set of tests of round iron, it is necessary to give the diameter of the rod tested, and not merely the breaking-strength per square inch.

4°. He gives evidence to show, that if a bar is under-heated, it will have an unduly high tenacity and elastic limit; and that if it is over-heated, the reverse will be the case.

5°. The discovery was made independently by Commander Beardslee and Professor Thurston, that wrought-iron, after having been subjected to its ultimate tensile strength without breaking it, would, if relieved of its load and allowed to rest, have its breaking-strength and its limit of elasticity increased.

He tried a considerable number of experiments, studying the action of this law under different periods of rest, from 1 minute to 3 days and upwards; and a great deal of valuable information is given in the tables of the report.

The most characteristic table is the following: —

EFFECT OF EIGHTEEN HOURS' REST ON IRONS OF WIDELY DIFFERENT CHARACTERS.

	Ultimate Strength per Square Inch.		Remarks.	
	First Strain.	Second Strain.		
Boiler iron . . .	48600	56500	Not broken.	
" " . . .	49800	57000	Broken	Average gain, 15.8%.
" " . . .	49800	58000	Broken	
" " . . .	48100	54400	Broken	
" " . . .	48150	55550	Broken	
Contract chain iron,	50200	54000	Broken	Average gain, 6.4%.
" " "	50250	53200	Not broken	
" " "	50700	55300	Not broken	
" " "	49600	52900	Not broken	
" " "	51200	52800	Not broken	
Iron K	58800	64500	Broken	Average gain, 9.4%.
" "	59000	65800	Broken	
" "	56400	60600	Broken	

His experiments show that the increase is in irons of a fibrous and ductile nature, rather than in brittle and steely ones: hence the latter class would be but little benefited by the action of this law.

The following table of results is given in the report; showing how the strength per square inch varies with the diameter of the piece, in consequence of the amount of reduction in the rolls.

Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.
¼	F	59885		1	D	52900		1¼	D	57977	31996
⅜	F	54090	40980	1	F	52819	32267	1¼	P	55782	35596
½	C	62700		1	F	51400	34600	1¼	Px	56334	33921
½	C	59000		1⅛	K	60458	37344	1¼	N	56478	33251
½	C	57700		1⅛	D	59582	33597	1¼	FxI	55253	34784
½	C	55400		1⅛	C	57470	31900	1¼	D	55550	28166
½	F	52275	39126	1⅛	FxI	56434	34682	1¼	E	53893	32712
⅝	F	55450		1⅛	P	57498	41311	1¼	Fx2	55132	38603
⅝	F	52050		1⅛	N	56143	32267	1¼	Fx3	53247	32520
⅝	F	57660		1⅛	Fx2	55927	37250	1¼	A	53897	27643
¾	F	51546	35933	1⅛	E	53097	33549	1¼	M	53752	–
⅞	F	50630	33931	1⅛	Fx3	54644	34695	1¼	M	54090	–
1	K	61727		1⅛	D	54687	28166	1¼	F	52970	32075
1	D	61115	33486	1⅛	A	53900	26787	1¼	F	52729	39608
1	O	57363	37415	1⅛	F	53850	33457	1¼	M	53022	–
1	FxI	55768	34729	1⅛	O	53035	32410	1¼	F	52620	33220
1	P	57807	39230	1⅛	F	50149	35493	1¼	O	50040	30730
1	A	54690	34881	1⅛	F	52267	32019	1 5/16	P	54518	35898
1	Fx2	56790	36885	1¼	K	59461	36501	1⅜	M	58926	37548
1	Fx3	53915	36336	1¼	P	56876	36868	1⅜	M	57649	38578
1	F	51921	31300	1¼	C	57897	32469	1⅜	D	58021	32152

Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.
$1\frac{3}{8}$	K	55790	31034	$1\frac{1}{2}$	M	54095	35544	$1\frac{5}{8}$	Fx2	53438	35870
$1\frac{3}{8}$	C	54949	31030	$1\frac{1}{2}$	E	54544	33027	$1\frac{5}{8}$	H	52314	29364
$1\frac{3}{8}$	M	54373	35820	$1\frac{1}{2}$	P	52868	29636	$1\frac{5}{8}$	E	51946	27695
$1\frac{3}{8}$	N	54277	33622	$1\frac{1}{2}$	M	53512	–	$1\frac{5}{8}$	O	52401	34012
$1\frac{3}{8}$	Fx1	52968	33275	$1\frac{1}{2}$	M	52941		$1\frac{5}{8}$	F	52163	33907
$1\frac{3}{8}$	Fx3	52733	34606	$1\frac{1}{2}$	Fx3	52819	34840	$1\frac{5}{8}$	G	51205	33318
$1\frac{3}{8}$	E	52254	25930	$1\frac{1}{2}$	Fx1	53491	34307	$1\frac{5}{8}$	F	50529	35390
$1\frac{3}{8}$	A	53557	33650	$1\frac{1}{2}$	M	52736	34901	$1\frac{5}{8}$	F	50970	33625
$1\frac{3}{8}$	P	52556	30802	$1\frac{1}{2}$	N	53555	34690	$1\frac{5}{8}$	C	49030	31099
$1\frac{3}{8}$	F	52537	34469	$1\frac{1}{2}$	C	52700	35880	$1\frac{11}{16}$	K	56595	38310
$1\frac{3}{8}$	F	52339	39103	$1\frac{1}{2}$	H	52462	29992	$1\frac{11}{16}$	B	54181	
$1\frac{3}{8}$	M	53016	35379	$1\frac{1}{2}$	D	52155	27708	$1\frac{11}{16}$	J	54114	
$1\frac{3}{8}$	Fx2	51487	35911	$1\frac{1}{2}$	A	51884	28794	$1\frac{11}{16}$	B	52895	33145
$1\frac{3}{8}$	F	51296	31992	$1\frac{1}{2}$	F	51994	32054	$1\frac{11}{16}$	E	52120	35549
$1\frac{3}{8}$	O	50594	34940	$1\frac{1}{2}$	O	50919	32312	$1\frac{11}{16}$	G	57789	34160
$1\frac{7}{16}$	P	53345	–	$1\frac{1}{2}$	F	51456	34591	$1\frac{11}{16}$	C	49821	33184
$1\frac{7}{16}$	E	53944	32542	$1\frac{1}{2}$	Fx2	51481	34917	$1\frac{3}{4}$	K	57874	
$1\frac{7}{16}$	G	53238	32534	$1\frac{1}{2}$	J	51047	–	$1\frac{3}{4}$	Px	54212	33908
$1\frac{7}{16}$	B	52287	32411	$1\frac{1}{2}$	M	49292	32597	$1\frac{3}{4}$	C	54410	31354
$1\frac{7}{16}$	C	51756	32655	$1\frac{5}{8}$	N	56344	35889	$1\frac{3}{4}$	P	52844	33842
$1\frac{7}{16}$	J	50400	–	$1\frac{5}{8}$	K	57132	35026	$1\frac{3}{4}$	Fx1	53846	36573
$1\frac{1}{2}$	M	57052	38417	$1\frac{5}{8}$	M	57402	35701	$1\frac{3}{4}$	H	53800	27856
$1\frac{1}{2}$	K	57317	33412	$1\frac{5}{8}$	P	55634	33522	$1\frac{3}{4}$	N	55018	34283
$1\frac{1}{2}$	D	56505	32496	$1\frac{5}{8}$	C	56227	33207	$1\frac{3}{4}$	D	53472	31892
$1\frac{1}{2}$	M	55466	34780	$1\frac{5}{8}$	Px	54689	33427	$1\frac{3}{4}$	J	53264	–
$1\frac{1}{2}$	M	55131	33771	$1\frac{5}{8}$	A	54334	32163	$1\frac{3}{4}$	D	52699	27817
$1\frac{1}{2}$	P	54159	33140	$1\frac{5}{8}$	D	53695	30087	$1\frac{3}{4}$	Fx3	53154	35323
$1\frac{1}{2}$	M	54540	–	$1\frac{5}{8}$	Fx3	53339	33540	$1\frac{3}{4}$	E	51606	26541
$1\frac{1}{2}$	C	55404	34770	$1\frac{5}{8}$	Fx1	53537	34335	$1\frac{3}{4}$	A	51509	29404
$1\frac{1}{2}$	E	55415	32869	$1\frac{5}{8}$	D	53614	30664	$1\frac{3}{4}$	F	50690	32229
$1\frac{1}{2}$	M	54816	34716	$1\frac{5}{8}$	J	52748	–	$1\frac{3}{4}$	G	50395	36254
$1\frac{1}{2}$	Px	54354	34617	$1\frac{5}{8}$	E	52675	33745	$1\frac{3}{4}$	C	50312	30852

Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.	Diameter, in inches.	Name of Iron.	Strength per Square Inch of Original Area.	Elastic Limit per Square Inch of Original Area.
$1\frac{3}{4}$	F	50547	35954	$1\frac{7}{8}$	F	49355	32855	2	O	48249	31413
$1\frac{3}{4}$	Fx2	52314	35320	$1\frac{7}{8}$	F	48670	23250	2	D	49146	33068
$1\frac{3}{4}$	E	49816	31214	$1\frac{7}{8}$	O	47478	30842	2	D	46151	36050
$1\frac{3}{4}$	F	49738	28907	$1\frac{15}{16}$	M	51474	–	$2\frac{1}{16}$	M	51559	–
$1\frac{3}{4}$	O	50129	32271	$1\frac{15}{16}$	M	51707	–	$2\frac{1}{16}$	M	49422	–
$1\frac{13}{16}$	K	56577	–	$1\frac{15}{16}$	M	51242	–	$2\frac{1}{8}$	M	50481	–
$1\frac{13}{16}$	B	53655	–	2	K	60213	31441	$2\frac{1}{8}$	M	51225	–
$1\frac{13}{16}$	C	50969	30814	2	K	57567	30839	$2\frac{1}{8}$	A	48382	30459
$1\frac{13}{16}$	G	50310	33565	2	Px	52914	31198	$2\frac{3}{16}$	M	51666	–
$1\frac{13}{16}$	E	50307	29767	2	M	52820	–	$2\frac{1}{4}$	M	51530	–
$1\frac{13}{16}$	J	48953		2	M	49164	–	$2\frac{1}{4}$	M	51296	–
$1\frac{7}{8}$	K	55803	31031	2	E	51818	27318	$2\frac{1}{4}$	F	48812	–
$1\frac{7}{8}$	C	54447	32334	2	P	51684	33104	$2\frac{1}{4}$	F	48894	–
$1\frac{7}{8}$	D	53100	32074	2	P	50834	31878	$2\frac{1}{4}$	F	49164	31966
$1\frac{7}{8}$	N	54004	33610	2	N	52127	32461	$2\frac{1}{4}$	F	49290	32163
$1\frac{7}{8}$	Fx1	52875	35641	2	N	51370	32460	$2\frac{1}{4}$	P	46866	28241
$1\frac{7}{8}$	Fx3	53361	35032	2	Fx1	52011	34702	$2\frac{1}{2}$	F	47344	29758
$1\frac{7}{8}$	P	52505	32312	2	C	51153	29335	$2\frac{1}{2}$	F	48475	28932
$1\frac{7}{8}$	E	50880	27100	2	D	51146	28567	$2\frac{1}{2}$	F	47428	29941
$1\frac{7}{8}$	D	51459	27816	2	P	49872	29953	$2\frac{3}{4}$	F	46446	26333
$1\frac{7}{8}$	Px	51762	32261	2	Fx2	50000	36184	3	F	47761	26400
$1\frac{7}{8}$	M	50363		2	Fx3	50763	33172	$3\frac{1}{4}$	F	47014	24591
$1\frac{7}{8}$	A	50584	28713	2	A	50171	28983	$3\frac{1}{2}$	F	47000	24961
$1\frac{7}{8}$	F	51039	33067	2	F	48596	27634	$3\frac{3}{4}$	F	46667	23636
$1\frac{7}{8}$	Fx2	51159	33970	2	F	47812	35864	4	F	46322	23430
$1\frac{7}{8}$	F	49744	35615	2	F	47569	28792				

TENSILE TESTS MADE SUBSEQUENTLY AT THE WATERTOWN ARSENAL.

Here will next be given, in tabulated form, the results of a number of tensile tests made on the government machine at the Watertown Arsenal.

The following tables of results on rolled bars, from the Elmira Rolling-Mill Company (mark L) *and from the Passaic Rolling-Mills (mark* S), *are given in Executive Document 12, 47th Congress, 1st session, and in Executive Document 1, 47th Congress, 2d session.*

SINGLE REFINED BARS.

Mark on Bar.		Sectional Area, in square inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation in 80 inches, %.	Contraction of Area, %.	Appearance of Fracture.		Modulus of Elasticity at Load of 20000 Lbs. per Square Inch.	
							Fibrous, %.	Crystalline, %.		
L	1	3.06	28500	52710	18.4	33.3	95	5	26981450	
L	2	3.06	29500	53630	16.4	36.0	92	8	27826036	
L	3	3.06	29000	52090	21.4	34.6	95	5	28419182	
L	4	3.06	29000	51440	15.0	20.3	90	10	30888030	
L	5	6.46	27500	50500	14.5	27.6	95	5	27826036	
L	6	6.40	27500	50530	17.3	22.3	70	30	27118644	
L	7	6.39	27000	50200	18.0	22.5	95	5	27444253	
L	8	3.24	–	51667	22.0	36.0	70	30	28318584	Round.
L	9	3.20	–	50844	16.3	22.0	15	85	27972027	"
L	10	3.20	–	53062	21.0	40.0	95	5	28119507	"
S	11	3.08	28500	48640	13.3	24.3	100	Slightly	27586206	
S	12	3.08	28000	50390	16.9	35.1	100	0	27586206	
S	13	3.05	28500	47050	9.0	22.0	100	0	27874564	
S	15	6.40	26000	49700	17.1	19.2	85	15	29906542	
S	16	6.40	24000	49280	15.7	17.7	85	15	26490066	
S	17	6.41	24500	48740	14.3	17.3	80	20	28119507	
S	18	3.17	24600	49680	19.5	32.0	100	Slightly	27972027	Round.
S	19	3.17	25870	49338	18.3	38.0	100	0	29357798	"
S	20	3.17	24920	48864	18.4	37.0	100	Cinder at centre	27729636	"

DOUBLE REFINED BARS.

Mark on Bar.	Sectional Area, in square inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation in 80 inches, %.	Contraction of Area, %.	Appearance of Fracture.		Modulus of Elasticity at Load of 20000 lbs. per Square Inch.	
						Fibrous, %.	Crystalline, %.		
L 201	3.06	29000	53560	15.3	37.9	100	0	27633851	
L 202	3.03	30000	52650	16.2	20.6	85	15	34042553	
L 203	3.06	32500	53500	16.5	27.5	100	0	28169014	
L 204	3.06	32500	54480	15.4	24.8	100	0	29090909	
L 205	6.33	27000	51230	17.8	24.2	80	20	28119507	
L 206	6.34	27500	50500	17.6	21.1	100	Slightly	29629629	
L 207	6.34	27000	51030	21.4	31.9	100	0	27826086	
L 208	3.20	–	50156	22.7	43.0	100	Cup-shaped	28021015	Round.
L 209	3.20	–	49937	22.6	45.0	100	"	28622540	"
L 210	3.20	–	50188	19.9	43.0	100	"	28985507	"
S 211	3.05	29500	51150	22.0	31.5	100	0	32989690	
S 212	3.05	28500	51110	22.0	36.1	100	0	25559105	
S 213	3.11	29500	51860	22.5	39.2	100	0	26446280	
S 215	6.31	27500	50980	19.1	23.6	95	5	29357798	
S 216	6.38	27000	50770	20.7	29.6	100	0	28268551	
S 217	6.33	27000	51340	19.3	35.2	100	0	28070175	
S 218	3.17	24610	50631	20.4	41.0	100	0	28622540	Round.
S 219	3.17	–	50915	25.5	44.0	100	Cup-shaped	28268551	"
S 220	3.17	–	50205	23.7	44.0	100	"	28070175	"

The moduli of elasticity had not been computed in the report, but have been computed in these tables from the elongations under a load of 20000 lbs. per square inch in each case, as recorded in the details of the tests.

In these reports are also to be found tensile tests of iron from other companies, as the Detroit Bridge Company, the Phœnix Company, the Pencoyd Company, etc. Some of these

tests were made to determine the effect of rest upon the bar after it had been strained to its ultimate strength. Some were made to determine the values of the modulus of elasticity of the same iron for tension and for compression; and these were found experimentally to be almost identical, as was to be expected. For these tests the student is referred to the reports themselves; and only certain tests on eye-bars of the Phœnix Company will be appended here. Some of them were tested only for modulus of elasticity, which are not calculated in the report, but are given here.

Arsenal Number.	Outside Length, inches.	Gauged Length, inches.	Sectional Area, square inches.	Modulus of Elasticity.	Ultimate Strength, lbs., per Sq. In.	Contraction of Area at Fracture, %.
509	95.90	70	0.765	25309000		
510	95.90	70	0.765	27035000		
511	67.75	50	1.478	22391000	40600	16.8
512	67.75	50	15.090	25075000		
513	67.80	50	1.940	24727000	39480	13.9
514	118.75	90	1.959	24590000		
515	118.75	90	1.959	25993000		
516	96.05	75	2.954	25388000		
517	109.72	75	3.800	25969000		
518	96.05	75	5.103	23338000	46720	8.1

Quite a number of tests of the iron of different American companies are to be found in the "Report on the Progress of Work on the Cincinnati Southern Railway," by Thomas D. Lovett, Nov. 1, 1875.

For these the student is referred to the report named.

SPECIAL MODULUS OF ELASTICITY.

In connection with Hodgkinson's experiment on a bar 50 feet long, attention was called to the fact that the modulus of elasticity remained more nearly constant in wrought than in cast iron for loads below the limit of elasticity, and that, after passing this limit, the modulus of elasticity decreases rapidly, inasmuch as the strain increases very much more rapidly than the stress.

This could be represented graphically by drawing a curve, having for abscissæ of its several points the loads per square inch applied to the piece, and for ordinates the elongations or shortenings. Such a curve would, in the case of wrought-iron, be nearly a straight line up to the limit of elasticity, and then would rise very rapidly.

If, now, we should plot another curve with the same abscissæ as before, but having for ordinates the permanent sets under the respective loads, this curve would start from the axis of abscissæ at the limit of elasticity, and rise rapidly from that point on.

Again: if another curve be drawn, having the same abscissæ as before, but having for ordinates in every case the differences of the ordinates of the other two curves, such a curve would represent, by its ordinate at any point, the recoil or the stretch of the piece over and above the permanent set; and this ordinate would, in the absence of further experiment, appear to represent the amount it would stretch if the load were removed and immediately applied again, over and above that part of the stretch which did not disappear on the removal of the load. The modulus of elasticity computed by means of the elongation above described, considered as the elongation of the piece, has been given the name "special modulus of elasticity" by Col. Rosset of the arsenal at Turin; and he has called attention to the fact, that in the case of wrought-iron,

and also of steel, this special modulus of elasticity is nearly constant, even though the load applied be far above the limit of elasticity.

This can be graphically shown from the fact that the third of the above-mentioned curves, if plotted, will be nearly a straight line almost up to the point of fracture in the case of wrought-iron and steel.

EXAMPLES.

1. Plot the curves referred to above, for Hodgkinson's test of a wrought-iron rod 50 feet long, recorded on p. 353.

2. Do the same for the table of experiments on cast-iron bars 10 feet long, recorded on p. 339.

WROUGHT-IRON BOILER-PLATE.

Some tests of iron boiler-plate have already been quoted; viz., Kirkaldy's. Some tables of tests made by Mr. C. B. Richards, and recorded in the "Transactions of the American Society of Civil Engineers," vol. ii., will next be given.

In these tables, L denotes that they were pulled lengthwise of the fibre, and C crosswise.

Long specimens are those whose smallest area of cross-section had a length of from 3 to 5 inches.

Short specimens are those whose smallest area of cross-section had no length.

No. of Specimen.	Kind of Iron.	Nominal Shape of Specimen.	Direction of Lamination.	Approximate Dimensions of Original Section.	Tensile Strength per Square Inch of Original Cross-Section.				Resistance per Square Inch of Fractured Area.	Contraction of Area, %.
					Strongest Specimen.	Weakest Specimen.	Averages.			
6	Bay State Flange	Long	L.	1.25 × 0.29	47785	46484	47017	47450	64411	27.0
3	" " "	"	C.	0.75 × 0.29	49113	46815	47884		56755	14.0
6	" " "	Short	L.	1.25 × 0.29	52993	50770	51943	52102	61295	15.3
3	" " "	"	C.	0.75 × 0.29	53161	51597	52262		58170	10.2
14	" " "	Long	L.	1.25 × 0.30	51378	44036	48098	47187	63596	24.8
12	" " "	"	C.	" "	49023	39898	46277		52349	10.7
4	Bay State C. No. 1	"	L.	" "	48819	46086	47725	46013	55967	14.5
4	" " " "	"	C.	" "	45240	42961	44301		48849	9.2
4	Bay State Homogeneous Metal,	"		" "	71139	70100	70672		136473	52.0
2	Thorneycroft English	Short	L.	0.87 × 0.27	47245	46410	46827	45293		
2	" " . . .	"	C.	" "	44355	43165	43760			
3	Pennsylvania Common . . .	"	L.	0.87 × 0.16	54699	44581	49227	48434		
3	" " . . .	"	C.	" "	54031	43436	47641			
1	Pennsylvania C. No. 1 . . .	"	L.	0.87 × 0.28			48660	51483		
2	" " " . . .	"	C.	" "	55218	53395	54306			
1	Pennsylvania Flange	"	L.	" "			54466	53733		
2	" "	"	C.	" "	54819	51184	53001			
4	Bay State C. No. 1	"	L.	" "	58450	51992	54264	53096		
4	" " " "	"	C.	" "	53145	50449	51928			
2	Bay State Flange	"	L.	" "	57934	54377	56165	54925		
2	" " "	"	C.	" "	53998	53395	53696			
2	Sligo Fire-Box	"	L.	" "	53791	52546	53168	52750		
2	" "	"	C.	" "	54394	50272	52333			
1	Specimen from different source.	"	L.	1.265 × 0.329			60911			

Also the following results on rivet-iron manufactured by Burden & Sons : —

Test No.	Nominal Shape of the Specimen.	Diameter, in inches.	Length, in inches.	Tensile Strength per Square Inch.		Limit of Elasticity, lbs., per Sq. Inch.	Contraction of Area, pr. cent.	Ultimate Elongation, per cent.
				Original Area.	Fractured Area.			
324	Long,	0.620	5.0	49206	93459	31450	47	33
325	"	0.619	5.0	49734	94218	26580	47	32
326	"	0.619	5.0	49535	93774	28240	47	29
327	"	0.619	5.0	49202	97434	26570	49	32
332	"	0.711	5.0	49345	100000	26460	50	34
333	"	1.000	5.0	49974	86454	–	42	28
334	"	1.000	5.0	49924	91186	–	45	29
335	"	1.000	5.0	49784	93430	–	47	26
328	Short,	0.614	–	62230	90620	–	30	–
329	"	0.616	–	61670	88040	–	31	–
330	"	0.616	–	60840	85358	–	29	–
331	"	0.615	–	60407	87941	–	31	–
336	"	1.002	–	63840	88783	–	28	–
337	"	1.000	–	63547	88025	–	28	–

Perhaps it ought to be stated that the boiler-plate tested and recorded in the first of these tables is boiler-plate of good quality in every case.

BRANDS OF PLATES.

The only safe way, in ordering plates for a boiler, as in ordering iron for any construction, is to prescribe the tests which it shall stand, as the same brands are used by different makers to denote very different qualities of iron.

As a rule, better iron will be obtained by purchasing at a mill where only superior qualities are manufactured, rather than at mills where all qualities are made.

In England the brands are very various, but are often graded "Best," "Best Best," and "Treble Best," the "Best" being the poorest of the three.

In the United States these brands are not used, and the usage varies with different mills. Some years ago the brands manufactured were as follows: —

1°. *Tank-Iron*, the lowest grade, not suitable for use in the shell of a boiler, and including all iron too poor for this purpose.

2°. *C. No. 1.* — This used to denote iron made by using only charcoal in the blast-furnace; and it furnished a good grade of iron, suitable to use in those parts of the shell of a steam-boiler which do not come in contact with the fire, and which do not have to be flanged.

3°. *C. H. No. 1*, or charcoal-hammered No. 1, which was also a charcoal iron, but was obtained by more working than C. No. 1, and was hence a better grade of iron.

4°. *Fire-box Iron*, suitable to use for sheets exposed to the fire.

5°. *Flange Iron*, or such as will flange without cracking.

These brands partly remain to-day, but they no longer bear the same meanings. Indeed, the name "C. H. No. 1" is, in many mills, given to iron in the manufacture of which no charcoal has been used.

There are a number of mills that use the following brands for boiler-plates; viz., —

Tank-Iron.	Extra Flange.
Refined Iron.	Shell Fire-Box.
Shell Iron.	C. H. No. 1 Fire-Box.
C. H. No. 1.	Extra Flange Fire-Box.

Nevertheless, if we buy any one of these grades at different mills, we have no certainty of obtaining the same quality of iron; and, moreover, there are other mills that use different brands. "Refined iron," for instance, is a term that includes a

large number of qualities, beginning at much too low a grade to put in a boiler-shell.

The only sure way to secure good iron is to prescribe the tests it shall stand: i.e., the tensile strength, which should be over 45000 or 46000 lbs.; the limit of elasticity, which should be as much as 27000 or 28000 lbs.; the contraction of area at fracture, which should be at least 30 per cent. It should also stand bending double cold, red-hot, and at a flanging heat. It would be well to test also the soundness of the plates by punching them.

BRANDS OF BARS AND SHAPES.

For these, each mill has its own peculiar brands, to which it attaches its own signification. The only way to secure iron of whose quality we are sure, is, therefore, to require that it shall stand the suitable tests before being accepted.

COMPRESSIVE STRENGTH OF WROUGHT-IRON.

As far as the compressive strength of short cylinders of wrought-iron is concerned, we have hardly any direct experimental evidence; because the iron, especially if it is soft, will gradually flatten out, and not give way by suddenly breaking. The evidence furnished by Fairbairn's experiments on transverse strength of wrought-iron goes to show that the tensile and the crushing strength are nearly equal, though the earlier experiments of Fairbairn had given a less value for the crushing than for the tensile strength; and for a long time the crushing-strength of wrought-iron has been given as 36000 lbs. per square inch, which value is still to be found in Gordon's formula for the strength of wrought-iron columns.

Leaving this matter to be explained under the head of "Transverse Strength of Wrought-Iron," we will next proceed to consider the strength of wrought-iron columns.

§ 223. **Wrought-Iron Columns.** — Until a very recent date, we have had no experimental knowledge on this subject beyond the experiments of Hodgkinson, which have furnished the constants for Hodgkinson's, and also for Gordon's, formula, as already given in § 209 and § 211.

These formulæ have been in very general use, and it is only very recently that we have been able to test their accuracy by tests on full-size wrought-iron columns. The disagreement of the formulæ already referred to, with the results of the tests, has led to the proposal of a large number of similar formulæ, each having its constants determined to suit a certain definite set of tests, and hence all these formulæ thus proposed must be classed as empirical formulæ, and can only be applied with safety within the range of the cases experimented upon.

A number of these will now be enumerated: and then will follow tables of the actual tests, which furnish the best means of determining the strength of these columns; and it would appear that it is these tables themselves which the engineer would wish to use in designing any structure.

On the 15th of June, 1881, Mr. Clark, of the firm of Clark, Reeves, & Co., presented to the American Society of Civil Engineers a report of a number of tests on full-size Phœnix columns, made for them at the Watertown Arsenal, together with a comparison of the actual breaking-weights with those which would have been obtained by using the common form of Gordon's formula for wrought-iron,

$$\frac{P}{A} = \frac{36000}{1 + \dfrac{l^2}{36000\rho^2}},$$

where $P =$ breaking-weight in lbs., $A =$ area of section in square inches, $l =$ length in inches, $\rho =$ least radius of gyration in inches. The table is as follows: —

No. of Experi-ment.	Length of Column.	Ratio of Diam-eter to Length.	Weight.	Sectional Area.	Total Compression under Loads.		Elastic Limit.		Ultimate Strength.		Total Ultimate Strength, in lbs., by Gordon's Formula.
					200000 lbs.	300000 lbs.	Total lbs.	Lbs. per Square Inch.	Total lbs.	Lbs. per Square Inch.	
	ft.		lbs.	sq. in.							
1	28	42	1142	12.062	0.190	–	–	–	424000	35150	330146
2	28	42	1153	12.181	0.186	–	–	–	416000	34150	333459
3	25	37½	1034	12.233	–	0.255	342000	27960	431500	35270	352013
4	25	37½	1023	12.100	0.168	0.264	–	–	424000	35040	348119
5	22	33	920	12.371	0.160	0.243	–	–	440000	35570	372837
6	22	33	–	12.311	0.152	0.236	–	–	423000	34360	371043
7	19	28½	773	12.023	–	0.198	–	–	425200	35365	377955
8	19		777	12.087	0.139	0.213	354000	29290	446000	36900	380197
9	16	24	650	12.000	0.120	–	–	–	439000	36580	391701
10	16		650	12.000	0.116	–	–	–	439000	36580	391701
11	13	19½	536	12.185	0.092	0.142	342000	28890	449000	36857	410660
12	13		531	12.009	0.091	–	–	–	449000	37200	406866
13	10	15	415	12.248	–	0.110	330000	26940	446800	36480	423886
14	10		418	12.339	–	0.109	350000	28360	449100	36397	427047
15	7	10½	291	12.265	0.054	–	360000	29350	468000	38157	433021
16	7		284	11.962	–	–	354000	29590	517000	43300	469324
17	4	6	164	12.081	0.031	–	–	–	598000	49500	432132
18	4	6	164½	12.119	0.025	0.042	340000	28050	621000	51240	433507
	in.										
19	8	1	27½	11.903	0.008	0.013	–	–	680000	57130	–
20	8	1	27½	11.903	0.007	0.011	–	–	682000	57300	–
21	25′ 2″.63	25½	1561	18.300	0.115	0.178	–	–	659000	36010	–
22	8′ 9″.50	9	544	18.300	0.045	0.067	540000	29510	772000	42180	–

On the other hand, if the breaking-weights of these columns were calculated from the formula

$$\frac{P}{A} = \frac{50000}{1 + \frac{l^2}{36000\rho^2}},$$

the results would be much nearer the actual results, though not very near. The Phœnix Company, in their handbook, give, for the strength of their columns,

$$\frac{P}{A} = \frac{50000}{1 + \frac{l^2}{3000h^2}},$$

where h = diameter in inches.

The set of tests above referred to, gave rise to considerable discussion, and the proposal of several modified forms of Gordon's formula, as well as other formulæ, which should agree more nearly with the results of the tests.

Mr. Bouscaren proposes the formula

$$\frac{P}{A} = \frac{38000}{1 + \frac{l^2}{100000\rho^2}},$$

and claims that this furnishes a very good degree of coincidence with the facts.

Mr. Theodore Cooper proposes, instead, a modified form of Gordon's formula, as follows: —

For square-ended columns . . . $\dfrac{P}{A} = \dfrac{f}{1 + \dfrac{\left(\frac{l}{\rho} - 80\right)^2}{18000}}.$

For pin-ended columns . . . . $\dfrac{P}{A} = \dfrac{f}{1 + \dfrac{\left(\dfrac{l}{\rho} - 33\right)^2}{18000}}.$

For round-ended columns . . . $\dfrac{P}{A} = \dfrac{f}{1 + \left(\dfrac{l}{\rho}\right)^2 \dfrac{1}{18000}}.$

And he gives, for the values of f,

For Phœnix columns . . . . $f = 36000.$
For American Company's columns . . . $f = 30000.$
For box and open columns . . . . . $f = 31000.$

He deduces these values of f from some tests made in 1875 by Mr. Bouscaren, combined with those, already referred to, made at the Watertown Arsenal. The box and open columns were made of channel-bars and latticing. The tables or diagrams presented to justify all these formulæ will not be given here, but any one can find them in the "Transactions of the American Society of Civil Engineers" for 1882. Mr. C. E. Emery proposed, at the same meeting, the formula

$$\frac{P}{A} = \frac{355063 + 30950x}{x + 6.175}$$

for the Phœnix columns tested, where x = number of diameters in the length.

Professor Burr proposes

$$\frac{P}{A} = \frac{42000\left[1 + \log_e\left(1 + \dfrac{\rho}{l}\right)\right]}{1 + \dfrac{1}{50000}\dfrac{l^2}{\rho^2}}.$$

Professor Merriman proposes

$$\frac{P}{A} = \frac{37200}{1 + \frac{1}{158500}\frac{l^2}{\rho^2}}.$$

The above are the formulæ evoked from the discussion on the set of tests of Phœnix columns above referred to.

In the "Report of Progress of Work of the Cincinnati Southern Railway," by Thomas D. Lovett (Nov. 1, 1875), is given an account of some tests made on full-size wrought-iron columns. He concludes that the formula

$$\frac{P}{A} = \frac{f}{1 + a'\frac{l^2}{\rho^2}}$$

will give results agreeing fairly well with the facts, provided we use

For flat ends. . . . . $a' = \frac{1}{36000}$,

For flat at one end and round at the other, $a' = \frac{2}{36000}$,

and for f we put the crushing-strength of the iron per square inch.

On the other hand, Professor Burr deduces from these same tests the following formulæ (see Burr's "Elasticity and Resistance of Materials," p. 441) : —

KEYSTONE COLUMNS.

Flat ends (swelled columns) . . . $\frac{P}{A} = \frac{36000}{1 + \frac{l^2}{18300\rho^2}}$.

Flat ends straight $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ . . . . $\dfrac{P}{A} = \dfrac{39500}{1 + \dfrac{l^2}{18300\rho^2}}$.

Pin ends swelled . . . . $\dfrac{P}{A} = \dfrac{36000}{1 + \dfrac{l^2}{15000\rho^2}}$.

SQUARE COLUMNS.

Flat ends . . . . $\dfrac{P}{A} = \dfrac{39000}{1 + \dfrac{l^2}{35000\rho^2}}$.

Pin ends . . . . $\dfrac{P}{A} = \dfrac{39000}{1 + \dfrac{l^2}{17000\rho^2}}$.

PHŒNIX COLUMNS.

Flat ends. . . . . $\dfrac{P}{A} = \dfrac{42000}{1 + \dfrac{l^2}{50000\rho^2}}$.

Round ends . . . . $\dfrac{P}{A} = \dfrac{42000}{1 + \dfrac{l^2}{12500\rho^2}}$.

Pin ends . . . . $\dfrac{P}{A} = \dfrac{42000}{1 + \dfrac{l^2}{22700\rho^2}}$.

AMERICAN BRIDGE COMPANY'S COLUMNS.

Flat ends . . . . $\dfrac{P}{A} = \dfrac{36000}{1 + \dfrac{l^2}{46000\rho^2}}.$

Round ends . . . . $\dfrac{P}{A} = \dfrac{36000}{1 + \dfrac{l^2}{11500\rho^2}}.$

Pin ends . . . . $\dfrac{P}{A} = \dfrac{36000}{1 + \dfrac{l^2}{21500\rho^2}}.$

Professor Burr has also determined the values of f and a in the formula

$$\frac{P}{A} = \frac{f}{1 + a\dfrac{l^2}{\rho^2}},$$

which should agree most nearly with the different sets of experimental results. He has also done the same for the values of a and n in the formula

$$\frac{P}{A} = a\left(\frac{\rho}{l}\right)^n,$$

this latter being a modification of Euler's formula; and he claims that the latter gives results agreeing better with experiment than the former.

The tests made at the Watertown Arsenal will next be given, together with cuts showing the form of the columns; these being taken from the government report, Executive Document 12, 47th Congress, first session.

LATTICE-COLUMNS AND CHANNEL-BARS FROM DETROIT BRIDGE AND IRON COMPANY.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS.

LATTICED COLUMNS.

Columns with Pin Ends tested, with Pins Vertical; Channel-Bars spaced 8 Inches apart.

No. of Test.	Kind.	Chan-nel-Bars.	Length.	Sectional Area.	Lattice Spacing	Ultimate Strength. Actual.	Ultimate Strength. Per Square Inch.	Manner of Failure.
		in.	ft. in.	sq. in.	in.	lbs.	lbs.	
1059	Flat ends	6	10 0	4.760	18	174800	36720	Channel-bars buckled.
1060	" "	6	10 0	4.670	18	165000	35330	" "
1095	One flat end, one pin end,	6	10 0	4.750	18	160000	33680	" "
1096	" " " " " "	6	10 0	4.580	18	154800	33800	" "
1107	Pin ends	6	12 0	4.600	18	159800	34740	Horizontal deflection.
1108	" "	6	12 0	4.570	18	156100	34160	" "
1	" "	6	12 6	4.560	18	163600	35880	" "
2	" "	6	12 6	4.740	18	153500	32380	" "
1231	" "	6	15 0	4.480	18	151500	33820	" "
1232	" "	6	15 0	4.560	18	157500	34540	" "
1229	" "	6	17 6	4.660	18	152600	32750	" "
1230	" "	6	17 6	4.740	18	147500	31120	" "
1117	" "	6	20 6	4.660	18	136000	29180	" "
1118	" "	6	20 0	4.630	18	143500	30990	" "
1119	" "	6	22 6	4.570	18	139800	30590	" "
1120	" "	6	22 6	4.660	18	144700	31050	" "
1121	" "	6	25 0	4.710	18	110000	23350	" "
1122	" "	6	25 0	4.630	18	117500	25360	" "
20	" "	6	27 6	4.690	18	102500	21850	" "
21	" "	6	27 6	4.670	18	97200	20810	" "
18	" "	6	30 0	4.700	18	69300	14740	" "
19	" "	6	30 0	4.730	18	75200	15900	" "
1111	" "	8	13 4	7.520	18	261800	34810	Defl. upward; chan.-bars buckled.
1112	" "	8	13 4	7.500	18	264300	35240	" horizon. " "

LATTICED COLUMNS. — *Concluded.*

No. of Test.	Kind.	Channel-Bars.	Length.		Sectional Area.	Lattice Spacing	Ultimate Strength.		Manner of Failure.
							Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	in.	lbs.	lbs.	
1113	Pin ends	8	16	8	7.480	18	254100	33970	Defl. horizon.; chan.-bars buckled.
1114	" "	8	16	8	7.480	18	251400	33610	" " " "
1115	" "	8	20	0	7.550	18	246200	32610	" " " "
1116	" "	8	20	0	7.510	18	241400	32140	" " and vertically.
1123	" "	8	23	4	7.990	18	257500	32230	" "
1124	" "	8	23	4	7.670	18	240600	31370	" "
24	" "	8	26	8	7.780	18	243900	31350	" "
25	" "	8	26	8	7.750	18	215800	27850	" "
22	" "	8	30	0	7.810	18	194100	24850	" "
23	" "	8	30	0	7.800	18	210000	26920	" "
13	" "	10	12	6	9.680	22	344100	35550	Channel-bars buckled.
14	" "	10	12	6	9.590	22	339000	35350	" "
11	" "	10	16	8	9.550	22	323200	33840	" "
12	" "	10	16	8	9.610	22	326700	34000	" "
3	" "	10	20	10	9.740	22	330000	33880	" "
4	" "	10	20	10	9.806	22	330100	33660	Defl. diagonally.
5	" "	10	25	0	10.040	22	342700	34130	" " chan.-bars buckled.
6	" "	10	25	0	10.000	22	319300	31930	" " " "
26	" "	10	29	2	9.300	22	299300	32180	" horizontally.
27	" "	10	29	2	9.570	22	281200	29380	" "
1109	" "	12	10	0	12.150	22	406000	33420	Channel-bars buckled.
17	" "	12	10	0	12.060	22	423000	35070	" "
15	" "	12	15	0	12.120	22	410000	33830	" "
16	" "	12	15	0	12.470	22	442600	35490	" "
9	" "	12	20	0	11.980	22	411600	34360	" "
10	" "	12	20	0	12.340	22	414800	33610	" "
7	" "	12	25	0	12.144	22	400000	32940	" "
8	" "	12	25	0	11.910	22	407800	34240	" "
28	" "	12	30	0	12.180	22	385000	31610	" "
29	" "	12	30	0	12.540	22	393000	31340	Deflected horizontally.

COMPRESSION TESTS OF CHANNEL-BARS; SIZES USED IN THE PRECEDING LATTICE-COLUMNS.

No. of Test.	Kind.	Size of Bars.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	in.	sq. in.	lbs.	lbs.	
1049	Flat ends . .	6	6.00	2.33	98530	42290	Flanges buckled inward.
1050	" " . .	6	6.00	2.33	99840	42695	" " "
1071	" " . .	6	17.58	2.37	86900	36670	Deflection.
1072	" " . .	6	17.70	2.23	82500	37000	"
1069	" " . .	6	23.83	2.23	78400	35160	"
1070	" " . .	6	23.90	2.37	77400	32660	"
1064	" " . .	6	48.00	2.38	66980	28140	"
1051	" " . .	8	8.00	3.85	164700	42780	Flanges buckled inward.
1052	" " . .	8	8.00	3.85	168690	43810	Web buckled outward, flanges inward.
1068	" " . .	8	17.90	3.73	131600	35280	Flanges bent outward.
1065	" " . .	8	23.85	3.73	136300	36540	Deflection.
1066	" " . .	8	23.85	3.73	132100	35410	"
1067	" " . .	8	29.90	3.73	124600	33400	"
1063	" " . .	8	48.00	3.73	114200	30620	"
1053	" " . .	10	10.00	4.78	166400	34810	Flanges buckled inward.
1054	" " . .	10	10.00	4.78	169000	35350	" " outward.
1074	" " . .	10	17.85	4.76	161000	33820	" " inward.
1075	" " . .	10	23.90	5.04	176800	35080	" " outward.
1076	" " . .	10	23.87	5.04	169500	33630	" " inward.
1073	" " . .	10	29.90	4.76	162100	34050	" " "
1062	" " . .	10	48.00	4.76	162250	34080	Deflection.
1055	" " . .	12	12.00	5.97	222300	37240	Web bent inward, flanges outward.
1056	" " . .	12	12.00	5.97	222300	37240	" " " " "
1079	" " . .	12	17.84	5.95	217700	36590	" " " " "
1077	" " . .	12	23.92	6.02	218800	36350	" " outward, " inward.
1078	" " . .	12	23.87	6.02	223000	37040	" " " " "
1080	" " . .	12	29.90	5.96	209500	35150	" " " " "
1061	" " . .	12	48.00	6.19	223100	36040	" " inward, " outward.

LATTICED COLUMNS BUILT BY THE DETROIT BRIDGE AND IRON COMPANY.

Columns tested with Pins Vertical; Channel-Bars spaced 6 Inches apart.

No. of Test.	Kind.	Channel-Bars.	Length.		Sectional Area.	Lattice Spacing.	Ultimate Strength.		Manner of Failure.
							Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	in.	lbs.	lbs.	
463	Pin ends . . .	6	20	0	4.68	16	117000	25000	Deflected horizontally.
464	" " . . .	6	25	0	4.68	16	71400	15260	" "
465	" " . . .	8	20	0	7.75	16	215100	27750	" "
466	" " . . .	8	25	0	7.75	16	201500	26000	" "
467	" " . . .	10	20	0	9.19	18	275500	29980	" upward.
468	" " . . .	10	25	0	9.19	18	294080	32000	" horizontally.
469	" " . . .	12	20	0	12.95	–	375200	28970	" upward.
470	" " . . .	12	25	0	12.95	–	388500	30000	" horizontally.

CIRCULAR COLUMNS, 4 SEGMENTS, BUILT BY THE PHŒNIX IRON COMPANY.

Tested with Diameter through Flanges 45° from Vertical ⌑.

No. of Test.	Kind.	Diameter.	Length.		Sectional Area.	Ultimate Strength.		Manner of Failure.
						Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	lbs.	lbs.	
325	Flat ends . .	8.04	0	30.000	11.610	651000	56070	Web bulged near end.
327	" " . .	8.04	0	29.940	11.902	628500	52800	" " " middle of column.
326	" " . .	8.00	11	10.625	12.181	466000	38256	Deflected downward.
31	" " . .	8.00	31	0.000	11.430	356000	31150	" upward and horizontally.
32	" " . .	8.00	31	0.000	11.310	370500	32760	" " " "
33	" " . .	8.04	31	6.000	11.660	363600	31180	" " " "
34	" " . .	8.04	31	6.000	11.510	373100	32220	" " " "

AMERICAN BRIDGE COMPANY'S STEEL COLUMNS.

Column with Pin Ends tested, with Pins Vertical.

No. of Test.	Kind.	Size.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	ft. in.	sq. in.	lbs.	lbs.	
1058	Flat ends	10.27 × 10.30	0 43.10	15.28	719000	47055	Flanges buckled.
30	Pin ends	10.27 × 80.35	30 0.44	15.28	290000	18980	Deflected horizontally.

LATTICED COLUMN BUILT BY THE KELLOGG BRIDGE COMPANY.

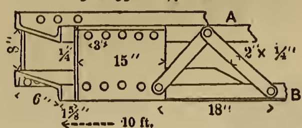
Column tested with Pin Vertical.

No. of Test.	Kind.	Channel-Bars.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	ft. in.	sq. in.	lbs.	lbs.	
492	One flat end, one pin end,	9.92	21 8	17.65	581000	32920	Deflected horizontally and upward.

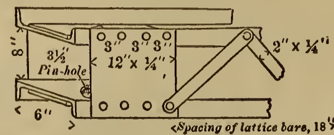
The following are the figures showing the columns of which the tests were recorded in the tables given above:—

1–2–1059–1060.

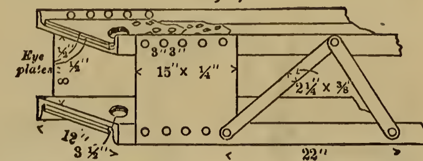
Latticing staggered ¾ inch rivets



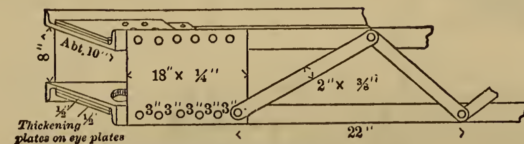
1095–1096–1107–1108.

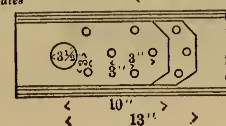


1109–17.



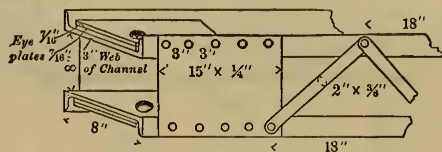
3–4–5–6–11–12–13–14.



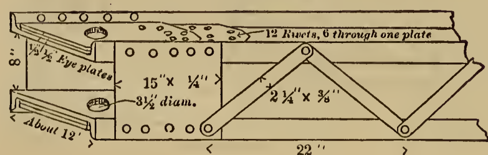


Sketch showing riveting of eye plates.

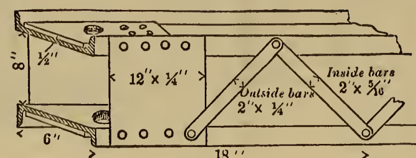
1111–1112–1113–1114–1115–1116–1117–1118–1119–1120–1121
1122–1123–1124–1229–1230–1231–1232.



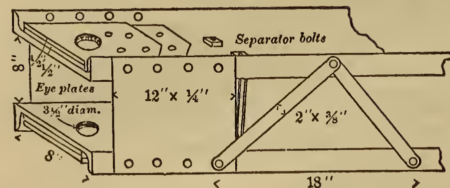
7–8–9–10–15–16–28–29.



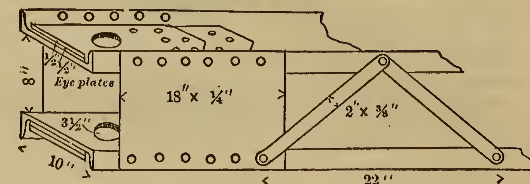
18–19–20–21.



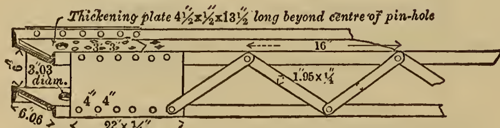
22–23–24–25.



26–27.

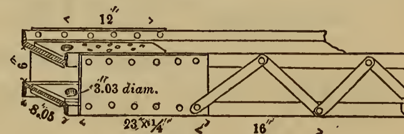


463.

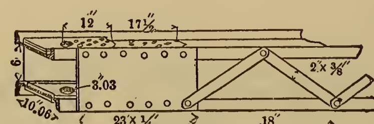


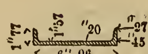
$r = 25.80$
$r_2 = 5.51$

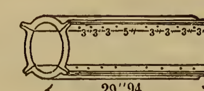
465–466.

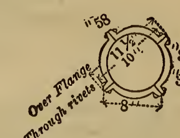


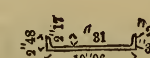
467–468.











In determining the strength of a bridge column made of channel-bars and latticing, these results of tests on full-size columns furnish us the best data upon which to base our

conclusions; and the formulæ proposed by the different persons who have deduced formulæ to cover special sets of values may be advantageously used for cases intermediate between those that have been experimented upon.

§ 224. **Transverse Strength of Wrought-Iron.** — In all probability wrought-iron owes its extensive introduction into construction as much or more to the efforts of Sir William Fairbairn than to any one else; and while he was furnishing the means to Eaton Hodgkinson to make extensive experiments on cast-iron columns, and while he made experiments himself on cast-iron beams, which were in use at that time, he also carried on a large number of tests on beams built of wrought-iron, more especially those of tubular form, and those having an **I** or a **T** section, and made of pieces riveted together. In his book on the "Application of Cast and Wrought Iron to Building Purposes" he gives an account of a large number of these experiments, including those made for the purpose of designing the Britannia and Conway tubular bridges, a fuller account of which will be found in his book entitled "An Account of the Construction of the Britannia and Conway Tubular Bridges." In the first-named treatise he urges very strongly the use of wrought-iron, instead of cast-iron, to bear a transverse load.

Fairbairn tested a number of wrought-iron built-up beams with different thicknesses of upper and lower flange. The first results he obtained showed, that, unless the upper plates were made very much thicker than the lower, the beam would invariably give way by crippling at the top.

At first he concluded that the tensile strength of wrought-iron is greater than its compressive strength, and it has often been so regarded.

Subsequent experiments made by him, however, showed, that, if the iron of the upper flange were distributed in the form of cells, the areas of the cross-sections required in order to

render the beam equally liable to give way by tearing or by crushing, were as 12 to 11, or nearly equal; thus tending to show that the tensile and compressive strengths of wrought-iron are nearly equal, and that the reason of the crippling in the first experiments was, that the iron between the rivets acted like a column, and bent, instead of bringing into play the entire compressive strength of the iron.

A summary of some of these first experiments, the details of which are in his "Application of Cast and Wrought Iron to Building Purposes," is given in the following table, taken from p. 116 of his book: —

No. of Experiment.	Distance between Supports.	Depth, in inches.	Width, in inches.	Thickness of Top Plate.	Thickness of Bottom Plate.	Ultimate Deflection, in inches.	Breaking-weight, in lbs.	Manner of Failure.
	ft. in.							
14	17 6	9.60	9.60	0.075	0.074	1.12	3738	Compression.
14 *a*	17 6	9.60	9.60	0.252	0.075	1.10	8273	Tension.
15	17 6	9.60	9.60	0.076	0.143	0.94	3788	Compression.
15 *a*	17 6	9.60	9.60	0.143	0.076	1.76	7148	Compression.
16	17 6	18.25	9.25	0.149	0.269	1.03	6812	Compression.
16 *a*	17 6	18.25	9.25	0.269	0.149	1.73	12188	Compression.
17	24 0	15.00	2.25	0.260	0.260	1.61	13120	Compression.
18	18 0	13.25	7.50	0.143	0.143	1.31	10880	Compression.
25	11 0	8.00	1.00	0.282	0.116	0.75	7146	Compression.
29	19 0	15.40	7.75	0.115	0.180	1.59	22469	Side plate tore.

Very few tests have been made which claim to give the modulus of rupture of wrought-iron, for the reason that wrought-iron will bend very much before breaking when it is subjected to a transverse load.

Some experiments made on small pieces claim to show a modulus of rupture considerably in excess of the tensile or

compressive strength, while the few experiments that have been made for this purpose on full-size beams have given a modulus of rupture about equal to the tensile or compressive strength of the iron per square inch.

Of the experiments on deflection, however, we generally have a very uniform result shown for the modulus of elasticity, both from small and from large beams. As to experiments on large beams, besides those of Fairbairn, already referred to, we have: —

1°. Those made at the Watertown Arsenal by Mr. William Sooy Smith and by Col. Laidley.

2°. Some tests made for the Phœnix Iron Company, and recorded in their published handbook.

3°. Some tests made for the New-Jersey Iron and Steel Company, and recorded in their handbook.

Those made at the Watertown Arsenal are recorded in Executive Document 23, 46th Congress, second session, and were made merely on the deflections under moderate loads, and on the elastic limit. From those made on the deflections under moderate loads, the moduli of elasticity are deduced; and they agree very closely with the moduli of elasticity that are deduced from experiments on the tension of wrought-iron. Because those on the elastic limit do not agree with the tensile elastic limit of the outside fibre, does not seem to the writer to justify Mr. Smith in looking for some other element in the resistance besides the direct tensions and compressions recognized in the common theory.

The table of these results will now be given, and then the results obtained from some of the Iron Company's beams.

The experiments were made on T-beams with the flange upwards.

These tests give a higher modulus of elasticity than would seem advisable to use in the present state of our knowledge of the subject.

No. of Beam.	Depth, in inches.	Moment of Inertia.	No. of Trials.	Span, in feet.	Elastic Limit: No. of Trials.	Elastic Limit: Weight applied.	Elastic Limit: One-half Weight of Span.	*f.* Ultimate Strain in Outer Fibre.	*E.* Co-efficient of Elasticity.
1	15	536.56	7	20	6	21225	486	18210	34328500
2	10½	221.86	24	22	19	18647	479	29866	26099400
3	10½	174.75	14	22	14	12741	383	26097	28903000
4	10½	174.75	17	22	15	13798	387	28721	29915000
5	10½	154.90	9	21	5	11476	322	25191	31167000
6	9	106.53	14	22	11	9973	327	28713	32628400
7	9	106.53	12	13	11	17701	192	29478	35409000
8	9	106.53	5	11	4–5	21000	163	29500	30720000
9	8	62.34	8	22	5	6962	241	30508	31859000
10	8	62.34	6	22	5				29270200
11	8	62.34	9	14	8	12237	165	33957	26863750
12	8	62.34	9	14	7	10961	165	30000	28816710
13	7	44.12	6	22	5				32834200
14	7	44.12	5	22	4	5346	236	27413	30836000
15	7	44.12	5	22	4				33333800
16	7	44.12	7	14	6	9704	173	32575	30128700
17	7	44.12	7	14	6				30099300
18	6	24.58	4	22	4	4189	175	34393	33799500
19	6	24.58	6	14	5	6364	114	33207	34057600
20	6	24.58	6	14	5	7631	114	36564	32453400
21	5	12.85	6	11	5	5351	60	34740	34064800
22	5	12.85	6	11	5	5877	60	38116	31631800
23	5	12.85	5	11	5	5877	60	38116	30976000
24	4	7.42	6	11	5	4189	54	37750	36064400
25	4	7.42	5	11	4				32687000
								*30117	†31128260
	Extremes .		. .	. .	. .	. . .	.	25191 38116	26099400 36664400

* Average of 24 tests. † Average of 25 tests.

The New-Jersey Iron and Steel Company give 12000 lbs. per square inch as the safe extreme fibre stress to be used in computing the safe load on any of their beams; and they claim, that, in so doing, they are using a factor of safety of four.

The rule which they give for computing deflections, on the other hand, corresponds to a modulus of elasticity of about 21000000 lbs. per square inch.

The Phœnix Company also give 12000 lbs. per square inch for extreme fibre stress, and claim, that, in so doing, they are using a factor of safety of about four. They give, for modulus of elasticity to be used in computing deflections, 24000000 lbs. per square inch.

Both companies claim to have had some of their beams tested, and that such tests support them in recommending the values given above.

Some of the tables of tests given in these handbooks will be placed here. The following table of tests, made for the New-Jersey Iron and Steel Company by a United-States government engineer, is given in their handbook; and it is stated that they are a few of the tests selected at random.

TESTS OF BEAMS OF NEW-JERSEY IRON AND STEEL COMPANY.

Size of Beam.	Clear Span, in feet.	How Loaded.	Safe Load, in lbs., as given by the Company's Tables.	Load actually applied.	Ratio of Actual to Safe Load.	Effect on Beam.	Calculated Deflection under Safe Load.	Actual Deflection under Safe Load.	Limit of Elasticity. Permanent Bending begins.
6-in. light .	12.00	At centre,	2608	11000	4.3	Failed .	0.27	0.30	7000
6-in. light .	11.93	" "	2624	17000	6.5	"	0.27	0.15	11000
9-in. heavy,	14.93	" "	6330	32000	5.1	"	0.28	0.16	22000
15-in. light .	21.00	Uniformly distributed,	25188	90000	3.6	Deflected 2.7 in.	0.42	0.36	–

The following tables are given by the Phœnix Company as the results of tests made at different times of their rolled beams:—

7-inch Beam. 60 lbs. per Yard. Clear Span, 21 Feet.		
Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.
2000	0.468	–
3000	0.743	0.275
4000	1.020	0.277
5000	1.298	0.278
0	0.029	–
6000	1.578	0.280
0	0.030	–
7000	1.887	0.309
0	0.060	–
8000	2.300	0.413
0	0.183	–
9000	3.540	1.240
9500	5.298	1.758
10000	Broke yielding	slowly, at top.

9-inch Beam. 187 lbs. per Yard. Clear Span, 21 Feet.			
Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.	Remarks.
2000	0.228	–	
4000	0.474	0.246	
6000	0.720	0.246	
8000	0.962	0.242	
10000	1.201	0.239	
0	0.048	–	
12000	1.432	0.231	
0	0.050	–	
13000	1.580	0.148	
0	0.117	–	
14000	1.863	0.283	
0	0.269	–	
16000	3.256	1.393	
17000	5.233	1.997	Side deflection begins.
17500	5.602	0.369	Beam yields slowly.

9-inch Beam. 150 lbs. per Yard. Clear Span, 14 Feet.		
Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.
5608	0.102	–
6720	0.126	0.024
7840	0.148	0.022
8960	0.170	0.022
10080	0.192	0.022
11200	0.214	0.022
12320	0.239	0.025
13440	0.261	0.022
14560	0.287	0.026
15680	0.310	0.023
16800	0.336	0.026
17920	0.359	0.023
19040	0.382	0.023
20160	0.409	0.027
21280	0.435	0.026
22400	0.458	0.023
23520	0.487	0.029
24640	0.516	0.029
25760	0.543	0.027
26880	0.572	0.026
28000	0.600	0.038
29120	0.633	0.033 } Load stood ½ hour.
29120	0.682	0.049 }
0	0.082	{ Weight removed.

15-inch Beam. 200 Lbs. per Yard. Clear Span, 14 Feet.		
Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.
6720	0.048	–
8960	0.060	0.012
11200	0.073	0.013
13440	0.090	0.017
15680	0.105	0.015
17920	0.120	0.015
20160	0.134	0.014
22400	0.148	0.014
24640	0.161	0.013
26880	0.178	0.017
29120	0.191	0.013
31360	0.206	0.015
33609	0.222	0.016
35840	0.234	0.012
38080	0.246	0.012
40329	0.258	0.012
42660	0.271	0.015
44800	0.287	0.016
47040	0.305	0.018

Weight removed; permanent set, 0.016. After lapse of 1 hour, the load of 15 tons was replaced, and caused a total deflection of 0.222 inches, as before.

12-inch Beam. 125 lbs. per Yard. Clear Span, 27 Feet.					
Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.	Centre Load, in lbs.	Deflection, in inches.	Increase, in inches.
6720	0.691	–	15680	1.800	0.170
7840	0.821	0.130	16800	1.976	0.176
8960	0.948	0.127	17920	2.228	0.252
10080	1.061	0.113	19040	2.455	0.227
11200	1.186	0.125	20160	2.742	0.287
12320	1.328	0.142	20720	2.900	0.158
13340	1.466	0.138	20720	2.965	0.065
14560	1.630	0.164			

Last load left on 15 minutes, deflection increasing to 2.965.

§ 225. **Steel.** — Steel is usually defined to be a compound of iron with a small percentage of carbon, this percentage varying from a very minute quantity up to one and a half, or at most two, per cent. Steel is made in one of the three following ways; viz., —

1°. By adding carbon to wrought-iron.

2°. By removing carbon from cast-iron.

3°. By melting together cast and wrought iron in suitable proportions.

These three processes are, as a rule, represented respectively by the following three kinds of steel: —

1°. Crucible steel,

2°. Bessemer steel,

3°. Open-hearth, or Siemens's Martin steel;

these being the three chief kinds that are made on the commercial scale.

Crucible Steel. — This should always be, and is by good makers, made by re-melting blister steel in crucibles; the latter being made by the cementation process, in which bars of wrought-iron are heated in contact with charcoal until they have absorbed the necessary amount of carbon.

Crucible steel is used for the finest cutlery, tools, etc., and wherever a very pure and homogeneous quality of steel is required.

Bessemer Steel is made by decarbonizing cast-iron by forcing a powerful blast through a mass of melted cast-iron, thus removing the greater part of its carbon, and then adding a small quantity of some very pure cast-iron which is rich in carbon, thus bringing up the percentage of carbon to the required amount.

Open-hearth Steel is made by fusing a charge consisting of the suitable proportions of cast-iron with wrought-iron scrap, or with Bessemer steel scrap.

Bessemer and open-hearth steel contain more impurities than crucible steel; but they are very much cheaper, and are just as suitable for many purposes. It is only in consequence of their introduction that steel can be extensively used on the large scale, as crucible steel would be too expensive for many purposes.

Steel is also made by puddling and by other processes.

Steel, unlike wrought-iron, is fusible; unlike cast-iron, it can be forged; and, with the exception of the higher grades, it can be welded, the welding of high-grade steel in large masses being a very uncertain operation, though small masses can be welded by taking proper care.

The special characteristic, however, is, that, with the exception of the lowest grades, when raised to a red heat and suddenly cooled, it becomes hard and brittle, and that, by subsequent heating and slow cooling, the hardness may be reduced to any desired degree. The first process is called hardening, and the second tempering.

Pure wrought-iron cannot be hardened by this means; and cast-iron can be hardened, but cannot be tempered.

Case Hardening is a process by which the outer coating of wrought-iron is turned into steel by heating it to a red heat in contact with bone-dust or some animal matter. This process gives the iron a hard surface combined with toughness.

Whereas the hardening element of steel should be only carbon, and whereas other substances should be absent as far as possible in the best steel, nevertheless phosphorus, silicon, and manganese, when present in small quantities, all have a hardening effect; and all these ingredients, and often sulphur, are generally found in Bessemer and open-hearth steel, sulphur, silicon, and phosphorus coming from the ore, the fuel, and the flux, and manganese being necessarily added, partly to counteract the effect of sulphur, partly, by its affinity for oxygen, to absorb any oxygen in the interior of the mass, and thus decrease the porosity, and partly to enable the steel to be welded by preventing the rapid oxidation of the surfaces at a high heat.

When Bessemer steel was first introduced, and for some time thereafter, it was chiefly used for rails. It is only since a more recent date that Bessemer and open-hearth steel have been made sufficiently homogeneous and reliable for use in construction generally; but at the present day it is displacing wrought-iron in many instances, as being more reliable, and it is likely to displace it even more.

In the construction of boilers, bridges, trusses, beams, etc., it will not do to use the higher grades of steel, but only the milder and more ductile kinds: thus steel with a tensile strength of more than 80000 or 90000 lbs. per square inch is generally too hard to use in construction; and for steam-boilers, if its strength exceed about 65000 lbs., it is liable to be too hard. It should also show a large percentage of contraction of area, as 30 per cent or upwards. Such steel contains but little carbon, generally not more than one-half per cent.

While there are a few isolated cases where it is claimed by some that the structure of iron or steel may be changed from fibrous to crystalline without over-heating, the greater part of the evidence tends to show, that, whenever crystallization has taken place, it has occurred at a temperature above a welding-heat; and in a great many instances where cold crystallization

has been claimed, it has been found on investigation that the piece has some time been over-heated.

Welding is a much more difficult operation in steel than in iron, as (1°) there is always danger of over-heating, and (2°) the metal does not unite as readily at a welding-heat; hence, in high grades of steel, welding is almost an impossibility, especially with large masses.

The injury done to steel plates by punching is greater than that done to iron plates: this injury can, however, be removed by annealing. Steel requires greater care in working it than iron, whether in punching, flanging, riveting, or other methods of working; otherwise it may, if over-heated, burn, or receive other injury from careless workmanship;

In regard to the term "temper," it should be observed, that by the steel-maker it is used to denote the percentage of carbon in the steel, a higher temper corresponding to a higher percentage of carbon. On the other hand, the term "temper" in common parlance refers to the degree of hardness as determined by tempering the steel.

The brands of steel are determined by each maker for himself, there being no uniformity in this regard.

The chemical composition of steel is one important element in its resisting properties; but, on the other hand, the mode of working also has a great influence on the quality.

The only means of securing good steel is, to prescribe the tests which it shall stand, and to reject all that does not fulfil the requirements.

Thus, good boiler-plate should have an ultimate strength of 55000 to 65000 lbs. per square inch, a limit of elasticity of about 30000 pounds, a contraction of area at fracture of about 30 per cent. It should not crack on (1°) being bent double cold, (2°) at a red heat, (3°) at a flanging-heat, and it should suffer but little injury by punching.

For other purposes, as in trusses, etc., it should be able to stand, without injury, the trials to which it has to be subjected

in construction, as bending, punching, riveting, etc. An account of the manner of applying such tests to angle irons, I-beams, etc., will be found in "Use of Steel for Constructive Purposes," by J. Barba.

Effect of Temperature upon the Resisting Properties of Iron and Steel. — The question of the effect of high and of low temperatures upon the resisting properties of iron and steel has been investigated by a number of different experimenters with seemingly discordant results. The following are the principal experimenters upon this subject: —

Sir William Fairbairn: Useful Information for Engineers.
Committee of Franklin Institute: Franklin Institute Journal.
Knutt Styffe and Christer P. Sandberg: Iron and Steel.
Kollman: Engineering, July 30, 1880.
Massachusetts Railroad Commissioners: Report of 1874.

The existing evidence on the subject has been very carefully collated, and compared by graphical means, by Professor Thurston. Comparing all these results and conclusions, it would seem, that, starting at the ordinary temperatures, —

1°. A decrease of temperature increases the tensile strength of iron and steel, but decreases its ductility; thus rendering it more brittle, and hence decreasing its resilience, or power of resisting shocks.

2°. An increase of temperature up to about 570° F. increases the tensile strength; but at a straw heat or a pale blue, almost all irons and steels are very brittle.

3°. An increase of temperature above 570° decreases the tensile strength, but increases the ductility.

Effect of Cold-Rolling on Iron and Steel. — It has already been stated, p. 370, that it was discovered independently by Commander Beardslee and Professor Thurston, that if a load were gradually applied to a piece of iron or steel which exceeded its elastic limit, and the piece then allowed to rest, the elastic limit and the ultimate strength would thus be increased. This

may be accomplished with soft iron and steel by cold-rolling or cold-drawing, but cannot be taken advantage of in hard iron or steel.

Professor Thurston, who has investigated this matter at great length, and made a large number of tests on the subject, gives the following as the results of cold-rolling : —

Increase in	Per Cent.
Tenacity	25 to 40
Transverse stress	50 to 80
Elastic limit (tension, torsion, and transverse),	80 to 125
Elastic resilience	300 to 400
Elastic resilience (transverse)	150 to 425

He also says, in regard to the modulus of elasticity, —

"Collating the results of several hundred tests, the author [Professor Thurston] found that the modulus of elasticity rose, in cold-rolling, from about 25000000 lbs. per square inch to 26000000, the tenacity from 52000 lbs. to nearly 70000, the elastic limit from 30000 lbs. to nearly 60000 lbs. ; and the extension was reduced from 25 to $10\frac{1}{2}$ per cent.

"Transverse loads gave a reduction of the modulus of elasticity to the extent of about 1000000 lbs. per square inch, an increase in the modulus of rupture from 73600 to 133600, and reduction of deflection at maximum load of about 25 per cent. The resistance of the elastic limit was doubled, and occurred at a much greater deflection than with untreated iron."

On the other hand, the two steel eye-bars referred to on p. 422 show a decrease of modulus of elasticity with increasing over-strain.

Whitworth's Compressed Steel. — Sir Joseph Whitworth produces steel of great strength by applying to the molten metal, directly after it leaves the furnace, a pressure of about 14000 lbs. per square inch ; this being sufficient to reduce the length

of an eight-foot column by one foot. He claims, according to D. K. Clark, to be able to obtain with certainty a strength of 40 English tons with 30 per cent ductility, and mild steel of a strength of 30 English tons with 33 or 34 per cent ductility.

The following table is taken from D. K. Clark's "Rules and Tables:"—

	Ultimate Tensile Strength, in lbs., per Square Inch.	Elongation, per cent.
Axles, boilers, connecting-rods, rivets, railway tires, guns, and gun-carriages,	89600	32
Cylinder linings, parts of large machines, hoops, and trunnions,	107520	24
Large planing and lathe tools, shears, smiths' punches, dies and sets, cold-chisels, screw tools, etc.,	129920	17
Boring-tools, finishing-tools for planing and turning,	152320	10
Alloyed with tungsten	161280	14

§ 226. **Tensile Strength of Steel.** — The older experiments on the strength of steel are of but little value; as we have very imperfect records of the kind of steel used in the tests, and its mode of manufacture. Hence only the more recent experimenters will be enumerated.

Sir William Fairbairn: Useful Information for Engineers.

David Kirkaldy: (*a*) Experiments on Wrought-Iron and Steel.

(*b*) An Experimental Inquiry into the Strength of Fagersta Steel.

W. E. Woodbridge: Report on the Mechanical Properties of Steel, Chiefly with Reference to Gun Construction on the Woodbridge System.

Government Machine: (*a*) Executive Document 23, 46th Congress, 2d session.

(*b*) Executive Document 5, 48th Congress, 1st session.

J. Barba: The Use of Steel for Constructive Purposes. By J. Barba.

Tests made for St. Louis Arch: Woodward's History of the St. Louis Arch.

Tests of Steel for the Brooklyn Bridge: Roebling's Report of the Brooklyn Bridge.

Charles B. Dudley: Franklin Institute Journal, 1881.

A. F. Hill: Proceedings of the Engineers' Society of Western Pennsylvania, vol. i.

Professor R. H. Thurston: Materials of Engineering.

The following are the summaries of Kirkaldy's tests on steel bars and steel plates as given in his book, "Experiments on Wrought-Iron and Steel," published in 1866, giving the tensile strength and contraction of area of such specimens as he tested:—

STEEL BARS.

No. of Experiments.	Names of Makers or Works.	Breaking-Weight per Square Inch of Original Area.	Contraction of Area at Fracture.	Breaking-Weight per Square Inch of Fractured Area.	Extreme Elongation.
		lbs.	%.	lbs.	%.
6	Turton & Sons, cast-steel for tools .	132909	4.7	139124	5.4
4	Jowitt, cast-steel for tools	132402	12.8	151857	5.2
8	Jowitt, cast-steel for chisels	124852	17.0	150243	7.1
4	Jowitt, cast-steel for drifts	115882	21.5	147570	13.3
4	Jowitt, double shear steel	118468	19.6	147396	13.5
8	Bessemer, Sheffield tool	111460	22.3	143327	5.5
4	Wilkinson, blister steel	104293	21.4	132472	9.7
4	Jowitt, cast-steel for taps	101151	28.8	142070	10.8
4	Krupp's cast-steel for bolts	92015	34.0	139434	15.3
4	Shortridge & Co.'s homogeneous metal,	90647	36.6	142920	13.7
4	" " " "	89724	26.0	121212	11.9
4	Jowitt's spring steel	72529	24.1	95490	18.0
6	Mersey Company's puddled steel . .	71486	35.3	110451	19.1
6	Blochairn puddled steel	70166	19.4	84871	11.3
6	" " "	65255	19.0	80370	12.0
4	" " "	62769	11.9	71231	9.1

STEEL PLATES.

No. of Experiments.	Names of Makers or Works.	Breaking-Weight per Square Inch of Original Area.	Contraction of Area at Fracture.	Breaking-Weight per Square Inch of Fractured Area.	Extreme Elongation.
		lbs.	%.	lbs.	%.
8	Turton & Son, cast-steel	95299	9.5	105937	7.68
10	Shortridge & Co., cast-steel	96715	15.2	114203	8.77
12	Naylor, Vicker, & Co., cast-steel . .	84435	21.7	108125	17.41
10	Moss & Gambles, cast-steel	72338	33.4	109050	19.73
1	Shortridge & Co., cast-steel	96989	14.4	113395	14.40
8	Mersey Company, puddled steel . .	93209	5.4	100649	2.02
5	Mersey Company, hard steel	93979	4.6	98472	4.08
10	Blochairn, hard steel	93316	4.8	97978	3.14
4	" " "	85010	7.1	92130	6.18
6	Shortridge & Co., hard steel	72994	8.6	80034	4.57
4	Mersey Company, mild	72366	10.5	80937	5.94
2	" " "	71532	7.5	77750	3.57

These tests were made before the use of steel in construction had become as general as it has at the present day.

The quality and strength of steel are affected very seriously by its chemical composition. In this regard we have the record of a series of tests made by the Committee on Chemical Research of the United-States Government Commission, and recorded in Executive Document 23, 46th Congress, second session.

The following summary of this report will be appended here, giving only the percentages of the most important ingredients besides the iron. This summary is as follows : —

Laboratory Number. 3 Samples of Each.	Average Elastic Limit, in lbs., per Sq. In.	Average Modulus of Elasticity.	Average Breaking-Strength per Sq. In. of Original Area.	Average Breaking-Strength per Sq. In. of Fractured Area.	Chemical Analysis. Combined Carbon.	Graphitic Carbon.	Silicon.	Sulphur.	Phosphorus.	Manganese.
					%	%	%	%	%	%
1518	21600	24648000	37933	96033	0.246	0.011	0.145	0.004	0.014	0.020
1519	40000	28010000	67666	109166	0.383 0.389	0.006	0.076	0.003	0.017	0.060
1520	37052	26578000	56659	123433	0.276 0.277	0.018	0.061	0.001	0.015	0.031
1521	44577	26750000	67817	139633	0.377 0.372	0.025	0.070	0.002	0.014	0.027
1522	35854	25799000	60214	121800	0.279 0.273	0.022	0.070	0.002	0.015	0.041
1523	41721	26507000	78119	114300	0.468 0.464	0.027	0.090	0.001	0.015	0.032
1524	45090	26013000	83497	129933	0.728 0.724	0.028	0.126	0.002	0.014	0.048
1525	49544	24816000	90940	122133	0.611 0.608	0.027	0 010	0.003	0.014	0.042
1526	51282	24588000	95340	130600	0 676 0.678	0.018	0.135	0 003	0.015	0.038
1527	51609	27049000	100719	121900	0.557 0.551	0.027	0.112	0.002	0.015	0.028
1528	51076	26587000	115169	138133	0.833 0.828	0.040	0.134	Trace	0.015	0.046
1529	52311	25979000	120275	141433	0.908 0.909	0.015	0.141	0.002	0.014	0.036
1530	48597	26648000	123697	144492	0.924	0.022	0.187	0.002	0.014	0.031
1531	55954	24682000	118156	140367	0.966	0.030	0.153	Trace	0.015	0.035
1532	54596	26819000	117660	141867	0.946	0.930	0.162	0.001	0.015	0.029
1533	60627	26250000	116636	138600	1.024	0.020	0.173	Trace	0.013	0.014
1534	59160	25576000	110823	118500	1.079	0.027	0.096	0.001	0.014	0.044
1535	66710	27091000	120602	132060	1.112	0.030	0.190	Trace	0 015	0.045
1536	63885	25186000	119514	128800	1.186	0.024	0.114	Trace	0.014	0.132
1537	70405	25824000	121253	128800	1.285	0.033	0.106	Trace	0.015	0.273
1053	58971	28169000	103761	149400	0.973	Total Carbon.	0.213	0.003	0.025	0.073
1054	59040	28608000	106205	167367	0 886		0.196	0.002	0.037	0.185
1055	51500	31368000	92081	151467	0.694		0 128	Trace	0.037	0.137
1056	57802	27668000	94529	137900	0.994		0.140	0.003	0.027	0.101
1057	50494	27776000	72979	151433	0.401		0.085	0.006	0.032	0.112
1058	54763	26574000	96373	152233	0.905		0.161	Trace	0.026	0.108
1059	54243	29028000	95612	127133	0.915		0.191	0.002	0.026	0.086
1060	50286	25538000	70066	102967	0.238		0.105	0.012	0.034	0.184
1061	40667	24536000	67000	120067	0.463		0.121	0.002	0.020	Trace
1065	35167	26379000	55000	140600	0.184	0.009	0.063	Trace	0.014	0.051
1066	48367	25637000	83660	139267	0.459	0.118	0.108	Trace	0.026	0.185
1067	50583	28546000	81216	150367	0 451	0.003	0.134	0.004	0.026	0.139

Laboratory Number. 3 Samples of Each.	Average Elastic Limit, in lbs., per Sq. In.	Average Modulus of Elasticity.	Average Breaking-Strength per Sq. In. of Original Area.	Average Breaking-Strength per Sq. In. of Fractured Area.	Chemical Analysis.					
					Combined Carbon.	Graphitic Carbon.	Silicon.	Sulphur.	Phosphorus.	Manganese.
					%	%	%	%	%	%
1068	50522	28755000	111910	137233	0.793	0.013	0.172	None	0.019	0.193
1069	51435	25627000	117668	143633	0.825	0.009	0.170	None	0.020	0.182
1070	55137	26498000	126577	151433	0.846	0 007	0.233	None	0.019	0.191
1071	62691	28219000	129837	151800	0.959	0.013	0.225	Trace	0 018	0.213
1072	61062	27643000	121416	148900	0 984	0.008	0.157	None	0.019	0.245
1073	68015	26805000	121920	132000	1.059	0.013	0.162	None	0.022	0.252
1074	69427	27552000	126036	144300	1.108	0.013	0.206	None	0.023	0.269
1075	72700	27760000	129293	145567	1.142	0.012	0.204	None	0.020	0.282
1076	75403	27264000	132010	138600	1.244	0.082	0.246	None	0.017	0.262
1077	50000	28102000	96699	101800	0.627	0.012	0.154	Trace	0.007	0.050
1078	49436	28701000	91538	169733	0.439	0.015	0.136	0.001	0.019	0.023
1079	36941	27525000	80945	135167	0.461	None	0.116	Trace	0.020	0.027
1080	25167	25953000	43417	88500	0.209 = total carbon.		0.163	0.009	0.084	0.020
1081	28333	29028000	51833	149200	0.116	0.014	0.011	0.029	0 045	0.192
1082	51609	28518000	102240	109333	0.675	0.016	0.028	0.028	0.065	0.459
1083	61388	26295000	128750	139833	0 670	0.011	0.043	0.014	0.063	0.625
1084	34333	24589000	54333	99800	0.049	0.008	0.219	0.007	0.179	0.063
1085	48350	28752000	85833	155267	0.433	0.014	0 071	0.008	0.062	0.493
1086	44667	25360000	65000	131833	0.111	0.012	0.006	0.031	0.123	0 103
1087	54217	25400000	88007	92967	0.348	0.007	0.028	0.041	0.125	0.404
1088	65190	23648000	101045	102300	0.744	0.012	0.074	0.043	0.104	0.465
1089	27617	23341000	48082	97033	0.042	None	0.168	0.011	0.109	0.051
1090	30000	24071000	51500	75833	0.012	0.003	0.214	0.048	0.315	0.081
1091	26000	25482000	59000	80033	0.230	0.004	0.084	Trace	0.039	None
1092	54869	25503000	79315	84733	0.314	0.012	0.016	0.066	0.099	0.525
1093	48733	29326000	78500	92167	0.243	0.011	0.013	0.058	0.128	0.341
1094	47833	26110000	68667	88467	0.135	0.004	0.007	0.056	0.113	0.165
1095	56505	25917000	95341	101633	0.374	0 008	0 018	0.062	0.138	0.584
1096	63126	24602000	111203	142133	0.375	0.012	0.070	0.038	0.092	0.685
1582	37919	28697000	55955	182533	-	-	-	-	-	-
1583	33247	28205000	50523	161700	-	-	-	-	-	-
1588	60627	27010000	110389	162900	-	-	-	-	-	-
1589	58562	25790000	106695	176467	-	-	-	-	-	-
1590	50305	25772000	79858	154267	-	-	-	-	-	-
1591	60627	27389000	123645	142533	-	-	-	-	-	-
1592	58345	25323000	103761	147167	-	-	-	-	-	-
1593	30933	24633000	49167	95600	-	-	-	-	-	-
1594	53565	25776000	78608	148300	-	-	-	-	-	-
1595	46394	25330000	64104	129200	-	-	-	-	-	-

Mr. A. F. Hill, in the "Transactions of the Engineers' Society of Western Pennsylvania" for 1880, gives an account of some tests of open-hearth steel manufactured by Anderson & Co. of Pittsburgh, taken from different runs, and ranging from 0.3 to 0.5 per cent of carbon; the steel being tested both in the form of specimens, and also in the form of eye-bars, plates, and riveted plate-girders.

The following account of some of these tests is condensed from his paper: —

I. SPECIMEN TESTS OF BAR-ENDS 30 INCHES LONG, DESIGNED FOR EYE-BARS.

Mark, and Carbon Percentage.		Tensile Stress per Square Inch at		Stretch at Fracture, %.	Reduction of Area at Fracture, %.
		Elastic Limit.	Rupture.		
1	0.3% C.	55712	94760	15.1	30
2	0.3% C.	56009	95380	12.9	26
3	0.3% C.	55120	95830	15.3	31
4	0.3% C.	55830	96020	14.5	27
5	0.3% C.	55512	94970	13.8	29
1	0.5% C.	65790	112340	10.8	19
2	0.5% C.	66040	112470	8.9	16
3	0.5% C.	66160	111980	10.5	22
4	0.5% C.	65550	113320	10.9	21
5	0.5% C.	65980	113040	9.4	20

II. EXPERIMENTS ON STEEL EYE-BARS WITH 0.3% CARBON.

Dimensions: Stem, 3 in. × $\frac{7}{8}$ in. × 10 ft. Head, $1\frac{1}{4}$ in. thick, $7\frac{1}{2}$ in. across the eye. Pin-hole, $3\frac{5}{8}$ in. diameter.

Mark.	Mode of Manufacture.	Tensile Stress per Square Inch at		Stretch at Fracture, %.	Reduction of Area at Fracture, %.	Remarks.
		Elastic Limit.	Rupture.			
B1	Upset.	54026	87400	8.2	44	Broke 1 ft. 3 in. from pin-hole.
B2		54113	94500	9.2	46	" 4 ft. $3\frac{3}{4}$ in. " "
B3		54113	89300	7.0	42	" 1 ft. $8\frac{1}{4}$ in. " "
F1	Rolled.	51762	92672	8.2	0	Broke in head into 3 pieces.
F2		54065	91570	9.3	29	" 5 ft. $2\frac{1}{4}$ in. from pin-hole.
F3		52518	94780	11.8	40	" 2 ft. $1\frac{3}{4}$ in. " "
A1	Welded.	58473	69140	2.0	Measurements not taken.	Broke $5\frac{1}{4}$ in. from pin-hole.
A2		56059	63000	2.6		" $5\frac{1}{2}$ in. " "
A3		55310	69400	2.2		" $5\frac{3}{8}$ in. " "

In those marked "upset," the bars were rolled to the required section of the stem, with sufficient surplus of length to form the heads by hydraulic upsetting.

In those marked "rolled," the heads were rolled by Kloman's patent process.

In those marked "welded," the heads were formed by welding pieces, and die forging.

Fractures of B1, B2, B3, F2, and F3 were fine, silky, and wedge-shaped.

F1 broke in head, and showed effect of over-heating; fracture coarse and granular.

A1, A2, and A3 broke in stem close to neck; fracture close-grained, 50 per cent granular, and showing effect of welding

heat. Weld was defective at junction of head and neck on account of welding-pieces having been too small.

Mr. Hill draws from these tests the following conclusions: —

1°. "The strength of the specimen exceeds, in each case, that of the manufactured bar."

2°. "The uniformity of the results obtained from the tests of the bar-ends shows conclusively that whatever difference in strength there is between these bar-ends and the manufactured eye-bar is properly ascribable to the mode of manufacture."

3°. "The results obtained from the rolled and the upset eye-bars approach nearest to the original bar strength, and give the best results. The difference between the results from these two methods is so trifling — and if any thing in the 0.3 per cent carbon group, slightly in favor of the upset bar — that it leaves no doubt in my mind that these two processes are equally good."

4°. "The results from the welded bars show, that, while steel can be perfectly welded, there is a loss of nearly 30 per cent of ultimate strength as compared with the original bar; moreover, the elastic limit is too near the ultimate strength, and the percentage of elongation too small, to give sufficient warning of impending failure."

"It will, therefore, be safe to conclude that welded members in steel construction, while no worse than welded iron ones, are not desirable, and, in fact, ought not to be admitted at all, except where the grade of steel used is very low; and then the greatest caution in working and annealing will be required."

Mr. Hill states also that his experiments with eye-bars containing 0.5 per cent carbon bear out these conclusions.

Tests of Steel Eye-Bars made on the Government Machine. — In Executive Document No. 5, 48th Congress, first session, is the record of the tests of six eye-bars of steel, presented by the president of the Keystone Bridge Company.

The following is an extract from the report in regard to these eye-bars: —

"The eye-bars were made of Pernot open-hearth steel, furnished by the Cambria Iron Company of Johnstown, Penn.

"The furnace charges, about .15 tons each of cast-iron, magnetic ore, spiegeleisen, and rail-ends, preheated in an auxiliary furnace, required six and one-half hours for conversion.

"All these bars were rolled from the same ingot.

"Samples were tested at the steel-works taken from a test ingot about one inch square, from which were rolled ¾-inch round specimens.

"The annealed specimen was buried in hot ashes while still red-hot, and allowed to cool with them.

"The following results were obtained by tensile tests: —

	Elastic Limit, in lbs., per Sq. In.	Ultimate Strength, in lbs., per Sq. In.	Contraction of Area.	Modulus of Elasticity.	Carbon.
			%.		%.
¾-inch round rolled bar .	48040	73150	45.7	28210000	0.27
¾-inch round rolled and annealed bar	42210	69470	54.2	29210000	0.27

"The billets measured 7 inches by 8 inches, and were bloomed down from 14-inch square ingot.

"They were rolled down to bar-section in grooved rolls at the Union Iron Mills, Pittsburgh.

"The reduction in the roughing-rolls was from 7 inches by 8 inches to 6¼ inches by 4 inches; and in the finishing-rolls, to 6½ inches by 1 inch.

"The eye-bar heads were made by the Keystone Bridge Company, Pittsburgh, by upsetting and hammering, proceeding as follows: —

"The bar is heated bright red for a length of (approximately) 27 inches, and upset in a hydraulic machine; after

which the bar is reheated, and drawn down to the required thickness, and given its proper form in a hammer-die.

"The bars are next annealed, which is done in a gas-furnace longer than the bars. They are placed on edge on a car in the annealing-furnace, separated one from another to allow free circulation of the heated gases. They are heated to a red heat, when the fires are drawn, and the furnace allowed to cool. Three or four days, according to conditions, are required before the bars are withdrawn.

"The pin-holes are then bored.

"The analyses of the heads before annealing were:—

"Carbon, by color	0.270	per cent.
Silicon	0.036	"
Sulphur	0.075	"
Phosphorus	0.090	"
Manganese	0.380	"
Copper	Trace.	

"The bars were tested in a horizontal position, secured at the ends, which were vertical.

"To prevent sagging of the stem, a counterweight was used at the middle of the bar.

"Before placing in the testing-machine, the stem from neck to neck was laid off into 10-inch sections, to determine the uniformity of the stretch after the bar had been fractured.

"A number of intermediate 10-inch sections were used as the gauged length, obtaining micrometer measurements of elongation, and the elastic limit for that part of the stem which was not acted upon during the formation of the heads. Elongations were also measured from centre to centre of pins, taken with an ordinary graduated steel scale.

"The moduli of elasticity were computed from elongations taken between loads of 10000 and 30000 lbs. per square inch, deducting the permanent sets.

"The behavior of bars Nos. 4582 and 4583, after having been strained beyond the elastic limit, is shown by elongations of the gauged length measured after loads of 40000 and 50000 lbs. per square inch had been applied; and with bar No. 4583, after its first fracture under 64000 lbs. per square inch, a rest of five days intervening between the time of fracture and the time of measuring the elongations.

"Considering the behavior between loads of 10000 and 30000 lbs. per square inch, we observe the elongations for the primitive readings are nearly in exact proportion to the increments of load.

"Loads were increased to 40000 lbs. per square inch, passing the elastic limit at about 37000 lbs. per square inch; the respective permanent stretch of the bars being 1.31 and 1.26 per cent.

"Elongations were then immediately redetermined, which show a reduction in the modulus of elasticity, as we advanced with each increment, of 5000 lbs. per square inch.

"Corresponding measurements after the bars had been loaded with 50000 lbs. per square inch reach the same kind of results.

"The first fracture of bar No. 4583, under 64000 lbs. per square inch, occurred at the neck, leaving sufficient length to grasp in the hydraulic jaws of the testing-machine, and continue observations on the original gauged length. This was done after the fractured bar had rested five days.

"The elongations now show the modulus of elasticity constant or nearly so, the only difference in measurements being in the last figures, up to 50000 lbs. The readings were then immediately repeated, and the same uniformity of elongations obtained.

"An illustration of the serious influence of defective metal in the heads is found in the first fracture of bar No. 4583.

"There was about 27 per cent excess of metal along the line of fracture over the section of the stem."

TABULATION OF STEEL EYE-BAR TESTS.

Number of Test.	Length, in inches.	Gauged Length, in inches.	Width, in inches.	Thickness, in inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation, per cent. In Gauged Length.	Elongation, per cent. Centre to Centre of Pins.	Contraction of Area, per cent.	Modulus of Elasticity.	Maximum Compression on Pin-Holes, in lbs., per Square Inch.	Position of Fracture.	Appearance of Fracture.
4582	221.88	160	6.48	0.98	37480	67800	15.80	15.9	33.7	29880000	86100	Broke in stem.	Silky and granular.
4583	221.60	160	6.46	0.98	36650	64000	-	-	-	30270000	79430	Broke across neck.	Granular.
4583*a*	-	160	-	-	-	71560	13.77	-	36.5	-	-	Broke in stem.	Fine, silky, and granular.
4583*b*	-	-	-	-	-	72050	-	-	38.4	-	-	" " "	Fine, silky trace of granulation.
4584	262.70	200	6.46	0.98	37600	68720	12.30	12.6	34.1	29630000	83650	" " "	Silky and granular.
4585	261.85	200	6.46	0.96	35810	65850	12.00	11.5	39.2	29960000	80000	" " "	Fine, silky, and granular.
4586	262.12	200	6.45	0.97	33230	64410	16.40	15.1	49 5	29670000	79060	" " "	Fine and silky.
4587	261.72	200	6.46	0.97	37640	68290	13.90	13.5	42.4	29960000	83960	" " "	Fine, silky, and granular.

ELONGATIONS OF No. 4582 FOR EACH INCREMENT OF 5000 LBS. PER SQUARE INCH.

Loads, in lbs., per Square Inch.	Elongations.		
	Primitive Loading.	After Load of 40000 lbs. per Square Inch.	After Load of 50000 lbs. per Square Inch.
10000	–	–	–
15000	0.0274	0.0300	0.0311
20000	0.0269	0.0305	0.0322
25000	0.0269	0.0320	0.0337
30000	0.0269	0.0330	0.0341

ELONGATIONS OF No. 4583 FOR EACH INCREMENT OF 5000 LBS. PER SQUARE INCH.

Loads, in lbs., per Square Inch.	Elongations.			Elongations after 64000 lbs. per Square Inch.	
	Primitive Loading.	After 40000 lbs. per Square Inch.	After 50000 lbs. per Square Inch.	First Reading.	Second Reading.
10000	–	–	–	–	–
15000	0.0272	0.0291	0.0302	0.0311	0.0310
20000	0.0272	0.0305	0.0315	0.0308	0.0310
25000	0.0268	0.0314	0.0325	0.0311	0.0310
30000	0.0267	0.0326	0.0340	0.0312	0.0310
35000	–	–	–	0.0311	–
40000	–	–	–	0.0312	–
45000	–	–	–	0.0310	–
50000	–	–	–	0.0315	–

STEEL PLATES.

Steel plates are used in making plate-girders and other forms for resisting load, and also for steam-boilers.

For the latter purpose the steel must be very low in carbon, and must stand very much more in the way of bending, punching, shearing, etc., than in the former case. Its tensile strength will not be high, but it must have great ductility. The first set of tests to which reference will be made is the set given in Mr. Hill's paper already referred to, this steel being too high to be suitable for boiler-work.

He had made 54 tests of rolled open-hearth steel plates.

The following 15 tests give the relative strengths of the plates, and the percentages of carbon.

The plates tested were all $\frac{3}{8}$ inch thick, 12 inches wide, and 6 feet long, tested as they came from the rolls; crop ends sheared, 50 inches between jaws of machine.

Mark.	Carbon, percentage.	Tensile Stress, in lbs., per Square Inch, in Direction of Rolling, at		Average, per cent, Elongation at Fracture.	Remarks.
		Elastic Limit.	Rupture.		
P1	0.3	43260	79120	19.3	Fractures fine and silky.
P2	0.3	44820	77840		
P3	0.3	45110	78390		
P4	0.3	43990	77970		
P5	0.3	44720	78280		
R1	0.4	51620	81990	13.9	Fractures very fine.
R2	0.4	50980	81720		
R3	0.4	51260	83730		
R4	0.4	51100	81830		
R5	0.4	50890	83130		
V1	0.5	58950	85790	10.5	Fractures good, slightly granular on edges.
V2	0.5	59200	86220		
V3	0.5	58540	85560		
V4	0.5	58880	86000		
V5	0.5	59330	86330		

He then gives the following table:—

COMPARATIVE RESULTS OF SHEARING, PUNCHING, ANNEALING, AND TEMPERING STEEL PLATE.

Carbon, per cent.	Treatment of Specimen.	Tensile Stress per Square Inch at	
		Elastic Limit.	Rupture.
0.3	Cut in planer	49431	94396
0.3	Sheared	32370	74980
0.3	Punched	0	63410
0.3	Punched and hammered cold . .	0	87540
0.3	Punched, hammered, and annealed,	55780	100410
0.4	Cut in planer	63475	87695
0.4	Sheared	46900	75330
0.4	Punched	0	68890
0.4	Sheared and annealed	59350	86160
0.4	Punched and tempered	52780	103560
0.5	Cut in planer	65186	84092
0.5	Sheared	51666	79000
0.5	Punched	0	78400
0.5	Sheared and tempered	60375	87293
0.5	Punched and annealed	57960	84900

From these tests he draws the following conclusions:—

1°. "That both shearing and punching are injurious to all grades of steel, and cold punching far more than shearing."

2°. "That both these operations affect the elastic limit far more than they do the ultimate strength."

3°. "That apparently the lower grades of steel are more injured than higher grades; but the evidence on this point is not certain, as the lower-grade plates were thicker than those of higher grade."

TRANSVERSE STRENGTH OF STEEL.

A number of tests of the transverse strength of small bars of steel 1 or 2 inches square have been made by Kirkaldy and others; and they have generally shown a modulus of rupture larger than the tensile strength of the steel, frequently 1.5 times the tensile strength.

It is only now that the attempt is being made to roll steel I-beams, and thus far no tests have been made on full-size steel beams. Before we can decide upon the value of the modulus of rupture suitable to use, we need such tests; and it would not be safe to assume for these beams the constants deduced from the small ones. The steel used may vary greatly from a very mild to a very hard steel. It is probable that it will be a mild steel that will be used for beams.

Until we have experiments upon full-size beams, it would hardly be safe to use, for modulus of rupture, much more than would hold for an iron beam, or from 50000 to 60000 or 70000 lbs. per square inch.

§ 227. **Factor of Safety.** — In order to determine the proper dimensions of any loaded piece, it becomes necessary to fix, in some way, upon the greatest allowable stress per square inch to which the piece shall be subjected.

The most common practice has been to make this some fraction of the breaking-strength of the material per square inch.

As to how great this factor should be, depends upon —

1°. The use to which the piece is to be subjected;

2°. The liability to variation in the quality of the material;

3°. The question whether we are considering, as the load upon the piece, the average load, or the greatest load that can by any possibility come upon it;

4°. The question as to whether the structure is a temporary or a permanent one;

5°. The amount of injury that would be done by breakage of the piece;
and other considerations.

The factors most commonly recommended are, 3 for a dead or quiescent load, and 6 for a live or moving load.

The American and English practice in the case of iron bridges is to use a factor of safety of 4 for both dead and moving load. In machinery a factor as large as 6 is desirable when there is no liability to shocks; and when there is, a larger factor should be used.

A method very rarely followed for tension and compression pieces is, to prescribe that the stretch under the given load should not exceed a certain fixed fraction of the length. This requires a knowledge of the modulus of elasticity of the material.

In the case of a piece subjected to a transverse load, it is the most common custom to determine its dimensions in accordance with the principle of providing sufficient strength; and for this purpose a certain fraction (as one-fourth) of the modulus of rupture is prescribed as the greatest allowable safe stress per square inch at the outside fibre. Thus, the two iron companies already referred to prescribe 12000 lbs. per square inch as the greatest allowable stress at the outside fibre for wrought-iron beams, this being about one-fourth of the modulus of rupture.

The other method for dimensioning a beam is, to prescribe its stiffness; i.e., that it shall not deflect under its load more than a certain fraction of the span. This fraction is taken as $\frac{1}{400}$ to $\frac{1}{600}$.

This latter method depends upon the modulus of elasticity of the beam; and while it is the most advisable method to follow, and as a rule would be safer than the other method, nevertheless, in the case of very stiff and brittle material it might be dangerous: hence we ought to know also the break-

ing-weight and the limit of elasticity of the beam we are to use, and not allow it to approach either of these. This precaution will be especially important to observe in the case of steel beams, which are only now being introduced.

On the other hand, in moving machinery a factor of safety of six is usually required when there is no unusual exposure to shocks, as in smooth-running shafting, etc.; and when there are irregular shocks liable to come upon the piece, a greater factor is used.

§ 228. **Wöhler's Results.** — The extensive experiments of Wöhler for the Prussian government, which were subsequently carried on by his successor, Spangenberg, were made to determine the effect of oft-repeated stresses, and of changes of stress, upon wrought-iron and steel.

In the ordinary American and English practice, it is customary, in determining the dimensions of a piece, as of a bridge member, to ascertain the greatest load which the piece can ever be called upon to bear, and to fix the size of the piece in accordance with this greatest load.

Wöhler called attention to the fact that the load that would break a piece depends upon both the greatest and least load that it would ever be called upon to bear. Thus, a tension-rod which is subjected to alternate changes of load extending from 20000 to 80000 lbs. would require a greater area for safety than one which was subjected to loads varying only between the limits of 60000 and 80000 lbs.; and this would require more area than one which was subjected to a steady load of 80000 lbs.

Wöhler expresses this law as follows, in his "Festigkeits versuche mit Eisen und Stahl."

"The law discovered by me, whose universal application for iron and steel has been proved by these experiments, is as follows: The fracture of the material can be effected by variations of stress repeated a great number of times, of

which none reaches the breaking-limit. The differences of the stresses which limit the variations of stress determine the breaking-strength. The absolute magnitude of the limiting stresses is only so far of influence as, with an increasing stress, the differences which bring about fracture grow less.

"For cases where the fibre passes from tension to compression and *vice versa*, we consider tensile strength as positive and compressive strength as negative; so that in this case the difference of the extreme fibre stresses is equal to the greatest tension plus the greatest compression."

Besides the ordinary tests of tensile, compressive, shearing, and torsional strength, he made his experiments mainly on the following two cases: —

1°. Repeated tensile strength; the load being applied and wholly removed successively, and the number of repetitions required for fracture counted.

2°. Alternate tension and compression of equal amounts successively applied, the number of repetitions required for fracture being counted.

In making these two sets of tests, he made the first set in two ways: —

(*a*) By applying direct tension.

(*b*) By applying a transverse load, and determining the greatest fibre stress.

The second set of tests was made by loading at one end a piece of shaft fixed in direction at the other, and then causing it to revolve rapidly, each fibre passing alternately from tension to an equal compression, and *vice versa*.

He also tried a few experiments where the lower limit of stress was neither zero nor equal to the upper limit, with a minus sign, also some experiments on torsion, on shearing, and on repeated torsion.

When Wöhler had made his experiments, and published his results, there were a number of attempts made by different

persons to deduce formulæ which should depend upon these experiments for their constants, and which should serve to determine the breaking-strength for any given variation of stresses.

Only two of these formulæ will be given here, viz., —

1°. That of Launhardt for one kind of stress,

2°. That of Weyrauch for alternate tension and compression, as these are the most used of the formulæ developed.

LAUNHARDT'S FORMULA.

The constants used in this formula are —

1°. t, the carrying-strength of the material per unit of area; this being the greatest load per square inch of which the piece can bear an unlimited application without breaking. Practically it is the breaking-strength per unit of area.

2°. u, the primitive safe strength; i.e., the greatest stress per unit of area of which the piece can bear, without breaking, an unlimited number of repetitions, the load being entirely removed between times. It is not a *safe* but a *breaking* strength. These two quantities have been determined experimentally by Wöhler; and it is the object of Launhardt's formula to deduce, in terms of t, u, and the ratio between the greatest and least loads to which the piece is ever subjected, the value a of the breaking-strength per unit of area when these loads are applied.

The formula and its deduction are as follows: —

Let the greatest stress per unit area be a.

the least stress per unit area be c.

their difference be $d = a - c$.

Now a depends on d for its value; and hence we may write

$$a = \alpha d,$$

where α is a function of a also.

Now, the two cases experimented upon, viz., —

1°. When $a = t$, $\quad c = t$, $\quad d = 0$,
2°. When $a = u$, $\quad c = 0$, $\quad d = u$,

must be satisfied by the value of α used: otherwise the value we use is wrong, as these are two particular cases. Now, Launhardt takes

$$\alpha = \frac{t - u}{t - a},$$

and hence

$$a = \frac{t - u}{t - a} d; \qquad (1)$$

and it will become evident that this value of α does satisfy the two conditions stated if we observe, that, in the first case, since

$$d = 0,$$

we must have $\alpha = \infty$ in order that we may have

$$a = t;$$

and if we put $a = t$ in the value of α, we obtain

$$\alpha = \infty:$$

also in the second case, since $d = u$ and $a = u$, we must have

$$\alpha = 1;$$

and if we put $a = u$ in the value of α, we obtain

$$\alpha = 1.$$

Hence in using (1) we are using a formula for a which satisfies the two cases of carrying-strength and of primitive safe strength; and the question as to its being a suitable value to use must depend upon how well it will satisfy intermediate values, i.e., cases where the two extremes are not those of the carrying-strength nor of the primitive safe strength.

In the few cases of intermediate values experimented upon by Wöhler, there is a very close agreement between the experimental results and those obtained by the formula.

Now, put $d = a - c$ in (1), and we have

$$a = \frac{t - u}{t - a}(a - c) \tag{2}$$

$$at - a^2 = at - au - c(t - u)$$

$$\therefore \quad a^2 = au + (t - u)c$$

$$\therefore \quad a = u + (t - u)\frac{c}{a}; \tag{3}$$

and if we denote by max L the greatest load on the entire piece, and by min L the least, we shall have

$$\frac{c}{a} = \frac{\min L}{\max L}.$$

Hence

$$a = u + (t - u)\frac{\min L}{\max L}, \tag{4}$$

this being in such a form as can be used. Or we may write it thus:

$$a = u\left\{1 + \frac{t - u}{u}\frac{\min L}{\max L}\right\}, \tag{5}$$

this being the more common form.

The values of the constants as determined by Wöhler's experiments, and the resulting form of the formula for Phœnix axle-iron and for Krupp cast-steel, have already been given in § 172.

In the same paragraph are given the corresponding values of b, the safe working-strength, the factor of safety being three.

WEYRAUCH'S FORMULA FOR ALTERNATE TENSION AND COMPRESSION.

The constants used in this formula are: —

1°. u, the primitive safe strength, which has been already defined.

2°. s, the vibration safe strength; i.e., the greatest stress per unit of area of which the piece can bear, without breaking, an unlimited number of applications, the other extreme stress which it bears alternately being $-s$.

He lets $a =$ greatest stress per unit of area, $c =$ greatest stress of the opposite kind per unit of area. If a is tension, c is compression, and *vice versa.*

Then, if d is the difference,

$$d = a + c.$$

Weyrauch writes, as before,

$$a = \alpha d,$$

and gives to α a value which will satisfy the two special cases experimented upon; viz., —

1°. When $a = u$,	$c = 0$,	$d = u$.
2°. When $a = s$,	$c = s$,	$d = 2s$.

He writes

$$\alpha = \frac{u - s}{2u - s - a}$$

$$\therefore \quad a = \frac{u - s}{2u - s - a} d. \qquad (6)$$

This value of α satisfies the two special cases referred to; for in the first case, since $d = u$ and $a = u$, we must have $\alpha = 1$; and if we write $a = u$ in the value of α, we obtain $\alpha = 1$. Also in the second case, since $d = 2s$ when $a = s$, we must have $\alpha = \frac{1}{2}$; and if we write $a = s$ in the value of α, we obtain $\alpha = \frac{1}{2}$.

This, however, has not been tested for intermediate values.

Now write $d = a + c$ in equation (1), and we shall have

$$a = \frac{u - s}{2u - s - a}(a + c),$$

$$2au - as - a^2 = au - as + (u - s)c$$

$$\therefore \quad a^2 = au - (u - s)c \tag{7}$$

$$\therefore \quad a = u - (u - s)\frac{c}{a}; \tag{8}$$

and if we write

$$\frac{c}{a} = \frac{\max L'}{\max L},$$

where $\max L$ = greatest load on the piece, and $\max L'$ = greatest load of opposite kind, so that, if L is tension, L' shall be compression, and *vice versa*, we shall have

$$a = u - (u - s)\frac{\max L'}{\max L}, \tag{9}$$

this being in a form suitable to use, the more common form being

$$a = u\left\{1 + \frac{u - s}{u}\frac{\max L'}{\max L}\right\}. \tag{10}$$

The values of the constants as determined from Wöhler's experiments, and the resulting form of the formulæ for Phœnix axle-iron and for Krupp cast-steel, are given in § 176.

The above demonstrations of Launhardt's and of Weyrauch's formulæ are substantially those given in the translation of Weyrauch's "Structures of Iron and Steel," by Professor Dubois, the explanations being somewhat changed.

GENERAL REMARKS.

In each case the value of a given by the formula (5) or (10) is the breaking-strength per unit of area for either tension or compression in Launhardt's formula; and in the case of Weyrauch's, the value of a is the breaking-strength per square inch of the same kind as max L; i.e., tension if this is tension, or compression if this is compression.

If either of these values of a be divided by 3, we have, according to Weyrauch, the safe working-strength.

If, now, the piece be a tension piece, its area will be found by dividing its greatest load by the safe working-strength; if, on the other hand, it be subjected to compression, and it be a short piece, its area will also be found by dividing the greatest compression by the safe working-strength per unit of area: but if it be a long column, and we wish to use Wöhler's results, we must merely use the value $\frac{a}{3}$ as the safe working-strength per unit of area, and use this in whatever formula we may employ for calculating a column.

WÖHLER'S EXPERIMENTAL RESULTS.

Wöhler himself made his tests upon the extremes of fibre stresses of which a piece could bear, without breaking, an unlimited number of applications. He gives, as a summary of these results, the following: —

In iron, —

Between	+16000	lbs.	per	sq. in.	and	−16000	lbs.	per	sq. in.
"	+30000	"	"	"		0	"		"
"	+44000	"	"	"		+24000	"		"

In axle-steel, —

Between	+28000	lbs.	per	sq. in.	and	−28000	lbs.	per	sq. in.
"	+48000	"	"	"		0	"		"
"	+80000	"	"	"		+35000	"		"

In untempered spring steel, —

Between	+50000	lbs.	per sq. in.	and	0	lbs.	per sq. in.
"	+70000	"	"	"	+25000	"	"
"	+80000	"	"	"	+40000	"	"
"	+90000	"	"	"	+60000	"	"

For shearing in axle-steel, —

Between	+22000	lbs.	per sq. in.	and	−22000	lbs.	per sq. in.
"	+38000	"	"	"	0	"	"

This table would justify the use, in Launhardt's and Weyrauch's formulæ, of the following values of u and s; viz., —

In iron, —

$$u = 30000 \text{ lbs. per sq. in.,}$$
$$s = 16000 \text{ lbs. per sq. in.}$$

In axle steel, —

$$u = 48000 \text{ lbs. per sq. in.,}$$
$$s = 28000 \text{ lbs. per sq. in.}$$

In untempered spring steel, —

$$u = 50000 \text{ lbs. per sq. in.}$$

And it would require, that if, with these values of u, and the values of t given in §§ 172 and 176, we put

$$c = 24000$$

in Launhardt's formula for iron, we ought to obtain approximately

$$a = 44000;$$

and if we put $c = 35000$ in that for steel, we should obtain approximately

$$a = 80000.$$

FACTOR OF SAFETY.

We have seen that Weyrauch recommends, to use with Wöhler's results, a factor of safety of three for ordinary bridge work and similar constructions.

Wöhler himself, however, in his "Festigkeits versuche mit Eisen und Stahl," says, —

1°. That we must guard against any danger of putting on the piece a load greater than it is calculated to resist, by assuming as its greatest stress the actually greatest load that can ever come upon the piece; and

2°. This being done, that the only thing to be provided for is the lack of homogeneity in the material.

3°. That any material which requires a factor of safety greater than two is unfit for use. This advice would hardly be accepted by engineers, however.

He also claims that the reason why it is safe to load car-springs so much above their limit of elasticity, and so near their breaking-load, is, that the variation of stress to which they are subjected is very inconsiderable compared with the greatest stress to which they are subjected.

GENERAL REMARKS.

It is to be observed, —

1°. The tests were all made on a good quality of iron and of steel, consequently on materials that have a good degree of homogeneity.

2°. The specimens were all small, and the repetitions of load succeeded each other very rapidly, no time being given for the material to rest between them.

3°. No observations were made on the behavior of the piece during the experiment before fracture.

4°. No experiments were made upon cast-iron and timber; and the results of such experiments, if made, could hardly be

expected to be of much value, as these materials are so lacking in homogeneity.

5°. As long as we are dealing only with tension, we can say without error that

$$\frac{c}{a} = \frac{\min L}{\max L};$$

but as soon as both stresses or either become compression, if the piece is long compared with its diameter, we cannot assert with accuracy that

$$\frac{c}{a} = \frac{\max L'}{\max L};$$

and hence results based on such an assumption must be to a certain extent erroneous.

6°. When a piece is subjected to alternate tension and compression, it must be calculated so as to bear either: thus, if sufficient area is given it to enable it to bear the tension, it may not be able to bear the compression unless the metal is so distributed as to enable it to withstand the bending that results from its action as a column.

EXAMPLES.

1. Determine the cross-section necessary for a wrought-iron tie where greatest load = 800000 lbs. and least load = 80000 lbs.

2. Determine the cross-section necessary for a wrought-iron strut to bear the same loads.

3. What is the greatest and what the least working-strength recommended by Wöhler for wrought-iron? What for steel? Compare with ordinary methods, using factor of safety of four.

§ 229. **Shearing-Strength of Iron and Steel.** — Some of the most common cases where the shearing resistance of iron and steel is brought into play are: —

1°. In the case of a torsional stress, as in shafting.

2°. In the case of pins, as in bridge-pins, crank-pins, etc.

3°. In the case of riveted joints.

In most cases the shearing is accompanied by tensions or compressions, or other stresses, and it is difficult to separate the effects; so that, in the present state of our knowledge, we cannot explain the fact, that, when the breaking shearing-strength of wrought-iron or steel is deduced from an experiment on torsional strength, it is found to be about equal to the tensile strength, while when deduced from experiments on riveted joints, it is found to be about $\frac{3}{4}$ or $\frac{4}{5}$ the tensile strength.

Experiments on this subject are to be found in —

1°. Steam-Boilers, by William H. Shock, U.S.N.

2°. Kirkaldy: Experimental Inquiry into the Properties of Fagersta Steel.

3°. A. B. W. Kennedy: Engineering. May 6, 1881.

In regard to cast-iron, Bindon Stoney found the shearing and tensile strength about equal.

As to shearing modulus of elasticity, Bauschinger's experiments on cast-iron give about two-fifths the tensile modulus of elasticity; and Wöhler's experiments gave for steel almost exactly two-fifths the tensile modulus, his value being 11236500 when the tensile modulus was 29600000. According to the theory of elasticity, the modulus of elasticity for shearing should be about two-fifths that for tension or compression, as will be shown later; and these experiments furnish a most beautiful confirmation of the theory of elasticity.

The cases where shearing comes in play in wrought-iron and steel will, therefore, be treated separately.

§ 230. **Torsional Strength of Wrought-Iron and Steel.** — The most common custom for computing the strength of a shaft has been to use one based upon its twisting-moment; and hence using the shearing breaking-strength of the material, as determined from an experiment on simple torsion, for our con-

stant. It is generally the fact, however, that, when shafting is running, the pulls of the belts create a bending backwards and forwards, bringing the same fibre alternately into tension and compression; and this is combined with the shearing-stresses developed due to the twisting-moment alone. No tests have been made upon the effect of the combination of these stresses under working conditions; and until we have a systematic set of tests of this character, we shall not be able to predict with certainty the behavior of a shaft under working conditions. At the two extremes of these general cases are:—

1°. The case when the portion of a shaft between two hangers has no pulleys upon it, and when the pulls on the neighboring spans are not so great as to deflect this span appreciably. That is a case of pure torsion: and if the shaft is running smoothly, with no jars or shocks, and no liability to have a greater load thrown upon it temporarily, we should properly compute it by the usual torsion formula, given in § 212; and we may use for breaking-strength of wrought-iron and steel the tensile strength, and use a factor of safety no greater than six.

2°. The case when, pulleys being placed otherwise than near the hangers, the belt-pulls are so great that the torsion becomes insignificant compared with the bending, and then it would be proper to compute our shaft so as not to deflect more than $\frac{1}{1200}$ of its span under the load, or better, not more than $\frac{1}{1600}$: of course we should compute also the breaking transverse load, and see that we have a good margin of safety.

In other cases, experiments upon shafting under working conditions are needed; as up to the present time all such matters have been decided in one of three ways, as follows:—

1°. By using the ordinary torsion formula combined with a large factor of safety.

2°. As recommended by D. K. Clark, by computing the shaft also for deflection, and providing that its deflection shall not exceed $\frac{1}{1200}$ or $\frac{1}{1600}$ of its span.

This, however, neglects the torsion, and also the rapid change of stress upon each fibre from tension to compression, and *vice versa.*

3°. By using the formula of Grashof or of Rankine for combined bending and shearing; but this has not had the constants worked out from a test on such combined stress, and it is seldom used.

In regard to experiments upon torsional strength, they are usually made by twisting a short piece of shaft until rupture occurs, measuring its twist under smaller loads, and from these computing the modulus of shearing elasticity.

Such tests have been made by —

1°. Kirkaldy: D. K. Clark's Rules and Tables.

2°. D. K. Clark: English Civil Engineers' Committee.

3°. Major Wade.

4°. United-States Board to test Iron and Steel: Executive Document No. 23, 46th Congress, 2d session.

5°. Professor Thurston: Materials of Engineering.

The general result has usually been, to obtain about the same shearing breaking fibre stress as the tensile strength of the material, and a modulus of elasticity about two-fifths the tensile modulus. Of course the values vary more or less, sometimes being greater and sometimes less than the values given above, according to the quality of the iron; so that we can determine by experiment only what any one iron will bear. None of these tests will be recorded here, but some of the rules in common use for figuring the strength will be given: they are merely some of the formulæ already referred to, with the constants changed.

Unwin gives the direction, —

1°. That the axle must be calculated as a beam to bear the weight of the pulley and the belt-pull that is to come upon it.

2°. For shafting transmitting power, and subject to torsion only, he gives

$$d = \alpha\sqrt[3]{PR} = \beta\sqrt[3]{\frac{HP}{N}},$$

where d = diameter in inches, PR = twisting-moment in inch-pounds, N = number of turns per minute, HP = number of horses-power to be transmitted; and he gives for α and β the following values: —

	α.	β.
Wrought-iron	0.08275	3.294
Cast-iron	0.1042	4.150
Steel	0.0723	2.877

these values assuming for safe greatest fibre stress, for wrought-iron, 9000; for cast-iron, 4500; and for steel, 13500 lbs. per square inch.

3°. For the crank-shaft of a steam-engine he advises us to consider the maximum twisting-moment as 1.3 times the mean.

4°. For combined twisting and bending, he gives the Rankine formula, equation (2), § 215, and puts it in the following form:

$$d = \sqrt[3]{K + \sqrt{K^2 + 1}}\sqrt[3]{\frac{5.1}{f}}\sqrt[3]{T},$$

where d = diameter in inches, f = outside fibre breaking shearing-strength per square inch, T = twisting-moment in inch-pounds, $K = \frac{M}{T}$ = ratio of bending to twisting moment.

Then, calling

$$n = \sqrt[3]{K + \sqrt{K^2 + 1}},$$

n will be the ratio of the diameter to be used, to the diameter needed to resist the torsion alone; and he gives, when

$K =$ 0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	3.00,
$n =$ 1.09	1.17	1.26	1.36	1.42	1.49	1.56	1.62	1.83.

For the propeller shaft of a steam-vessel, he advises to use

$$K = 0.25 \text{ to } 0.50 \qquad \therefore \quad n = 1.09 \text{ to } 1.17.$$

For line shafting in mills, he advises

$$K = 0.75 \text{ to } 1.00 \qquad \therefore \quad n = 1.26 \text{ to } 1.34.$$

For crank-shafts and heavy shafting subject to shocks,

$$K = 1.00 \text{ to } 1.50 \qquad \therefore \quad n = 1.34 \text{ to } 1.49.$$

5°. For forged crank-shafts, he gives

$$d = 4.55\sqrt[3]{\frac{HP}{N}}.$$

To control the deflection of shafting 10 or 11 feet between hangers, he gives

$$d = \left(\frac{L}{\gamma}\right)^{\frac{3}{2}},$$

where L = span in inches, and $\gamma = 54$ to 60.

Professor Thurston gives, —

For head shafts well supported against springing, —

Wrought-iron, $d = \sqrt[3]{\dfrac{125HP}{N}}$. Cold-rolled iron, $d = \sqrt[3]{\dfrac{75HP}{N}}$.

For line-shafting, hangers 8 feet apart, —

Wrought-iron, $d = \sqrt[3]{\dfrac{90HP}{N}}$. Cold-rolled iron, $d = \sqrt[3]{\dfrac{55HP}{N}}$.

For transmission simply, no pulleys, —

Wrought-iron, $d = \sqrt[3]{\dfrac{62.5HP}{N}}$. Cold-rolled iron, $d = \sqrt[3]{\dfrac{35HP}{N}}$.

Mr. James B. Francis gives the following table for distance between bearings of shafting carrying no side strain :—

Diameter.	Wrought-Iron.	Steel.	Diameter.	Wrought-Iron.	Steel.
in.	ft.	ft.	in.	ft.	ft.
2	15.5	15.9	6	22.3	22.9
3	17.7	18.2	7	23.5	24.1
4	19.5	20.0	8	24.6	25.2
5	20.9	21.6	9	25.5	26.2

It should be observed, that it is only a mild steel that can be used for shafting, — steel of about 60000 lbs. tensile strength per square inch: hence the calculations cannot differ very greatly from those for wrought-iron.

§ 231. **Bridge-Pins.** — In the case of a bridge-pin or other pin, the shearing-stress is always accompanied by a very large element of bending; so that the principles of transverse strength come into play very largely, and perhaps in many cases wholly.

The rules for proportioning bridge-pins used by different engineers and constructors have been various, but have generally been founded upon the theory of transverse strength; for those given by Charles Bender, the student is referred to his treatise on "Bridge-Pins" (Van Nostrand's Science Series). No attempt will be made to give any of the rules used; but, inasmuch as there were quite a number of such pins tested at the Watertown Arsenal, there will be given a summary of the tests, which are to be found in detail in Executive Document 12, 47th Congress, first session. All the pins tested were of wrought-iron, and had semicircular seats at middle and ends.

From this table the modulus of elasticity can be figured, and the deflection of the pin under any given load and span;

or the table may be employed to interpolate, to find the sizes to be used without causing too much deflection.

Diameter, in inches.	Clear Span.	Width of Middle Bearing.	Load nearest one-half Maximum Load, in lbs.	Deflection, in inches.	Maximum Load applied, in lbs.	Deflection, in inches.	Amount of Indentation, in inches.
2.500	24	1	10000	0.0517	20000	1.2700	0.0070
2.497	24	1	8000	0.0420	15000	0.3290	0.0020
2.500	18	1	10000	0.0238	21000	0.1660	0.0025
2.500	18	1	10000	0.0248	18000	0.1000	< 0.0010
2.500	12	–	15000	0.0133	30000	0.0367	0
2.500	12	–	14000	0.0129	27000	0.0437	< 0.0010
2.500	6	1	33000	0.0066	65000	0.0263	0.0030
2.500	6	1	30000	0.0064	60000	0.0184	–
3.000	24	–	11000	0.0278	22000	0.1260	–
3.000	24	–	11000	0.0278	22000	0.1231	–
3.000	12	$1\frac{5}{8}$	25000	0.0101	50000	0.0414	–
3.000	12	$1\frac{5}{8}$	25000	0.0102	50000	0.0464	–
3.000	6	–	62000	0.0077	125000	0.0240	–
3.000	6	$1\frac{5}{8}$	60000	0.0070	125000	0.0250	–
3.495	24	$1\frac{5}{8}$	20000	0.0302	40000	0.1681	–
3.495	24	$1\frac{5}{8}$	20000	0.0310	35000	0.0923	–
3.495	12	–	40000	0.0121	80000	0.0330	–
3.495	12	–	40000	0.0124	80000	0.0340	–
3.496	6	–	90000	0.0083	170000	0.0222	0.0040
3.496	6	–	90000	0.0087	160000	0.0171	–
4.000	24	$1\frac{3}{4}$	25000	0.0227	50000	0.0667	–
4.000	24	$1\frac{3}{4}$	25000	0.0234	50000	0.0682	–
4.000	12	$1\frac{3}{4}$	55000	0.0097	110000	0.0258	–
4.000	12	$1\frac{3}{4}$	55000	0.0120	110000	0.0255	–
4.500	24	$1\frac{3}{4}$	38000	0.0240	75000	0.0648	–
4.500	24	$1\frac{3}{4}$	40000	0.0260	75000	0.0630	–
4.500	12	–	80000	0.0099	160000	0.0245	–
4.500	12	–	80000	0.0096	170000	0.0281	–
4.877	24	$1\frac{3}{4}$	50000	0.0240	100000	0.0912	–
4.877	24	$1\frac{3}{4}$	40000	0.0190	85000	0.0460	–
4.877	12	$1\frac{3}{4}$	100000	0.0155	200000	0.0324	–
4.877	12	$1\frac{3}{4}$	100000	0.0152	200000	0.0322	–

§ 232. **Riveted Joints.** — The most common way of uniting plates of wrought-iron or steel is by means of rivets. It is, therefore, a matter of importance to know the strength of such joints, and also the proportions which will render their efficiencies greatest; i.e., that will bring their strength as near as possible to the strength of the solid plate.

In § 177 was explained the mode of proportioning riveted joints usually taught, based upon the principle of making all the resistances to giving way equal, and assuming, as the modes of giving way, those there enumerated. This theory does not, however, represent the facts of the case, as —

1°. The stresses which resist the giving-way are of a more complex nature than those there assumed, so that the efficiency of a joint constructed in the way described above may not be as great as that of one differently constructed;

2°. The effects of punching, drilling, and riveting, come in to modify further the action; and

3°. The purposes for which the joint is to be used, often fix some of the dimensions within narrow limits beforehand.

In order to know, therefore, the efficiency of any one kind of joint, we must have recourse to experiment. And here again we must not expect to draw correct conclusions from experiments made upon narrow strips of plate riveted together with one or two rivets; but we need experiments upon joints in wide plates containing a sufficiently long line of rivets to bring into play all the forces that we have in the actual joint. The greater part of the experiments thus far made have been made upon narrow strips, with but few rivets, — there have been but few of the latter class of tests, — and no complete and systematic series of tests has thus far been carried out, though such a series has been begun on the government testing-machine at the Watertown Arsenal.

The only tests that will be quoted here, are: —

1°. The summary of the tests of this series that have been

made thus far, and are recorded in Executive Document No. 1, 47th Congress, second session, and Executive Document No. 5, 48th Congress, first session.

2°. A few tests made recently by David Kirkaldy.

While it is from these tests upon long joints that we can derive correct and reliable information to use in practice, and hence while the experiments already made give us some information in regard to such joints as were tested, nevertheless, as these tests have not yet been carried far enough to furnish all the information we need, and to settle cases that we are liable to be called upon to decide, therefore, before quoting the above experiments, some of the rules and proportions in use in practice at the present time, and the modes of determining them, will be first explained.

In this regard we must observe that practical considerations render it necessary to make the proportions different when the joint is in the shell of a steam-boiler, from the case when it is in a girder or other part of a structure.

In the case of boiler-work, the joint must be steam-tight, and hence the pitch of the rivets must be small enough to render it so: whereas in girder-work this requirement does not exist; and hence the pitch can, as far as this requirement goes, be made greater.

It is probable, that, with good workmanship, we might be able to secure a steam-tight joint with considerably greater pitches than those commonly used in boiler-work; and now and then some boiler-maker is bold enough to attempt it.

Tables of usual dimensions employed and recommended by Robert Wilson, Thomas Box, and Unwin respectively, will now be given.

The following tables give the proportions recommended by Robert Wilson for boiler-work, and by Thomas Box for girder-work:—

PROPORTIONS GIVEN BY ROBERT WILSON FOR USE IN BOILER-WORK.

Thickness of Plates.	Diameter of Rivets. All Lap-Joints and Butt-Joints with Single Strips.	Pitch of Rivets. Single-Riveted Lap-Joint.	Pitch of Rivets. Double-Riveted Lap-Joints, and Butt-Joints with Single Strips.	Double-Riveted Butt-Joints with Double Strips.	
				Diameter of Rivet.	Pitch of Rivets.
$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{3}{4}$	–	–
$\frac{5}{16}$	$\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{1}{4}$	–	–
$\frac{3}{8}$	$\frac{11}{16}$	$1\frac{3}{4}$	$2\frac{1}{4}$	$\frac{5}{8}$	$2\frac{1}{2}$
$\frac{7}{16}$	$\frac{3}{4}$	$1\frac{7}{8}$	$2\frac{1}{2}$	$\frac{5}{8}$	$2\frac{5}{8}$
$\frac{1}{2}$	$\frac{3}{4}$	$1\frac{7}{8}$	$2\frac{1}{2}$	$\frac{5}{8}$	$2\frac{3}{4}$
$\frac{9}{16}$	$\frac{7}{8}$	$2\frac{1}{8}$	$2\frac{3}{4}$	$\frac{11}{16}$	$2\frac{7}{8}$
$\frac{5}{8}$	$\frac{7}{8}$	$2\frac{1}{8}$	$2\frac{3}{4}$	$\frac{3}{4}$	3
$\frac{11}{16}$	$\frac{7}{8}$	$2\frac{1}{8}$	$2\frac{7}{8}$	$\frac{3}{4}$	$3\frac{1}{8}$
$\frac{3}{4}$	1	$2\frac{1}{4}$	$3\frac{1}{4}$	$\frac{7}{8}$	$3\frac{1}{4}$
$\frac{13}{16}$	1	$2\frac{1}{4}$	$3\frac{1}{4}$	$\frac{7}{8}$	$3\frac{1}{2}$
$\frac{7}{8}$	1	$2\frac{1}{4}$	$3\frac{1}{4}$	1	$3\frac{5}{8}$
$\frac{15}{16}$	$1\frac{1}{8}$	$2\frac{1}{2}$	$3\frac{1}{2}$	1	$3\frac{3}{4}$
1	$1\frac{1}{8}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$1\frac{1}{8}$	4

He gives, for the lap in single riveting, 3 times the diameter of the rivet, and never more than 3.3 times.

PROPORTIONS RECOMMENDED BY THOMAS BOX FOR USE IN GIRDER-WORK.

SINGLE-RIVETED LAP-JOINTS.

Thickness of Plates.	Diameter of Rivets.	Pitch of Rivets.	Lap.
$\frac{1}{8}$	$\frac{5}{16}$	$1\frac{1}{8}$	1
$\frac{3}{16}$	$\frac{7}{16}$	$1\frac{1}{2}$	$1\frac{3}{8}$
$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{3}{8}$
$\frac{5}{16}$	$\frac{9}{16}$	$1\frac{5}{8}$	$1\frac{1}{2}$
$\frac{3}{8}$	$\frac{11}{16}$	2	$1\frac{3}{4}$
$\frac{7}{16}$	$\frac{3}{4}$	$2\frac{1}{8}$	$1\frac{7}{8}$
$\frac{1}{2}$	$\frac{13}{16}$	$2\frac{3}{16}$	2

DOUBLE-RIVETED JOINTS.

Thickness of Plates.	Diameter of Rivets.	Pitch of Rivets.	Lap.
$\frac{3}{8}$	$\frac{11}{16}$	$2\frac{13}{16}$	$2\frac{7}{8}$
$\frac{7}{16}$	$\frac{3}{4}$	3	3
$\frac{1}{2}$	$\frac{13}{16}$	$3\frac{1}{16}$	$3\frac{1}{4}$
$\frac{9}{16}$	$\frac{7}{8}$	$3\frac{1}{4}$	$3\frac{3}{8}$
$\frac{5}{8}$	$\frac{15}{16}$	$3\frac{3}{8}$	$3\frac{5}{8}$
$\frac{11}{16}$	$1\frac{1}{16}$	$3\frac{15}{16}$	$4\frac{1}{8}$
$\frac{3}{4}$	$1\frac{1}{8}$	$4\frac{1}{16}$	$4\frac{1}{4}$

1°. In regard to the diameter of the rivets, Robert Wilson first shows, that by taking the crushing-strength in front of the rivet at 89600, and the shearing-strength of the rivet at 47040 lbs. per square inch, we should find that the joint would be safe against crushing the metal in front of the rivet with a diameter of rivet equal to $2\frac{1}{2}$ times the thickness of the plate; he recommends the diameters given in the table, which for plates $\frac{1}{4}$ inch and $\frac{5}{16}$ inch thick are double the thickness of the plate (this being a rule frequently used), and for thicker plates they grow gradually less.

Thomas Box, in deducing his diameters for girder-work, gives the formula

$$d = \tfrac{5}{4}t + \tfrac{3}{16},$$

from which he calculates the diameters given in his table, giving this as an empirical rule.

2°. In regard to the pitch of the rivets, Robert Wilson, by calling the shearing-strength of the rivets per square inch equal to the tensile strength of the plate per square inch, deduces the formula

$$p = \frac{\pi d^2}{4t} + d.$$

The values recommended by him differ, however, somewhat from the results of this formula, in order to retain a larger section of plate between rivets.

Thomas Box deduces his values of the pitch by considering the shearing-strength of the rivet per square inch as equal to $\frac{3}{4}$ the tensile strength of the plate per square inch, and then calculating the joint so as to give equal strength for tearing and shearing.

3°. In regard to the lap, Robert Wilson computes it so that there shall be strength enough to resist breaking through: his formula has been given in § 177. This would give, for the lap, the formula for single riveting,

$$l = 0.81d.$$

Hence he concludes that the common rule of making the distance between the hole and edge of the plate equal to the diameter of the rivet is to provide sufficient resistance in this regard: this rule, however, he contradicts by his empirical rule given just after the table.

Thomas Box, on the other hand, gives a graphical construction for the lap, which practically accounts the shearing and the tensile strength of the plate per square inch the same, and provides sufficient strength to prevent the rivet from shearing out the plate in front of it. His results are a little larger than three times the diameter of the rivet.

Next, as to Unwin's recommendations, —

1°. In regard to the diameter of rivet, he says that the diameters used in practice range from

$$d = \tfrac{3}{4}t + \tfrac{3}{8}$$

to

$$d = \tfrac{7}{8}t + \tfrac{3}{8},$$

and recommends, as a good rule to follow,

$$d = 1.2\sqrt{t}.$$

It will be seen that he is thus recommending a little larger diameters than Wilson or Box.

2°. As to the pitch, he determines it from the formula

$$(p - d)tf_t = \frac{\pi}{4} d^2 f_s,$$

where p = pitch, t = thickness of plate, d = diameter of rivet, f_t = tensile strength of plate between rivet-holes after the riveting has been done, f_s = shearing-strength of rivet per square inch, using such values of f_t and f_s as he considers suitable. The result of all this will be shown in the following tables, given by Unwin.

Ratio of Tearing and Shearing Strength $\frac{f_t}{f_s}$ *in Riveted Joints,* f_t *being Tensile Strength per Square Inch after the Plate has been Punched or Drilled.*

	Iron Plates, Iron Rivets.		Steel Plates, Steel Rivets.	
	Drilled or Punched, and Annealed or Rymered.	Punched.	Drilled or Punched, and Annealed or Rymered.	Punched.
Single-riveted	0.94	0.77	1.26	1.05
Double-riveted	1.02	0.85	1.34	1.17
Treble-riveted	1.05	–	1.36	–

Values of Pitch for Single Riveting for Various Values of $\frac{f_t}{f_s}$.

Thickness of Plates.	Nominal Diameter of Rivets.	Real Diameter of Rivets.	Single Riveting.							
			Iron Rivets.				Steel Rivets.			
			Iron Punched Plates.		Iron Drilled or Punched, and Annealed or Rymered Plates.		Steel Punched Plates.		Steel Punched and Annealed, or Rymered Plates.	
			Pitch for values of $\frac{ft}{fs}$ =							
			0.75	0.85	0.95	1.0	1.05	1.15	1.25	1.35
$\frac{5}{16}$	$\frac{11}{16}$	0.72	2.46	2.25	2.09	2.02	1.96	1.85	1.77	1.69
$\frac{3}{8}$	$\frac{3}{4}$	0.78	2.48	2.28	2.12	2.06	1.99	1.89	1.81	1.72
$\frac{7}{16}$	$\frac{13}{16}$	0.85	2.58	2.38	2.22	2.15	2.09	1.98	1.90	1.81
$\frac{1}{2}$	$\frac{7}{8}$	0.92	2.69	2.48	2.32	2.25	2.19	2.08	2.00	1.90
$\frac{5}{8}$	$\frac{15}{16}$	0.98	2.69	2.40	2.25	2.19	2.13	2.03	1.95	1.87
$\frac{3}{4}$	$1\frac{1}{16}$	1.10	2.79	2.59	2.43	2.37	2.31	2.20	2.12	2.04
$\frac{7}{8}$	$1\frac{1}{8}$	1.17	2.81	2.62	2.46	2.40	2.34	2.24	2.16	2.08
1	$1\frac{1}{4}$	1.30	3.07	2.86	2.70	2.63	2.56	2.45	2.36	2.28

By "real diameter" he means the diameter after riveting.

Thickness of Plates.	Nominal Diameter of Rivets.	Real Diameter of Rivets.	Double Riveting.				
			Iron Rivets.			Steel Rivets.	
			Iron Punched Plates.	Iron Drilled or Punched, and Ry-mered Plates.		Steel Punched Plates.	Steel Drilled or Punched, and Ry-mered Plates.
			Pitch of rivets for value $\frac{ft}{fs} =$				
			0.85	1.00	1.10	1.20	1.35
$\frac{5}{16}$	$\frac{11}{16}$	0.72	3.78	3.33	3.12	2.91	2.66
$\frac{3}{8}$	$\frac{3}{4}$	0.78	3.78	3.33	3.12	2.91	2.66
$\frac{7}{16}$	$\frac{13}{16}$	0.85	3.91	3.45	3.24	3.03	2.77
$\frac{1}{2}$	$\frac{7}{8}$	0.92	4.05	3.58	3.37	3.16	2.88
$\frac{5}{8}$	$\frac{15}{16}$	0.98	3.82	3.39	3.18	3.00	2.76
$\frac{3}{4}$	$1\frac{1}{16}$	1.10	4.08	3.63	3.42	3.22	2.98
$\frac{7}{8}$	$1\frac{1}{8}$	1.17	4.06	3.63	3.42	3.23	2.99
1	$1\frac{1}{4}$	1.30	4.42	3.95	3.74	3.52	3.26

It will be noticed, that, having used a larger rivet than Wilson or Box, he naturally used a larger pitch.

For lap, he gives the following values, computed by the same rule as Wilson computes his, but with a different constant; and he then compares them with values of $1.5d$, which he states to be an ordinary rule.

d	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$
For iron, $l =$	1.00	1.14	1.29	1.41	1.55	1.67	1.80
For steel, $l =$	0.86	0.98	1.12	1.22	1.35	1.46	1.57
$1.5d$	0.75	0.94	1.12	1.31	1.50	1.69	1.88

Efficiency of the Joint. — The efficiency of the joint is an item of great importance. Sir William Fairbairn's experiments gave the following efficiencies: —

Entire plate	100 per cent,
Double-riveted joint	76 "
Single-riveted joint	56 "

and these efficiencies have been very much quoted and used.

Mr. Forney, in his "Catechism of the Locomotive," gives the following: —

Solid plate	100.0
Single-riveted seam, punched holes, $\frac{11}{16}$ rivets	50.5
" " " drilled " $\frac{7}{8}$ "	60.0
Double-riveted seam, drilled and zigzag holes, $\frac{3}{4}$ rivets .	70.0
" " " " " " $\frac{7}{8}$ " .	72.0
Single riveted seam, punched holes, with covering-plate .	65.3

On the other hand, Unwin gives the following tables for the efficiencies in different cases: —

Thickness of Plates.	Nominal Diameter of Rivets.	Real Diameter of Rivets.	Single Riveting.							
			Iron Punched Plates.		Iron Drilled or Punched, and Annealed or Rymered Plates.		Steel Punched Plates.		Steel Punched and Annealed, or Rymered Plates.	
			Efficiency of joints for values of $\frac{f_t}{f_s}$ =							
			0.77		0.88		0.9		1.0	
$\frac{5}{16}$	$\frac{11}{16}$	0.72	0.55	0.52	0.58	0.56	0.57	0.55	0.59	0.57
$\frac{3}{8}$	$\frac{3}{4}$	0.78	0.53	0.51	0.55	0.54	0.55	0.53	0.57	0.55
$\frac{7}{16}$	$\frac{13}{16}$	0.85	0.52	0.49	0.55	0.54	0.53	0.51	0.55	0.53
$\frac{1}{2}$	$\frac{7}{8}$	0.92	0.51	0.49	0.52	0.52	0.52	0.50	0.54	0.52
$\frac{5}{8}$	$\frac{15}{16}$	0.98	0.48	0.45	0.49	0.48	0.49	0.47	0.50	0.48
$\frac{3}{4}$	$1\frac{1}{16}$	1.10	0.47	0.44	0.48	0.47	0.47	0.45	0.48	0.46
$\frac{7}{8}$	$1\frac{1}{8}$	1.17	0.45	0.42	0.46	0.45	0.45	0.43	0.46	0.44
1	$1\frac{1}{4}$	1.30	0.42	0.40	0.46	0.45	0.45	0.43	0.45	0.43

Thickness of Plates.	Nominal Diameter of Rivets.	Real Diameter of Rivets.	Double Riveting.				
			Iron Punched Plates.	Iron Drilled or Punched, and Rymered Plates.		Steel Punched Plates.	Steel Drilled or Punched, and Rymered Plates.
			Efficiency of joints for values of $\frac{ft}{fs}$ =				
			0.85	0.95	1.00	1.00	1.06
$\frac{5}{16}$	$\frac{11}{16}$	0.72	0.69	0.74	0.77	0.75	0.77
$\frac{3}{8}$	$\frac{3}{4}$	0.78	0.68	0.73	0.75	0.73	0.75
$\frac{7}{16}$	$\frac{13}{16}$	0.85	0.66	0.71	0.74	0.72	0.74
$\frac{1}{2}$	$\frac{7}{8}$	0.92	0.65	0.70	0.73	0.71	0.73
$\frac{5}{8}$	$\frac{15}{16}$	0.98	0.63	0.67	0.69	0.67	0.69
$\frac{3}{4}$	$1\frac{1}{16}$	1.10	0.62	0.66	0.68	0.66	0.68
$\frac{7}{8}$	$1\frac{1}{8}$	1.17	0.60	0.65	0.66	0.64	0.65
1	$1\frac{1}{4}$	1.30	0.60	0.63	0.65	0.63	0.64

Punching and Drilling. — One matter in this connection that has occupied a good deal of attention is the relative advantages of punched and drilled holes. The usual practice is to punch the holes, and it is less expensive than drilling them.

On the other hand, it is generally acknowledged, and has been shown by a number of experimenters, that punching in most cases injures the metal around the hole: the amount of this injury may vary from a very small quantity up to 20 per cent in iron, and up to 35 per cent in steel, plates. The amount of injury will vary according to the quality of the plate, and also according to the amount of clearance between the punch and the die; the injury being less in plates of good quality and ductile, and greater in hard and brittle plates, also less when the clearance between the punch and the die is ample than when it is too small.

Accounts of tests on this subject are to be found in —

Sir William Fairbairn: Application of Iron to Building Purposes.
J. Barba: Use of Steel in Construction.
A. F. Hill: Paper in the Proceedings of Society of Engineers of Western Pennsylvania.
Executive Document No. 1, 47th Congress, 2d session, p. 142 *et seq.*
Executive Document No. 5, 48th Congress, 1st session.

There are two other matters that sometimes appear to modify these statements as far as breaking-strength is concerned; i.e., —

1°. When the specimen is a grooved one, as it must be when punched or drilled, its apparent ultimate strength in the testing-machine is greater than it would be if it had any opportunity to stretch.

2°. Cold-punching has, to a small extent, a similar effect to cold-rolling; i.e., it may harden the metal a little, and increase its breaking-strength on this account, while rendering it less ductile, and hence more brittle.

The injury done by punching may be almost entirely removed in either of the following ways: —

1°. By annealing the plate.

2°. By reaming out the injured portion of the metal around the hole; i.e., by punching the hole a little smaller than is desired, and then reaming it out to the required size. This removing the injurious effect of punching is more needed in steel than in iron plates.

There is a certain friction developed by the contraction of the rivets in cooling, tending to resist the giving-way of the joint; but this is likely to disappear before fracture takes place, and cannot, therefore, be depended upon.

The shearing-strength of the rivets would appear to be about $\frac{3}{4}$ or $\frac{4}{5}$ the tensile strength of the plate.

Government Experiments. — These experiments form, as has been already stated, the first portion of a systematic series; and the tables of results are given here, because it seems to the author, that, although the series is not yet completed, yet these tests themselves furnish more reliable information in regard to the behavior and the strength of joints made somewhat in these proportions than any other experiments that have been made, and that the figures themselves furnish the engineer with the means of using his judgment in many cases where he had no reliable data before.

Thus it is plain, that, when we compute the crushing-strength in front of the rivet, it is not this direct crushing action that is produced, as the small piece of plate that lies in front of a rivet would not bear so much unless re-enforced by the metal on the sides.

Nevertheless, the compression per square inch on the bearing-surface of the rivet (i.e., the pull on one rivet divided by the longitudinal projection of its bearing on one plate, however it be resisted) apparently forms an important element in the strength of the joint, and shows the advantage of large rivets; this being one of the deductions made by Mr. James E. Howard (who conducted the tests) from these tests, inasmuch as some of the experiments tried with the same section of plate between the holes, and with rivets of different size, showed a greater efficiency for the larger rivets.

In calculating the resistance to tearing out of the plate in front of the rivet by the rules of Wilson or Unwin, we must observe that there is not the action that is considered in the formulæ.

The forces brought into play are more complex, and it is only experiment that can show us what they are.

A perusal of the tables will give a good idea of the shearing-strength per square inch of the rivet iron, which is seen to be less than the tensile strength of the solid plate; also the

loss of strength of the plates due to the entire process of riveting, punching, drilling, and driving the rivets; also the efficiencies of the joints tested.

One of the strongest single-riveted joints tested was a single-riveted lap-joint with a single covering-strip.

The apparent anomaly of the punched plates in a few cases, showing a greater strength than the drilled plates, is explained by Mr. Howard to be due to the strengthening effect of cold-punching combined with smallness of pitch, inasmuch as then the masses of hardened metal on the two sides re-enforce each other.

Further than this, the student is left to study the figures themselves as to the effect of different proportions, etc.

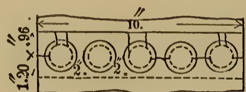
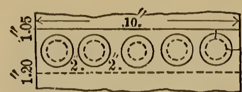
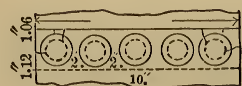
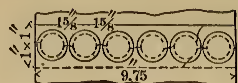
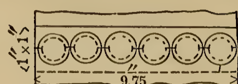
In regard to these tests, it is stated in the report that —

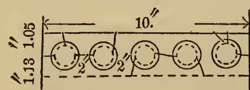
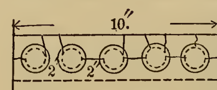
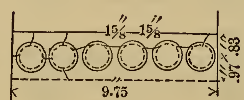
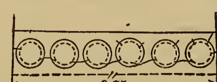
1°. "The wrought-iron plate was furnished by one maker out of one quality of stock."

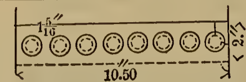
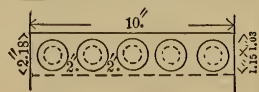
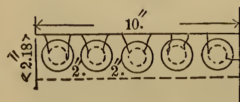
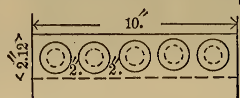
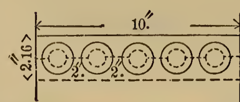
2°. "The steel plates were supplied from one heat, cast in ingots of the same size; the thin plates differing from the thicker plates only in the amount of reduction given by the rolls."

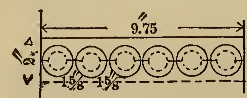
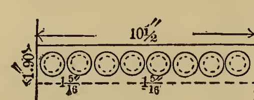
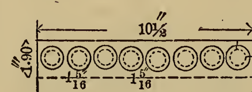
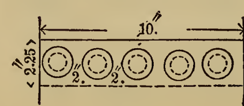
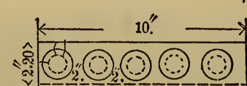
The modulus of elasticity of the metal was, iron plate, 31970000 lbs.; steel plate, 28570000 lbs.

In the tabulated results, the manner of fracture is shown by sketches of the joints, and is further indicated by heavy figures in columns headed "Maximum Strains on Joints, in lbs., per Square Inch."

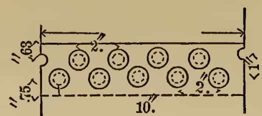
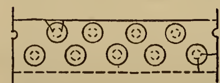
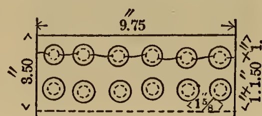
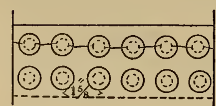
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
35		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. punched holes	47925	27630	43230	76140	34900	57.7
36		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. punched holes	47925	29444	45520	82910	38640	61.4
37		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. drilled holes	47925	25270	38580	73260	34870	52.8
38		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. drilled holes	47925	27380	41790	79360	38660	57.1
39		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. punched holes	47925	29020	52160	65420	33420	60.6
40		$\frac{5}{8}$-in. iron rivets; $1\frac{1}{16}$-in. punched holes	47925	30680	54930	68890	35200	64.0

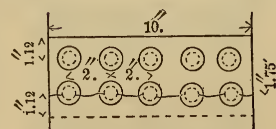
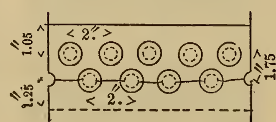
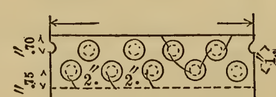
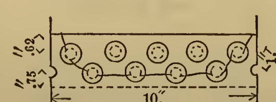
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
41		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. punched holes	47925	31580	49420	87670	39640	65.9
42		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. punched holes	47925	30220	47260	83940	40610	63.1
43		$\frac{1}{2}$-in. steel rivets $\frac{9}{16}$-in. punched holes	47925	28875	45890	78220	45300	60.3
44		$\frac{1}{2}$-in. steel rivets $\frac{9}{16}$-in. punched holes	47925	31390	49720	84660	48420	65.5
45		$\frac{7}{16}$-in. iron rivets $\frac{1}{2}$-in. drilled holes	47925	25440	41095	66778	44204	53.1

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
46		$\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	23158	37500	60886	42038	48.3
		$\frac{1}{4}$-Inch Steel Plates.						
426		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	29690	46340	82480	37890	53.2
427		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	29430	46010	81780	37860	52.8
436		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. punched holes	55765	38610	60250	107260	49270	69.2
437		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. punched holes	55765	37895	59240	105290	48750	68.0
428		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	55765	26870	40950	77870	36350	48.2
429		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	55765	27680	42370	80200	36710	49.6
438		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	41460	63190	120160	56100	74.3
439		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	40040	61310	116090	52460	71.8

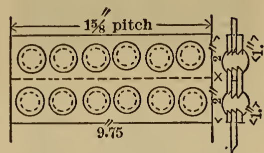
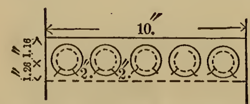
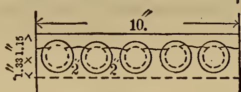
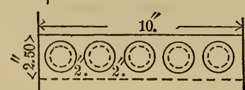
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
430		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	38360	66860	90000	41790	68.8
431		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	40160	70000	94230	43750	72.0
47		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	38496	62496	101180	65220	69.0
48		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	36114	58338	94800	60382	64.8
49		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	39400	60184	114603	52742	70.6
50		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	37680	57439	109650	50645	67.6

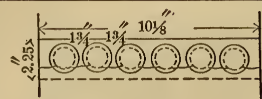
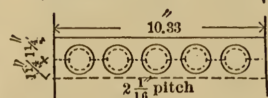
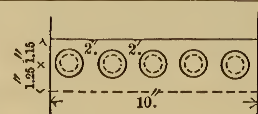
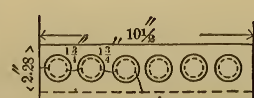
DOUBLE-RIVETED LAP-JOINTS, $\frac{1}{4}$-INCH IRON AND STEEL PLATE.

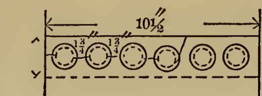
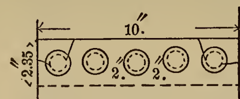
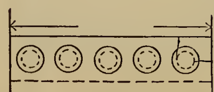
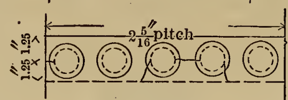
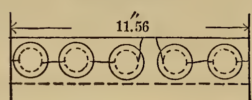
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
85		*Iron Plate.* $\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	28900	38535	64120	43110	60.3
86		*Iron Plate.* $\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	31314	41750	69710	41750	65.3
617		*Iron Plate.* $\frac{1}{2}$-in. iron rivets, $\frac{9}{16}$-in. punched holes	47925	31550	50592	42118	28691	65.8
618		*Iron Plate.* $\frac{1}{2}$-in. iron rivets, $\frac{9}{16}$-in. punched holes	47925	31282	49950	41660	28660	65.3

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
432		*Steel Plate.* $\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	39280	61510	54640	25400	70.4
433		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	38700	60300	53715	25530	69.4
434		*Steel Plate.* $\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	41785	65400	64600	30430	74.9
435		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	41460	64600	63430	30430	74.3
87		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	42530	56944	94910	57910	76.3
88		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	44344	59130	98360	61130	79.5

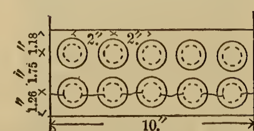
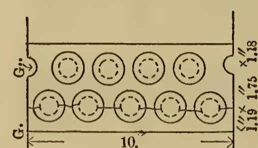
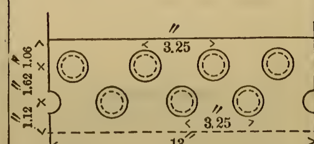
DOUBLE-WELT BUTT-JOINTS, $\frac{1}{4}$-INCH IRON PLATE.

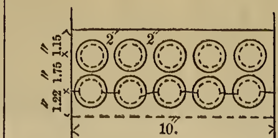
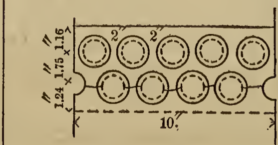
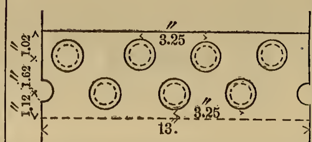
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
615		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	29800	53475	67321	16944	62.2
616		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	28397	50959	64138	16719	59.3
		SINGLE-RIVETED LAP-JOINTS, $\frac{3}{8}$-INCH IRON PLATE.						
62		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23110	37460	60340	38280	49.0
63		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	22280	36130	58150	35520	47.2
64		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23445	38190	60730	37530	49.7
65		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	22220	36210	57530	36050	47.1

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
66		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23570	41750	54130	34230	50.0
67		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23280	41290	53400	34150	49.3
720		1-in. iron rivets, $1\frac{1}{16}$-in. punched holes	47180	28500	61700	52970	26180	60.4
721		1-in. iron rivets, $1\frac{1}{16}$-in. punched holes	47180	26960	58510	50220	24830	57.1
SINGLE-RIVETED LAP-JOINTS, $\frac{3}{8}$-INCH STEEL PLATE.								
51		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	24180	39220	63210	39740	45.4
52		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	23240	37700	60760	38190	43.6
53		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. punched holes	53330	34160	55215	89580	56430	64.1
54		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. punched holes	53330	33840	54740	88660	55460	63.5
55		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	35590	63650	80930	50650	66.7

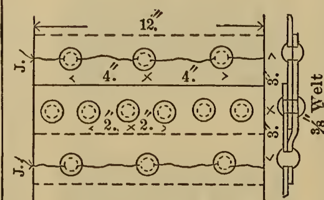
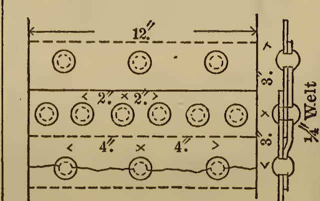
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
56		$\frac{11}{16}$-in. steel rivets $\frac{3}{4}$-in. drilled holes	53330	35860	63976	81600	50900	67.2
238		$\frac{3}{4}$-in. steel rivets $\frac{13}{16}$-in. punched holes	53330	37810	65460	89490	53560	70.9
239		$\frac{3}{4}$-in. steel rivets $\frac{13}{16}$-in. punched holes	53330	37630	65210	88990	53600	70.6
718		1 -in. iron rivets 1$\frac{1}{16}$-in. punched holes	53330	38075	73394	79510	36614	71.4
719		1 -in. iron rivets 1$\frac{1}{16}$-in. punched holes	53330	38390	73970	80200	36590	72.0

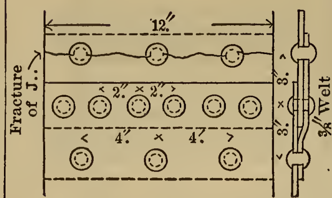
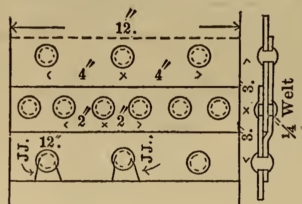
DOUBLE-RIVETED LAP-JOINTS, $\frac{3}{8}$-INCH IRON AND STEEL PLATE.

No. of Test.	Style of Joint.		Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
68		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	29970	48450	39160	24760	63.5
69			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	31320	50730	41070	26150	66.4
58			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	31000	50220	40640	25330	65.7
70		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28530	46255	41480	27550	60.5
71			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28475	46110	41270	27010	60.4
81		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23750	30920	58700	39130	50.4
82			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23180	30130	57340	38410	49.1

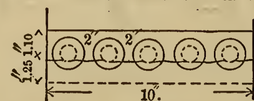
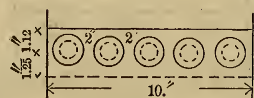
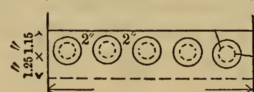
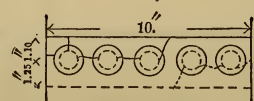
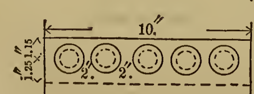
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
57		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	39020	62800	50760	32310	73.2
59		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	40080	64720	52450	32930	75.2
60		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	39010	63210	56860	34710	73.2
61		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	34000	54930	49530	30830	63.8
83		*Steel plate.* $\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	34316	44660	84460	52750	64.4
84		*Steel plate.* $\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	33585	43650	83000	51845	63.0

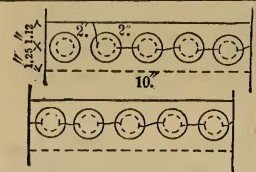
RIVETED JOINTS, $\frac{3}{8}$-INCH IRON AND STEEL PLATE, RE-ENFORCED LAP-JOINTS.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
244		*Iron plate.* $\frac{5}{8}$-in. iron rivets, 0.72-in. drilled holes	47180	31900	38870	59080	40360	67.0
245		*Iron plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	34900	43770	56640	34460	74.0
296		*Iron plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	35700	44840	57910	33890	75.7
297		*Iron plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	33950	42680	55350	31810	71.9

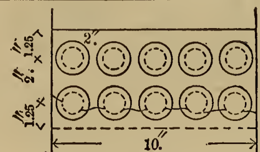
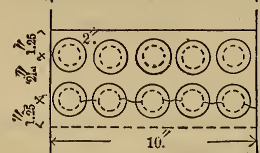
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
246		Steel Plate. $\frac{7}{8}$-in. steel rivets, $\frac{15}{16}$-in. drilled holes	53330	47465	62050	67320	32960	89.0
247		Steel Plate. $\frac{7}{8}$-in. steel rivets, $\frac{15}{16}$-in. drilled holes	53330	48050	62880	68135	33900	90.1
298		Steel Plate. $\frac{7}{8}$-in. iron rivets, $\frac{15}{16}$-in. drilled holes	53330	46820	61020	67300	34250	87.8
299		Steel Plate. $\frac{7}{8}$-in. iron rivets, $\frac{15}{16}$-in. drilled holes	53330	47390	61710	68040	34750	88.9

SINGLE-RIVETED LAP-JOINTS, $\frac{1}{2}$-INCH IRON AND STEEL PLATE.

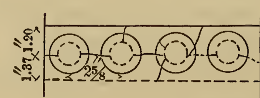
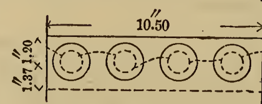
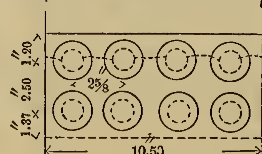
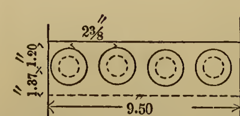
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
240		*Iron Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	17760	31100	41500	34280	39.8
241		*Iron Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	17690	31395	41955	34960	39.7
292		*Iron Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	44615	19130	32376	47850	38020	42.9
293		*Iron Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	44615	19765	33180	48890	39220	44.3
327		*Iron Plate.* $\frac{3}{4}$-in. steel rivets, $\frac{13}{16}$-in. drilled holes	44615	23740	39900	58880	47020	52.2
328		*Iron Plate.* $\frac{3}{4}$-in. steel rivets, $\frac{13}{16}$-in. drilled holes	44615	24200	40590	59900	47830	54.2
242		*Steel Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	57215	21830	38204	50940	41100	38.2
243		*Steel Plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	57215	20523	35915	47890	38636	35.9

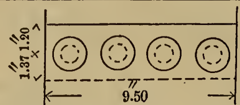
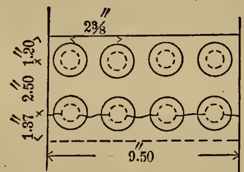
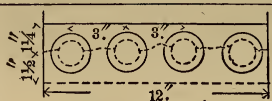
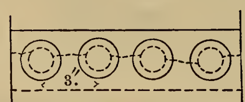
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
294		*Steel plate.* $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	29310	60210	56980	36770	51.2
295		$\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	24145	49590	47060	30540	42.2

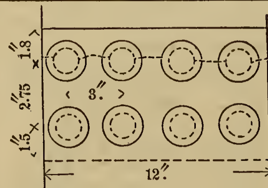
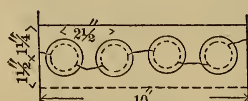
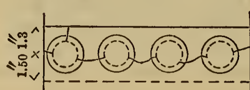
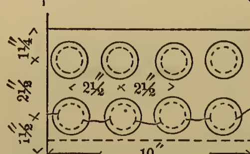
Double-Riveted Lap-Joints, $\frac{1}{2}$-Inch Iron and Steel Plate.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
329		*Iron plate.* $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	25430	44320	59640	25280	57.0
635		$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	24660	42920	57950	24560	55.2
619		*Steel plate.* $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	30757	64602	29354	19670	53.8
620		$\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	30778	64519	29371	19644	53.8

SINGLE AND DOUBLE RIVETED LAP-JOINTS, $\frac{5}{8}$-INCH IRON PLATE.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
730		1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	20020	34680	47510	35460	44.9
731		1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	19750	34230	46790	34930	42.0
732		1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	25150	43580	29740	22990	56.3
733		1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	26470	45850	31310	23670	59.3
734		1 -in. steel rivets $1\frac{1}{16}$-in. punched holes	52445	26480	49650	56760	43490	50.5

No. of Test.	Style of Joint.		Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
					Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
735			1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes	52445	28110	52770	60150	46080	58.6
736			1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes	52445	37180	69680	39780	30470	70.9
737			1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes	52445	35800	67100	38300	29340	68.3
Single and Double Riveted Lap-Joints, $\frac{3}{4}$-Inch Iron and Steel Plate.									
722		*Iron plate.*	$1\frac{1}{8}$ -in. iron rivets; $1\frac{3}{16}$-in. punched holes	46590	17230	29290	41980	32600	37.0
723		*Iron plate.*	$1\frac{1}{8}$ -in. iron rivets; $1\frac{3}{16}$-in. punched holes	46590	18080	30730	44040	34200	38.8

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
724		*Iron plate.* $1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	24720	42000	30040	23460	53.1
725		*Iron plate.* $1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	25870	43950	31460	23860	55.5
726		*Steel Plates.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	20280	39970	41180	33840	39.3
727		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	23940	47370	48540	39890	46.4
728		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	24720	48970	24990	20450	48.0
729		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	24050	47510	24340	20060	46.7

No. of Test.	Style of Joint.	Kind and Thickness of Plate.	Size and Kind of Rivets and Holes.	Tensile Strength per Square Inch.	Maximum Stresses on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets	Efficiency of Joint.
4418	Single-riveted lap	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	47180	23170	39300	50850	33710	47.0
4419	Single-riveted lap	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	47180	23130	41000	53050	35170	49.0
4428	Single-riveted lap	$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	44615	20340	35650	47350	37300	45.6
4429	Single-riveted lap	$\frac{3}{8}$-in. iron	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	44615	20050	35150	46640	36780	44.9
764	Single-riveted butt	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	47180	28260	46360	72390	25380	59.9
765	Single-riveted butt	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	47180	28550	46875	73050	25450	60.5
768	Single-riveted butt	$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	44615	26530	46400	61940	24630	59.4
769	Single-riveted butt	$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	44615	26400	46140	61740	24310	59.2
772	Single-riveted butt	$\frac{3}{8}$-in. iron	1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	25530	44260	60330	23010	57.2
773	Single-riveted butt	$\frac{3}{8}$-in. iron	1 -in. iron rivets $1\frac{1}{16}$-in. punched holes	44635	24490	42350	58080	22310	54.9
776	Single-riveted butt	$\frac{3}{4}$-in. iron	$1\frac{1}{8}$-in. iron rivets $1\frac{3}{16}$-in. punched holes	46590	24290	42310	57000	21870	52.1
777	Single-riveted butt	$\frac{3}{4}$-in. iron	$1\frac{1}{8}$-in. iron rivets $1\frac{3}{16}$-in. punched holes	46590	24073	41920	56540	22140	51.7

No. of Test.	Style of Joint.	Kind and Thickness of Plate.	Size and Kind of Rivets and Holes.	Tensile Strength per Square Inch.	Maximum Stresses on Joints, in lbs., per Square Inch.				Efficiency of Joint.
					Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
4420	Single-riveted lap	$\frac{3}{8}$-in. steel	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	53330	31720	61270	65760	40390	59.5
4427		$\frac{3}{8}$-in. steel	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	53330	31500	60830	65320	39990	59.1
4430		$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	57215	33000	47530	44590	29390	40.2
4431		$\frac{3}{8}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	57215	24180	49840	46960	31070	42.3
766	Single-riveted butt	$\frac{3}{8}$-in. steel	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	53330	38240	62770	97940	31240	71.7
767		$\frac{3}{8}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	53330	37260	61210	95210	31020	69.8
770		$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	57215	32690	68920	62220	20370	57.1
771		$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	57215	31470	66710	59580	19890	55.0
774		$\frac{5}{8}$-in. steel	1-in. steel rivets $1\frac{1}{16}$-in. punched holes	52445	33250	62180	71450	27750	63.4
775		$\frac{5}{8}$-in. steel	1-in. steel rivets $1\frac{1}{16}$-in. punched holes	52445	33470	62590	71930	27940	63.8
778		$\frac{3}{4}$-in. steel	$1\frac{1}{8}$-in. steel rivets $1\frac{3}{16}$-in. punched holes	51545	27810	54650	55610	23190	54.0
779		$\frac{3}{4}$-in. steel	$1\frac{1}{8}$-in. steel rivets $1\frac{1}{16}$-in. punched holes	51545	27500	54200	55840	22810	53.4

The following tests, made recently by David Kirkaldy, are also quoted from Executive Document No. 48, 47th Congress, second session, p. 24 *et seq.*

These tests were made on joints of $\frac{7}{8}$-inch steel plate. The rivet-holes of 1418, "4," 1406, "1," 1410, "2," and 1414, "3," were drilled separately with a $1\frac{1}{16}$-inch drill. The plates were then heated, as the boiler-plates themselves would be for rolling, and, after cooling, were put together, and the holes reamed out to $1\frac{1}{8}$ inch.

The rivet-holes of the remainder were punched separately with a $\frac{7}{8}$-inch punch on a $1\frac{1}{16}$-inch die. The plates were then treated like the others. All samples were riveted together the full width of the plates, 24 inches.

The efficiencies deduced from the Watertown experiments are not summed up here, as they can be obtained from the tables.

On page 480 will be given the tests of grooved specimens of wrought-iron and steel made at the arsenal on plates used in the joints already referred to. These plates include both punched and drilled plates, and vary in thickness from $\frac{1}{4}$ inch to 1 inch.

Test number and mark	N 1418, "4"		N 1426, "B"		N 1422, "A"		N 1406, "1."	
Description of joint	Lap, triple riveted		Lap, triple riveted		Lap, quadruple riveted		Lap, quadruple riveted.	
Machine or hand riveted	Machine		Machine		Machine		Machine.	
Plates: width and thickness	13.50 × 0.875		13.50 × 0.860		13.50 × 0.865		13.60 × 0.875.	
Plates: sectional area, gross	11.812		11.610		11.677		11.900	
	Elastic.	Ultimate.	Elastic.	Ultimate.	Elastic.	Ultimate.	Elastic.	Ultimate.
Stress, total	197000	391540	208000	416720	265000	496830	296000	518370
Stress per square inch of gross area, joint	16670	33148	17910	35893	22690	42548	24870	43559
Stress per square inch of gross plates, solid	33300	61400	33650	61167	33500	60752	35900	61500
Ratio of joint to solid plate, %	50.06	53.99	53.22	58.68	67.73	70.04	69.27	70.83
Where fractured	9 rivets sheared		9 rivets sheared		Plate at rivet-holes, 100 % silky		12 rivets sheared.	
Rivets: diameter, area, and number	Steel, 1.13 = 1.002 × 9 × 1		Steel, 1.13 = 1.002 × 9 × 1		Steel, 1.13 = 1.002 × 12 × 1		Steel, 1.13 = 1.002 × 12 × 1.	
Rivets: sectional area, total square inches	9.018		9.018		12.024		12.024	
Shearing-stress per square inch of rivet area	43417		46263		41319		43111	
Tensile stress per square inch of rivet area	64840		64840		64840		64840	
Ratio of shearing to tensile percentage	66.96		71.35		Rivets not sheared		66.49	

Test number and mark	N 1410, "2"		N 1430, "C"		N 1414, "3."	
Description of joint	Butt double riveted		Butt double riveted		Butt double riveted.	
Machine or hand riveted	Machine		Machine		Machine.	
Plates: width and thickness	13.50 × 0.885		13.53 × 0.880		13.52 × 0.865	
Plates: sectional area, gross	11.947		11.906		11.694	
	Elastic.	Ultimate.	Elastic.	Ultimate.	Elastic.	Ultimate.
Stress, total	294000	441820	298000	454190	335000	534110
Stress per square inch of gross area, joint	24610	36981	25030	38147	28650	45673
Stress per square inch of plates, solid	33600	62895	33150	60697	35350	63382
Ratio of joint to solid plate, %	73.24	58.80	75.54	62.85	81.06	72.06
Where fractured	Plate at rivet-holes, 100% silky		Plate at rivet-holes, 25% silky, 75% granular		6 rivets sheared.	
Rivets: diameter, area, and number	Steel, 1.13 = 1.002 × 6 × 2		Steel, 1.13 = 1.002 × 6 × 2		Steel, 1.13 = 1.002 × 6 × 2	
Rivets: sectional area, total square inches	12.024		12.024		12.024	
Shearing-stress per square inch of rivet area	36745		37773		44420	
Tensile stress per square inch of rivet area	64840		64840		64840	
Ratio of shearing to tensile percentage	Rivets not sheared		Rivets not sheared		68.51	

GOVERNMENT TESTS OF GROOVED SPECIMENS.

Tensile Tests of ¼-in. Grooved Specimens Wrought-Iron Punched.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.48	0.240	48090
0.46	0.235	46940
0.46	0.241	49280
0.49	0.240	55340
0.44	0.239	51520
0.47	0.241	49910
0.97	0.247	49540
0.98	0.247	49960
0.94	0.249	50128
0.96	0.248	46900
0.98	0.250	46980
0.96	0.251	46350
1.47	0.250	37636
1.50	0.252	37326
1.48	0.249	41030
1.48	0.247	39480
1.47	0.250	37446
1.45	0.251	39533
1.96	0.281	43194
1.95	0.274	47490
1.95	0.282	41360
1.92	0.279	43080
2.03	0.250	41140
1.99	0.248	39575
2.42	0.280	36210
2.40	0.245	42245
2.47	0.243	42233
2.46	0.285	42712
2.48	0.245	38125
2.44	0.248	41620
2.97	0.247	38964
2.98	0.241	41540
2.96	0.241	39972
2.92	0.240	41712
2.98	0.250	40430
2.95	0.247	40850

Tensile Tests of ¼-in. Grooved Specimens Wrought-Iron Drilled.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.51	0.249	55787
0.52	0.245	55905
0.52	0.275	57480
0.52	0.276	56000
0.49	0.248	49600
0.50	0.248	56700
0.47	0.275	54880
0.51	0.276	57800
1.00	0.276	54300
1.02	0.273	57700
1.00	0.276	53800
1.00	0.280	52430
1.00	0.252	49400
1.02	0.275	54060
1.01	0.247	52770
1.00	0.278	54600
1.50	0.276	49130
1.52	0.273	51300
1.48	0.251	47220
1.51	0.273	53400
1.52	0.275	54180
1.50	0.276	54600
1.48	0.274	56250
1.50	0.249	46260
2.01	0.275	45900
2.05	0.279	46820
2.00	0.275	47950
2.00	0.278	49640
2.00	0.286	44650
2.00	0.275	50780
2.02	0.279	48850
2.00	0.277	49840
2.51	0.244	44980
2.52	0.280	40150
2.51	0.282	43150
2.50	0.244	45500
2.51	0.285	46500
2.49	0.242	49520
2.49	0.242	–
2.50	0.280	44780
3.02	0.250	45700
3.02	0.249	44870
3.00	0.240	46760
3.00	0.250	45700
2.93	0.242	47950
2.99	0.250	48740
2.98	0.279	45900
3.01	0.281	44410

Tensile Tests of ¼-in. Grooved Specimens Steel Plate Punched.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.49	0.250	65120
0.47	0.249	67010
0.48	0.249	63420
0.48	0.248	66550
0.48	0.247	67060
0.47	0.248	65300
0.99	0.249	59840
1.00	0.250	62160
1.01	0.249	68246
0.96	0.250	67330
0.96	0.248	65966
0.95	0.245	62700
1.45	0.248	64080
1.45	0.252	64000
1.45	0.249	61025
1.51	0.251	59420
1.96	0.250	59900
1.93	0.252	63500
1.98	0.250	59350
1.96	0.251	59060
2.49	0.249	58100
2.47	0.249	63900
2.43	0.250	61640
2.95	0.251	56530
3.01	0.249	58780
3.04	0.253	55500
2.97	0.252	60060
2.98	0.251	54050
2.97	0.249	56040

Tensile Tests of ¼-in. Grooved Specimens Steel Plate Drilled.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.52	0.246	67890
0.54	0.248	67160
0.53	0.247	66870
0.50	0.247	65610
0.51	0.249	66370
0.51	0.250	67420
0.52	0.248	67750
0.52	0.252	61910
1.03	0.247	67090
1.02	0.250	66390
1.02	0.246	66770
1.02	0.250	67730
1.01	0.247	66020
1.00	0.251	67010
1.00	0.247	64450
1.01	0.250	66090
1.54	0.250	64390
1.52	0.251	63350
1.50	0.253	64370
1.54	0.248	64895
2.02	0.252	64320
2.00	0.251	62970
2.00	0.251	60910
2.50	0.248	59260
2.50	0.252	63250
2.53	0.248	59390
3.03	0.251	61577
3.00	0.249	59080
3.02	0.251	59550
3.02	0.250	59700
3.00	0.250	63370
3.00	0.251	58630
3.03	0.252	63940

IRON PUNCHED.

Tensile Tests of Grooved Wrought-Iron Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.01	0.373	47000
0.98	0.370	47520
2.00	0.382	39760
2.02	0.383	36630
2.39	0.390	37600
2.98	0.395	36340
2.98	0.392	39210
3.47	0.390	37680
3.47	0.389	38340
0.97	0.467	50820
1.48	0.506	45090
1.49	0.506	45050
1.91	0.513	42500
1.97	0.512	43430
2.47	0.516	39410
2.41	0.513	39720
3.00	0.515	38950
2.90	0.517	37290
3.50	0.520	37800
3.49	0.513	37770
4.00	0.515	35730
4.03	0.516	36690
3.99	0.511	37000
4.03	0.508	37420
0.97	0.614	49770
1.01	0.619	52960
1.48	0.618	46320
1.52	0.620	46750
2.99	0.614	40140
3.50	0.615	37480
3.50	0.616	36940
4.04	0.619	37310
0.98	0.678	50840
1.01	0.682	46590
1.49	0.688	45970
3.48	0.691	40350
3.53	0.692	39380

IRON DRILLED.

Tensile Tests of Grooved Wrought-Iron Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch	
0.98	0.376	50870
0.98	0.377	52660
1.98	0.379	49710
2.00	0.380	49830
2.50	0.390	50250
3.00	0.392	45150
3.00	0.393	47540
3.50	0.392	43940
3.49	0.390	46490
0.99	0.477	47140
1.00	0.479	48370
1.49	0.510	51240
1.49	0.512	51510
1.98	0.514	50050
1.98	0.516	47790
2.51	0.520	45580
2.52	0.516	44960
3.00	0.515	44980
3.01	0.519	47030
3.51	0.513	46170
3.49	0.514	44760
3.99	0.510	45330
3.98	0.513	45000
4.00	0.506	46100
0.97	0.628	47220
1.00	0.626	48350
1.52	0.625	47170
1.49	0.629	46530
2.98	0 613	48220
3.46	0.616	47770
3.47	0.617	44900
3.91	0.625	44840
3.96	0.626	45100
0.99	0.695	47500
0.99	0.691	52780
1.51	0.692	48470
3.44	0.700	47750
3.49	0.692	46350

STEEL PUNCHED.

Tensile Tests of Grooved Steel Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.99	0.365	61890
0.99	0.494	70080
1.00	0.492	68130
1.50	0.497	66340
1.51	0.494	63810
1.99	0.499	55930
1.97	0.500	64260
2.43	0.502	52050
2.51	0.504	64360
3.00	0.503	60320
2.99	0.503	62430
3.50	0.503	49430
3.50	0.505	48270
4.00	0.497	48010
4.00	0.499	55190
3.99	0.501	55780
3.99	0.498	46250
1.01	0.613	66720
1.52	0.612	64800
1.50	0.615	64400
2.50	0.618	58060
2.52	0.619	58780
2.99	0.617	57180
3.46	0.615	58410
3.51	0.615	57190
4.04	0.612	54450
4.03	0.614	57380
1.01	0.721	67930
1.00	0.718	67620
1.50	0.719	62890
3.50	0.735	56730
3.51	0.733	54220

STEEL DRILLED.

Tensile Tests of Grooved Steel Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.97	0.369	63620
1.00	0.498	66220
0.99	0.495	66800
1.00	0.500	67000
1.53	0.497	65930
1.50	0.498	66270
1.98	0.504	67510
2.03	0.502	66730
2.50	0.497	67950
2.52	0.501	67440
3.01	0.502	66310
3.01	0.503	66190
3.49	0.504	64920
3.50	0.502	65210
3.99	0.499	64470
4.00	0.498	64810
4.00	0.503	64690
4.00	0.498	64140
0 99	0.619	60290
1.49	0.614	63610
1.49	0.616	63450
2.49	0.620	59170
2.50	0.619	59600
3.01	0.617	59270
3.50	0.614	61610
3.49	0.617	62060
4.00	0.615	60330
4.01	0.617	61120
0.96	0.726	58480
1.01	0.727	58790
1.51	0.726	59290
3.50	0.736	58700
3.49	0.729	59180

§ 233. **Chain Cable.**—The most thorough set of tests of the strength of chain cable is that made by Commander Beardslee for the United-States government, an account of which may be found either in the report already referred to, or in the abridgment by William Kent.

In this report are to be found a number of conclusions, some of which are as follows: —

1°. That cables made of studded links (i.e., links with a cast-iron stud, to keep the sides apart) are weaker than open-link cables.

2°. That the welding of the links is a source of weakness; the amount of loss of strength from this cause being a very uncertain quantity, depending partly on the suitability of the iron for welding, and partly on the skill of the chain-welder.

3°. That an iron which has a high tensile strength does not necessarily make a good iron for cables. Of the irons tested, those that made the strongest cables were irons with about 51000 lbs. tensile strength.

4°. The greatest strength possible to realize in a cable per square inch of the bar from which it is made being 200 per cent of that of the bar-iron from which it was made, the cables tested varied from 155 to 185 per cent of that of the bar-iron.

5°. The Admiralty rule for proving chain cables, by which they are subjected to a load in excess of their elastic limit, is objected to, as liable to injure the cable: and the report suggests, in its place, a lower set of proving-strengths, as given in the following table; the Admiralty proving-strengths being also given in the table.

In these recommendations, account is taken of the different proportion of strength of different size bars as they come from the rolls, also no proving-stress is recommended greater than 50 per cent of the strength of the weakest link, and 45.5 per cent of the strongest; whereas in the Admiralty tests, 66.2

per cent of the strength of the weakest, and 60.3 per cent of the strongest, is sometimes used.

For the details of this investigation, see the report, Executive Document No. 98, 45th Congress, second session, or the abridgment already referred to.

Diameter of Iron, in inches.	Recommended Proving-Strains.	Admiralty Proving-Strains.	Diameter of Iron, in inches.	Recommended Proving-Strains.	Admiralty Proving-Strains.
2	121737	161280	$1\frac{7}{16}$	66138	83317
$1\frac{15}{16}$	114806	151357	$1\frac{3}{8}$	60920	76230
$1\frac{7}{8}$	108058	141750	$1\frac{5}{16}$	55903	69457
$1\frac{13}{16}$	101499	132457	$1\frac{1}{4}$	51084	63000
$1\frac{3}{4}$	95128	123480	$1\frac{3}{16}$	46468	56857
$1\frac{11}{16}$	88947	114817	$1\frac{1}{8}$	42053	51030
$1\frac{5}{8}$	82956	106470	$1\frac{1}{16}$	37820	45517
$1\frac{9}{16}$	77159	98437	1	33840	40320
$1\frac{1}{2}$	71550	90720			

§ 234. **Iron and Steel Wire.** — It has long been known that the process of cold-drawing, by which wrought-iron and steel are made into wire, greatly increases its strength per square inch; and usually the smaller the wire, the greater the strength per square inch. Thus iron wire varies in strength from about 70000 to about 90000 lbs. per square inch when unannealed, its strength being reduced to 45000 or 50000 lbs. by being annealed.

In steel wire unannealed, the strength runs up even to 200000 lbs. per square inch in some cases; and Fairbairn even records strengths as great as 275000 lbs. per square inch.

Accounts of experiments upon the strength and elasticity of wrought-iron and steel wire will be found in —

W. E. Woodbridge: Report on the Mechanical Properties of Steel.
Professor Thurston: Materials of Engineering and other papers.
Pocket-Book of the New-Jersey Iron and Steel Company.

The strength of wire becomes of special importance in connection with wire rope, and also with wire-wound guns. Dr. Woodbridge has made a great number of tests of steel wire for wire-wound guns.

Wire Rope. — Inasmuch as wire rope is extensively used in the transmission of power, it may be a matter of convenience to have here a table giving the strength of the rope as claimed by some of the makers. There will follow, therefore, the table of strengths of the different sizes as given by Mr. John A. Roebling for the rope manufactured by him, and also some of his remarks in regard to its use: —

Rope of 133 Wires, or 19 Wires to the Strand.

Trade Number.	Circumference, in inches.	Ultimate Strength, in lbs.	Circumference of Hemp Rope of the Same Strength.
1	$6\frac{3}{4}$	148000	$15\frac{1}{2}$
2	6	130000	$14\frac{1}{2}$
3	$5\frac{1}{2}$	108000	13
4	5	87200	12
5	$4\frac{3}{8}$	70000	$10\frac{3}{4}$
6	4	54400	$9\frac{1}{2}$
7	$3\frac{1}{2}$	40400	8
8	$3\frac{1}{8}$	32000	7
9	$2\frac{3}{4}$	22800	6
10	$2\frac{1}{4}$	17280	5
$10\frac{1}{4}$	2	10260	$4\frac{1}{2}$
$10\frac{1}{2}$	$1\frac{5}{8}$	8540	4
$10\frac{3}{4}$	$1\frac{1}{2}$	6960	$3\frac{3}{4}$

Rope of 49 Wires, or 7 Wires to the Strand.

Trade Number.	Circumference, in inches.	Ultimate Strength, in lbs.	Circumference of Hemp Rope of the Same Strength.
11	$4\frac{5}{8}$	72000	$10\frac{3}{4}$
12	$4\frac{1}{4}$	60000	10
13	$3\frac{3}{4}$	50000	$9\frac{1}{2}$
14	$3\frac{3}{8}$	40000	$8\frac{1}{4}$
15	3	32000	$7\frac{1}{4}$
16	$2\frac{5}{8}$	24600	$6\frac{1}{4}$
17	$2\frac{3}{8}$	16600	$5\frac{1}{2}$
18	$2\frac{1}{8}$	15200	5
19	$1\frac{7}{8}$	11600	$4\frac{3}{4}$
20	$1\frac{5}{8}$	8180	4
21	$1\frac{3}{8}$	5660	$3\frac{1}{4}$
22	$1\frac{1}{4}$	4260	$2\frac{3}{4}$
23	$1\frac{1}{8}$	3300	$2\frac{1}{2}$
24	1	2760	$2\frac{1}{4}$
25	$\frac{7}{8}$	2060	2
26	$\frac{3}{4}$	1620	$1\frac{3}{4}$
27	$\frac{5}{8}$	1120	$1\frac{1}{2}$

Notes by Mr. Roebling. — "Two kinds of wire rope are manufactured. The most pliable variety contains nineteen wires in the strand, and is generally used for hoisting and running rope. The ropes with twelve wires and seven wires in the strand are stiffer, and are better adapted for standing-rope, guys, and rigging. Ropes are made up to three inches in diameter, both of iron and steel, upon special application.

"For safe working-load, allow one-fifth to one-seventh of the ultimate strength, according to speed, so as to get good wear from the rope. When substituting wire rope for hemp rope, it is good economy to allow for the former the same weight per foot which experience has approved for the latter.

"Wire rope is as pliable as new hemp rope of the same strength: the former will therefore run over the same size sheaves and pulleys as the latter. But the greater the diameter of the sheaves, pulleys, or drums, the longer wire rope will last. In the construction of machinery for wire rope, it will be found good economy to make the drums and sheaves as large as possible.

"Experience has demonstrated that the wear increases with the speed. It is therefore better to increase the load than the speed.

"Wire rope is manufactured either with a wire or a hemp centre. The latter is more pliable than the former, and will wear better where there is short bending.

"*Wire rope must not be coiled or uncoiled like hemp rope.* When mounted on a reel, the latter should be mounted on a spindle or flat turn-table, to pay off the rope. When forwarded in a small coil without reel, roll it over the ground like a wheel, and run off the rope in that way. All untwisting or kinking must be avoided."

In the case of wire rope it is true, as in all other cases, the only way to secure certainty in regard to the strength is to test it. The author has tested wire rope which bore no more than

two-thirds the strength claimed for it; though it is but justice to say, in this connection, that the few samples of Roebling rope tested in his laboratory bore out very fairly the results given for the sizes tested in the table.

A number of tests of wire rope have been made on the government testing-machine at Watertown Arsenal and elsewhere.

Wire rope is generally used with some kind of holder, and it is generally the case that the breakage of the rope occurs at the holder.

Another matter worth noticing is, that it yields in detail, and hence that new rope is not as strong as that which has been under tension for some time, provided the tension has not been excessive,

§ 235. **Other Metals and Alloys.** — Copper is, next to iron and steel, the metal most used in construction, sometimes in the pure state, especially in the form of sheets or wire, but more frequently alloyed with tin or zinc; those metals where the tin predominates over the zinc being called bronze, and those where zinc predominates over tin, brass. Copper in the pure state was used not long ago for the fire-box plates of locomotive and other steam-boilers, as it was believed to stand better the great strains due to the changes of temperature that come upon these plates, than iron or steel; but now steel or iron has almost entirely superseded it for this purpose, except in some cases where the feed-water is very impure, and where the impurities are such as corrode iron.

The alloys of copper, tin, and zinc which are used most where strength and toughness are needed, are those where the tin predominates over the zinc; and the composition, mode of manufacture, and resisting properties of these metals, together with the effect of other ingredients, as phosphorus, have been very extensively investigated with reference to their use as a material for making guns, instead of cast-iron.

Accounts of tests made on these alloys will be found as follows: —

Major Wade: Ordnance Report, 1856.
T. J. Rodman: Experiments on Metals for Cannon.
Executive Document No. 23, 46th Congress, 2d session.

No attempt will be made to give a complete account of the results of these tests; but a table will be given for convenience of use, showing rough average values of the resisting powers of some metals and alloys other than iron.

	Specific Gravity.	Tensile Strength per Sq. In.	Modulus of Elasticity.
Brass cast	8.396	18000	9170000
Brass wire	–	49000	14230000
Bronze unwrought:			
84.29 copper + 15.71 tin (gun metal) .	8.561	36060	–
82.81 " + 17.19 " " .	8.462	34048	–
81.10 " + 18.90 " " .	8.459	39648	–
78.97 " + 21.03 " (brasses) . . .	8.728	30464	–
34.92 " + 65.08 " (small bells) .	8.056	3136	–
15.17 " + 84.83 " (speculum metal)	7.447	6944	–
Tin	7.291	5600	–
Zinc	6.861	7500	–
Copper cast	8.712	24138	–
Copper bolts	8.878	33000	–
Copper wire	–	60000	17000000
Gold cast	19.258	20000	–
Silver cast	10.476	40000	–
Platinum wire	22.069	56000	–
Lead cast	11.352	1800	–

Professor Thurston gives, for the ultimate tensile strength per square inch of a compound of copper and tin and zinc,

$$T = 30000 + 1000t + 500z,$$

where $t =$ percentage of tin, and not over 15 per cent; and $z =$ percentage of zinc, and not over 50 per cent.

Some specimens of phosphor bronze tested by Kirkaldy gave for tensile strengths from 22000 to 50000 for cast, and 98000 to 159000 for phosphor bronze wire 0.06 inch to 0.11 inch diameter. If the student is to use alloys, he should ascertain their strength, elastic limit, and modulus of elasticity, and he should observe that the mode of manufacture has a great influence on the strength and ductility of any alloy.

§ 236. **Timber.** — However extensively iron and steel may have superseded timber in construction, nevertheless, there are many cases in which iron is entirely unsuitable, and where timber is the only material that will answer the purpose; and in many cases where either can be used, timber is much the cheaper. Hence it follows that the use of timber in construction is even now, and as it seems always will be, a very important item.

Another advantage possessed by timber is, that, on yielding, it gives more warning than iron, thus affording an opportunity to foresee and to prevent accident.

If we make a section across any of the exogenous trees, as the oak, pine, etc., we shall find a series of concentric layers; these layers being called annual rings, because one is generally deposited every year.

Radiating from the heart outwards will be found a series of radial layers, these being known as the medullary rays.

Of the annual rings, the outer ones are softer and lighter in color than the inner ones; the former forming the sap-wood, and the latter the heart-wood. When the log dries, and thus tends to contract, it will be found that scarcely any contraction takes place in the medullary rays; but it must take place along the line of least resistance, viz., along the annual rings, thus causing radiating cracks, and drawing the rays nearer together on the side away from the crack. This action is exhibited in

Fig. 241, where a log is shown with two saw-cuts at right angles to each other; when this log becomes dry, the four right angles all becoming acute through the shrinkage of the rings.

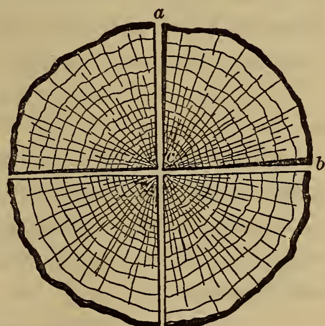


Fig. 241.

If the log be cut into planks by parallel saw-cuts, the planks will, after drying, assume the forms shown in Fig. 242, as is pointed out in Anderson's "Strength of Materials," from which these two cuts are taken.

This internal construction of a plank has an important influence upon the side which should be uppermost when it is used for flooring; for, if the heart side is uppermost, there will be a liability to having layers peel off as the wood dries: indeed, boards for flooring should be so cut as to have the annual rings at right angles to the side of the plank. Before discussing any other considerations which affect the adaptability of timber to use in construction, we will consider the question of its strength.

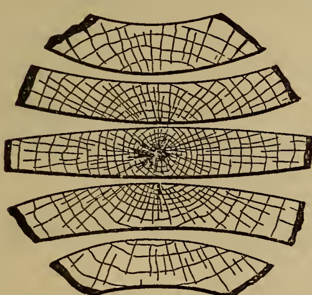


Fig. 242.

§ 237. **Strength of Timber.** — In this regard we must observe, that, whereas the strength and elasticity and other properties of iron and steel vary greatly with its chemical composition and the treatment it has received during its manufacture, the strength, etc., of timber is much more variable, being seriously affected by the soil, climate, and other accidents of its growth, its seasoning, and other circumstances; and that over many of these things we have no control: hence we must not

expect to find that all timber that goes by one name has the same strength, and we shall find a much greater variation and irregularity in timber than in iron. The experiments that have been made on strength and elasticity of timber may be divided into the following classes: —

1°. Those of the older experimenters, except those made on full-size columns by P. S. Girard, and published in 1798. A fair representation of the results obtained by them, all of which were deduced from experiments on small pieces, is to be found in the tables given in Professor Rankine's books, "Applied Mechanics," "Civil Engineering," and "Machinery and Millwork."

2°. Tests made by modern experimenters on small pieces. Such tests have been made by —

(*a*) Trautwine: Engineers' Pocket-Book.
(*b*) Hatfield: Transverse Strains.
(*c*) Laslett: Timber and Timber Trees.
(*d*) Thurston: Materials of Construction.
(*e*) A series of tests on small samples of a great variety of American woods, made for the Census Department, and recorded in Executive Document No. 5, 48th Congress, 1st session.

3°. Tests made by Capt. T. J. Rodman, U.S.A., the results of which are given in the "Ordnance Manual."

4°. All tests that have been made on full-size pieces.

In regard to tests on small pieces, such as have commonly been used for testing, it is to be observed, that, while a great deal of interesting information may be derived from such tests as to some of the properties of the timber tested, nevertheless, such specimens do not furnish us with results which it is safe to use in practical cases where full-size pieces are used. Inasmuch as these small pieces are necessarily much more perfect (otherwise they would not be considered fit for testing), having less defects, such as knots, shakes, etc., than the full-size pieces,

they have also a far greater homogeneity. They also season much more quickly and uniformly than full-size pieces. In making this statement, I am only urging the importance of adopting in this experimental work the same principle that the physicist recognizes in all his work; viz., that he must not apply the results to cases where the conditions are essentially different from those he has tested.

Moreover, it will be seen in what follows, that, whenever full-size pieces have been tested, they have fallen far short of the strength that has been attributed to them when the basis in computing their strength has been tests on small pieces; and, moreover, the irregularities do not bear the same proportion in all cases, but need to be taken account of.

The results of the first class of experiments named in the following table are taken from Rankine's "Applied Mechanics;" and, inasmuch as the table contains also the strengths of some other organic fibres, it will be inserted in full. The student may compare these constants with those that will be given later.

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Ash	17000	1600000	9000	12000 / 14000	1400	76000
Bamboo	6300	–	–	–	–	–
Beech	11500	1350000	9360	9000 / 12000	–	–
Birch	15000	1645000	6400	11700	–	–
Blue gum	–	–	8800	16000 / 20000	–	–
Box	20000	–	10300	–	–	–
Bullet-tree	–	–	14000	15900 / 16000	–	–
Cedar of Lebanon	11400	486000	5860	7400	–	–

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Chestnut	10000 to 13000	1140000	–	10660	–	–
Cowrie	–	–	–	11000	–	–
Ebony	–	–	19000	27000	–	–
Elm	14000	700000 to 1840000	10300	6000 to 9700	1400	76000
Fir, Red pine	12000 to 14000	1460000 to 1900000	5375 to 6200	7100 9540	500 800	62000 116000
" Yellow pine (Am.)	–	–	5400	–	–	–
" Spruce	12400	1400000 to 1800000	–	9900 12300	600	–
" Larch	9000 to 10000	900000 to 1360000	5570	5000 10000	970 1700	–
Hoxen yarn	25000	–	–	–	–	–
Hazel	16000	–	–	–	–	–
Hempen rope	12000 to 16000	–	–	–	–	–
Ox-hide, undressed	6300	–	–	–	–	–
Hornbeam	20000	–	–	–	–	–
Lancewood	23400	–	–	–	–	–
Ox-leather	4200	24300	–	–	–	–
Lignum-vitæ	11800	–	9900	12000	–	–
Locust	16000	–	–	–	–	–
Mahogany	8000 to 21800	1255000	8200	7600 11500	–	–
Maple	10600	–	–	–	–	–
Oak, British	–	–	10000	10000 13600		
" Dantzic	–	–	7700	8700		
" European	10000 to 19800	1200000 1750000	–	–	2300	82000
" American red	10250	2150000	6000	10600		

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Silk fibre	52000	1300000	–	–	–	–
Sycamore	13000	1040000	–	9600	–	–
Teak, Indian . . .	15000	2400000	12000	{ 12000 19000 }	–	–
" African . . .	21000	2300000	–	14980	–	–
Whalebone	7700	–	–	–	–	–
Willow	–	–	–	6600	–	–
Yew	8000	–	–	–	–	–

In regard to the tests of the second class, a few comments are in order:—

1°. These experiments, like those of the first class, were all made upon small pieces; and the results are correspondingly high.

The usual size of the specimens for crushing being one or two square inches in section, and of those for transverse strength being about two inches square in section and four or five feet span, those for tension had even a much smaller section than those for compression; as it is necessary, in order to hold the wood in the machine, to give it very large shoulders.

The only exception to this is the tests of Sir Thomas Laslett, an account of which is given in his "Timber and Timber Trees," and also in D. K. Clark's "Rules and Tables." In these tests he gives very much lower tensile strengths than those given above; and he states that his specimens were three inches square, but does not say how he managed to hold them in such a way as to subject them to a direct tensile stress. His results for crushing and transverse strength are about as great as

those given in Rankine's tables, and as were obtained by the other experimenters on small pieces, as his specimens were of about the same dimensions as those used by the others. The figures obtained by these experimenters will only be given incidentally, as —

(*a*) They are very similar to those given in Rankine's table.

(*b*) They are not suitable for practical use on the large scale.

(*c*) While they have been used, it has only been done by employing a very large factor of safety for timber.

The series of tests made for the Census Department, and recorded in Executive Document No. 5, 48th Congress, first session, form a very interesting series of experiments upon small specimens of an exceedingly large number of American woods. In order to have working figures, we should need to test large pieces of the same; as the proportion between the strengths of the different kinds would be liable to be different in the latter case.

We will next consider Rodman's experiments. The only record of them available is a table of results in the "Ordnance Manual," and this table is appended here. It will be seen, on comparing it with the former table, that the results are lower as a rule than those obtained by the experimenters of the first or second class. This is to be accounted for by the fact, that, while he did not experiment on full-size pieces, he used much larger pieces than those heretofore employed; his specimens for transverse strength, many of which are still stored at the Watertown Arsenal, being $5\frac{3}{4}$ inches deep, $2\frac{7}{8}$ inches thick, and 5 feet span.

In the table, the modulus of rupture will be given, instead of one-sixth of its value, which is the constant given in the "Ordnance Manual."

Material.	Locality.	Length of Seasoning.	Crushing Force per Square Inch.	Tensile Strength per Square Inch.	Modulus of Rupture, in lbs., per Square Inch.
		years.	lbs.	lbs.	
Ash	Ohio	15	8783	24033	12708
"	Pennsylvania . . .	3	4475	14266	8796
"	Canada	9	5571	15000	–
"	New York	7	4783	11786	–
"	Vermont	2	5858	10893	15984
"	Virginia	1	6663	23167	9168
"	Oregon	1	5789	14700	8628
Birch	Maine	4	7969	15333	16776
Bass	"	12	5271	12600	11478
"	Canada	9	4609	14953	–
Box	Africa	5	10513	23600	–
Fir, White . . .	Oregon	2	6644	14533	4194
Gum, Black . . .	Alabama	1	6703	15860	8886
Hickory	Ohio	13	9887	25900	16362
"	North Carolina . .	3	6125	18000	–
"	Eastern Virginia .	1	5492	35500	–
" Red . .	Massachusetts . .	7	10942	27133	17400
" " . . .	New York	7	7725	12866	16536
" White . .	Massachusetts . .	7	8925	38700	17316
" " . .	Alabama	1	11213	40067	16818
" " . .	Virginia	1	9733	36666	20352
Holly	"	1	5246	18567	3384
Hemlock	Oregon	1	6817	16533	7752
Hackmatack . . .	Maine	1	–	–	7860
Lignum-vitæ . . .	South America . .	4	9854	16000	16080
Locust	Pennsylvania . . .	1	9113	27517	14478
Mahogany	San Domingo . .	4	7390	12350	9996
Maple	Canada	9	7716	22933	–
"	Maine	4	8621	21720	11574
"	Oregon	1	4443	10400	5838
Oak, White . . .	New England . .	18	6668	19600	10980
" " . . .	Western New Jersey,	12	6620	19166	11256
" " . . .	Ohio	13	6258	19066	8754
" " . . .	Monongahela . .	13	6592	20333	12216

Material.	Locality.	Length of Season-ing.	Crushing Force per Square Inch.	Tensile Strength per Square Inch.	Modulus of Rupture, in lbs., per Square Inch.
		years.	lbs.	lbs.	
Oak, White . . .	Ohio.	5	9108	19466	17340
" " . . .	New York. . . .	11	4691	12300	10668
" " . . .	Maryland	19	6092	17666	14556
" " . . .	Massachusetts . .	43	5800	16766	14658
" " . . .	" . .	7	7292	19200	11700
" " . . .	" (pasture),	7	6962	16200	13596
" " . . .	Canada	9	6000	16646	–
" " . . .	Connecticut . . .	14	5199	13333	–
" " . . .	" . . .	18	7089	21000	–
" " . . .	North Carolina . .	8	6550	21100	–
" " . . .	Alabama	2	5744	18307	9912
" " . . .	Virginia	1	6902	19033	10758
" " . . .	Oregon	1	6072	18467	9432
" " . . .	James River, Va. .	13	6667	25222	10938
" Yellow . . .	New Hampshire .	13	6279	25000	11490
" Live	Alabama	3	6531	16383	9780
" "	–	–	7279	15800	7998
Pine, Pitch . . .	North Carolina . .	3	8947	11400	–
" White . . .	Alleghany River .	4	5017	11433	6798
" " . . .	New York. . . .	5	5775	11933	6912
" " . . .	Maine	13	5617	11960	7092
" Yellow . . .	Florida	6	8350	18000	8796
" " . . .	North Carolina . .	–	7836	12600	11676
" " . . .	Alabama	1	8201	17946	10254
" " . . .	Virginia	2	7867	19200	9168
" Sugar . . .	Nevada Co., Cal. .	1	–	–	5322
" " . . .	Humboldt Co., Cal.,	1	–	–	5658
Poplar	Ohio	3	5742	14933	7260
"	New York . . .	2	6075	9066	5856
"	Virginia	1	6579	8200	7782
Redwood	California	1	6083	10833	4518
Spruce	Maine	1	6862	13666	6168
"	Oregon	1	5092	10867	5964
Teak	East India. . . .	4	10819	33800	18558

Material.	Locality.	Length of Season-ing.	Crushing Force per Square Inch.	Tensile Strength per Square Inch.	Modulus of Rup-ture, in lbs., per Square Inch.
		years.	lbs.	lbs.	
Walnut, Black . .	Western States . .	7	7471	16633	12318
" " . .	Virginia	1	7500	16300	8190
" " . .	Michigan	2	5782	17580	–
" " . .	Canada	9	5989	16133	–

The fourth class of tests are those which furnish reliable data for use in construction; and we will proceed to a consideration of these, taking up (1°) tension, (2°) compression, (3°) transverse strength, and (4°) shearing along the grain.

TENSION.

In all cases where the attempt has been made to experiment upon the tensile strength of timber, a great deal of difficulty has been encountered in regard to the manner of holding the specimens. In all cases it has been found necessary to provide them with shoulders, each shoulder being five or six times as long as the part of the specimen to be tested, and to bring upon these shoulders a powerful lateral pressure, to prevent the specimen from giving way by shearing along the grain, and pulling out from the shoulder, instead of tearing apart.

The specimens tested have generally had a sectional area less than one square inch, and it seems almost impossible to provide the means of holding larger specimens. This being the case, it is plain, that, whenever timber is used as a tie-bar in construction (except in exceedingly rare and out-of-the-way cases), it will give way by some means other than direct

tension ; i.e., either by the pulling-out of the bolts or fastenings, and the consequent shearing of the timber, or else by bending if there is a transverse stress upon the piece ; and, this being the case, these other resistances should be computed, instead of the direct tension. Hence, while the direct tensile strength of timber may be an interesting subject of experiment, it can serve hardly any purpose in construction ; and the conclusion follows, that the resistances of timber to breaking we may expect to meet in practice are its crushing, transverse, and shearing strength. Indeed, the use of timber for a tie-bar should be avoided whenever it is possible to do so ; and, when it is used, the calculations for its strength should be based upon the pulling-out of the fastenings, the shearing or splitting of the wood, etc., and not on the tensile resistance of the solid piece.

Moreover, when a wooden tie-bar is loaded, in addition, with a transverse load, and when the magnitude of this transverse load is so great as to render the piece more liable to give way by cross-breaking than by the pulling-out of the fastenings, we must then compute the greatest tension per square inch at the outside fibre due to the bending, and to that add the direct tension per square inch : and this sum must be less than the modulus of rupture if the piece is not to give way ; i.e., the modulus of rupture, and not the ultimate tensile strength per square inch, must be our criterion of breaking in such a case, the working-strength per square inch being the modulus of rupture divided by a suitable factor of safety.

COMPRESSIVE STRENGTH.

Tests of the compressive strength of full-size wooden columns are, with the exception of one set of tests, of very recent date, and have not yet found their way, to any extent, into our text-books and engineers' handbooks. It is therefore only

suitable, that, before enumerating them, we should observe what formulæ have been given in these books for the computation of timber columns in practice, what experimental basis these formulæ rest upon, and how they coincide with the facts.

The oldest formulæ are those of Euler. His formulæ for the strength of such circular columns as yield by bending are as follows: —

For ends fixed in direction,

$$P = \frac{(3.1416)^3}{16} E \frac{d^4}{l^2},$$

For rounded ends,

$$P = \frac{(3.1416)^3}{64} E \frac{d^4}{l^2};$$

where P = breaking-weight in pounds, and E = modulus of elasticity of the material. For wood, Weisbach gives for use in Euler's formulæ, $E = 1664000$, and crushing-strength per square inch = 6770.

According to Hodgkinson, we should take one-ninth of the breaking-strength of a cast-iron pillar, as given by equations (1), (2), (3), (4), (5), § 211; this being the rule given by Mr. James B. Francis in his book, where he has computed and tabulated the strength of pillars of the ordinary sizes used in practice, by means of Mr. Hodgkinson's formulæ.

In Gordon's formula the constants used were determined in such a way as to make the results agree as nearly as possible with Hodgkinson's experiments. The formulæ devised by Gordon himself refer only to cylindrical and hollow cylindrical columns. A formula devised by the same course of reasoning, and also depending for its constants on Hodgkinson's experiments, but so arranged as to be applicable to any form of

section whatever, is given by Professor Rankine. For wood it is as follows:

$$P = \frac{7200S}{1 + \frac{l^2}{3000r^2}},$$

where P = breaking-strength in pounds, S = sectional area in square inches, l = length in inches, r = least radius of gyration in inches.

Besides the above, we have formulæ which are practically Rankine's formulæ with the constants changed. One of these is that of Mr. C. Shaler Smith as given in Trautwine's "Pocket-Book," and applicable, as he claims, to a square or rectangular column of white or yellow pine. It is as follows:

$$P = \frac{5000S}{1 + 0.004\frac{l^2}{d^2}},$$

where P = breaking-weight in pounds, S = sectional area in square inches, l = length in inches, d = least side of rectangle.

I have computed, by means of these formulæ, the breaking-weights of certain oak columns, with the following results:—

Length = 14 feet.

		Diameter 10.5 Inches.	Diameter 9.5 Inches.
Euler	Flat ends	586214	479858
	Rounded ends . . .	347147	232623
Rankine . . .	Flat ends	465261	360164
	Rounded ends . . .	263615	191286
Francis	Flat ends	420000	312795
	Rounded ends . . .	140000	104265

If we use the average of the results obtained from the oak columns tested at Watertown, which will be referred to later, we should find, for the breaking-weights of the above columns with flat ends, about 277000 and 227000 lbs. respectively.

A glance at the above results will show that they differ very much from each other, and the question naturally arises as to the trustworthiness of the experimental data on which they are based.

The constants used in Euler's formulæ are not deduced from any experiment on the breaking of a column.

Rankine's and Francis's have, for experimental basis, the experiments of Hodgkinson. He made a very large number of tests of cast-iron columns, none of which were as large as those used in practice. On oak columns, he made seventeen experiments on as many columns, all cut from one good plank of Dantzic oak, the largest of which was five feet long and two inches square. Of these seventeen, only seven were used in deducing his formulæ.

It is plain that such data are insufficient, and cannot furnish us reliable information as to the strength of a column.

As to Mr. C. Shaler Smith's formula, it is based upon some experiments made by himself, which have never been published. The student can easily satisfy himself as to how accurately it represents the facts on the large scale, by computing by it the strength of some of the columns tested, of which an account will shortly be given. Trautwine, in his "Handbook," states that it is "for the breaking-loads of either square or rectangular pillars or posts, of moderately seasoned white and common yellow pine, with flat ends, firmly fixed, and equally loaded."

Mr. Smith, as I understand, however, intended it to meet the case of such ill-fitting joints as occur in practice, and not for perfectly even bearings.

TESTS OF FULL-SIZE COLUMNS.

The only tests of full-size columns of which I have any knowledge are: —

1°. Trautwine, in his "Handbook," speaks of some tests of wooden pillars 20 feet long and 13 inches square, made by David Kirkaldy, which, as he says, gave results agreeing with Mr. C. Shaler Smith's rule.

2°. A series of tests made at the Watertown Arsenal for the Boston Manufacturers' Mutual Fire Insurance Company, under the direction of the author.

3°. The tests that have been made at the Watertown Arsenal on the government testing-machine.

4°. Besides these, the writer has very recently had his attention called to a series of tests of full-size columns of oak and fir, made by P. S. Girard in 1798, which give results agreeing very well with the modern results on full-size columns. Why these tests should have been lost sight of, and Hodgkinson's always used instead, is incomprehensible.

In regard to the first, no details or results are given: hence nothing will be said about them.

In regard to the second, a summary only will be presented here, the reader being referred for details to my published report entitled "Strength of Wooden Columns."

TESTS OF YELLOW-PINE POSTS AND BLOCKS.

TESTED WITH FLAT ENDS.

Set	Distinguishing No.	Weight, in lbs.	Length, ft.	Length, in.	Diameter of Small End, in inches.	Diameter of Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength.	Ultimate Strength, in lbs., per Sq. In.	Modulus of Elasticity, in lbs., per Square Inch.	
1st set.	Post 1	320	12	0.15	9.31	10.55	1.67	65.90	270000	4097	1885264	Flat ends.
	Post 2	235	12	0.20	8.30	10.07	1.70	51.90	190000	3661	1631035	Flat ends.
	Post 3	211	12	0.15	7.54	8.99	1.70	42.50	200000	4706	2087347	Flat ends.
	Post 4	164	12	0.20	6.40	7.79	1.67	30.00	138000	4600	2204585	Flat ends.
2d set.	Post 1	342	12	0.00	10.45	–	1.60	83.75	390000	4657	–	Tested with round pintles.
	Post 3	249	11	11.20	8.96	–	1.54	61.20	250000	4085	2169882	Flat ends.
	Post 4	180	11	11.70	7.70	–	1.53	44.72	205000	4584	2081321	Flat ends.
2d set.	Block 1	62	2	0.13	10.46	–	–	85.90	380000	4424	1657425	Flat ends.
	Block 2	52	2	0.00	9.98	–	–	78.23	368000	4704	2443411	Flat ends.
	Block 3	41	2	0.33	8.91	–	–	62.35	270000	4330	1644453	Flat ends.
	Block 4	29	2	0.00	7.79	–	–	47.66	200000	4511	1900252	Flat ends.
3d set.	Post 2	428	12	10.25	10.05 × 10.13	Rectangular	–	99.80	510000	5220	–	One flat end, one with rectangular pintle.
	Post 3	213	13	11.90	7.90	–	1.60	47.01	200000	4254	–	Flat ends.
	Post 4	193	11	11.04	8.00	–	1.60	48.26	225000	4662	–	One flat end, one rounded bearing.
3d set.	Block 1	58	2	0.00	8.98 × 9.02	Rectangular	–	81.00	482000	5350	–	Flat ends.
	Block 2	62.5	2	0.06	10 20 × 10.07	Rectangular	–	102.70	560000	5452	–	Flat ends.
	Block 3	26	1	11.95	7.70	–	1.60	44.56	218000	4892	–	Flat ends,
	Block 4	29.5	1	11.80	7.98	–	1.60	48.00	173000	3604	–	Flat ends.
	Average	. .	. .	. .	. . .	. . .	. . .	. . .	. . . . .	4544	1996351	

TESTS OF WHITE-OAK POSTS AND BLOCKS.

Tested with Flat Ends.

Set	Distinguishing No.	Weight, in lbs.	Length, in feet and inches. ft.	in.	Diameter of Small End, in inches.	Diameter of Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength.	Ultimate Strength per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	
1st set.	Post 1	395	12	0.17	9.15	10.15	1.67	63.2	190000	3006	1222222	Flat ends.
	Post 2	325	12	0.20	8.37	10.23	1.67	52.8	200000	3788	1633987	Flat ends.
	Post 3	289	12	0.20	7.55	9.05	1.67	42.6	160000	3756	1504389	Flat ends.
	Post 4	215	12	0.20	6.60	8.06	1.67	32.0	110000	3438	1748817	Flat ends.
2d set.	Post 2	352	12	0.00	10.00	–	1.53	76.7	210000	2738	–	Tested with round pintles.
	Post 4	200	12	0.00	7.74	–	1.60	45.04	145000	3219	1508906	Flat ends.
2d set.	Block 1	72	2	0.00	10.91	–	–	93.48	416000	4450	–	Flat ends.
	Block 2	59	2	0.06	9.98	–	–	78.23	245000	3132	1165382	Flat ends.
	Block 3	41	2	0.00	8.18	–	–	52.55	165000	3139	1104938	Flat ends.
	Block 4	33	1	11.88	7.73	–	–	46.93	155000	3303	1302623	Flat ends.
	Average	. .	. .	. .	. . .	. . .	. . .	. . .	. . . . .	3470		

TESTS OF OLD AND SEASONED WHITE-OAK POSTS.

Distinguishing Mark.	Weight, in lbs.	Length, in feet and inches. ft.	in.	Diameter of the Small End, in inches.	Diameter of the Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength.	Ultimate Strength per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	
1914	113	12	0.92	5.85	6.84	1.95	23.90	110000	4602	1843725	Ends brought to even bearing.
1915	116	12	0.92	5.85	6.85	1.90	24.00	145000	6032	2138804	Ends brought to even bearing.
1916	118	12	1.40	5.87	6.70	2.00	23.90	112000	4682	1657824	Ends not brought to even bearing.
1917	113	12	0.60	6.02	6.75	1.95	25.50	75000	2943	2052454	Tested with maple cap and oak base.
1918	108	12	0.80	5.75	6.88	1.95	22.98	75000	3263	–	Tested with maple cap and oak base.
1931	118	12	0.10	5.97	6.74	1.92	25.10	106000	4223	–	Tested with maple cap and oak base.
1932	118	11	8.65	6.10	6.83	2.00	26.08	90000	3450	–	Tested with maple cap and oak base.
A	398	13	10.70	10.54	–	1.90	84.40	360000	4265	–	Ends not brought to even bearing.
B	360	13	9.40	10.50	–	1.90	83.76	325000	3880	–	Ends not brought to even bearing.
C	383	14	0.00	10.54	–	1.95	87.26	420000	4084	2053024	Ends not brought to even bearing.
D	388	13	10.50	10.56	–	1.90	84.70	390000	4604	1448964	Ends brought to even bearing.
E	310	13	11.04	10.20	–	1.80	79.17	370000	4662	1574555	Ends not brought to even bearing.
F	378	13	11.63	10.40	–	1.70	82.68	400000	4838	1790964	Iron cap and pintle at one end, iron base at the other.
G	301	13	8.20	9.30	–	1.95	64.94	317000	4881	–	Ends not brought to even bearing.
H	293	13	8.08	9.25	–	1.90	64.37	221000	3433	–	Ends not brought to even bearing.
I	267	13	7.80	9.40	–	1.90	60.56	265000	3981	–	Ends not brought to even bearing.
J	310	13	7.80	9.35	–	1.90	65.83	215000	3266	–	Tested with rectangular pintles.
K	121	11	6.20	5.98	–	1.62	26.03	160000	6147	–	
Average .		. . .		. . .	. . .	. .	. . .	. . .	3957		

In all the experiments enumerated in the tables given above, the columns gave way by direct crushing, and hence the strength of columns of these ratios of length to diameter can properly be found by multiplying the crushing-strength per square inch of the wood by the area of the section in square inches.

This conclusion is deduced from the fact that the deflections were measured in every case, and found to be so small as not to exert any appreciable effect.

In regard to other tests of this same set, there were eight tests made, in addition to those already enumerated; and of these, five were loaded off centre. A summary of the results is appended, together with a comparison of their actual strength with that which would be computed on the basis of 4400 per square inch for yellow pine, and 3000 for oak.

	Weight, in lbs.	Length, in feet and inches.	Diameter of Column.	Diameter of Core.	Sectional Area, in square inches.	Eccentricity, in inches.	Ultimate Strength.	Computed Ultimate Strength.
		ft. in.						
2, 2d series	320	11 11.27	9.92	1.53	75.45	2.33	265000	331980
5, 3d series	298	12 6.8	8.30 × 7.60	–	63.1	2.07	240000	277640
1, 3d series	386	12 9.3	8.75 × 8.92	–	76.04	2.25	280000	334576
1, 2d series	451	11 11.4	10.95	1.80	92.16	2.75	170000	276480
3, 2d series	236	11 11.2	8.2	1.55	50.92	1.91	100000	152760

These results exhibit a great falling-off of strength due to the eccentricity of the load; and if we observe, that, whenever the beam on one side of a column is loaded differently from that on the other side, we have an eccentric loading, and hence

a falling-off in strength, we must conclude that this should be taken into account. Probably the best way to proceed in the matter is to compute always the greatest eccentricity possible in any given case, and to compute from that the additional stress on the column due to the bending consequent on this eccentricity by the principles of the short strut, already explained in § 207. This will always be on the safe side. The three remaining experiments of the set were, (1°) Two tests of whitewood columns: these gave a crushing-strength of 3009 lbs. per square inch. The columns were, however, brittle, and did not give warning of fracture. (2°) One yellow-pine square column of a sectional area of 68.8 square inches, and length 12 feet 6.85 inches, tested by resting one end on a thick yellow-pine bolster, crushing this bolster at right angles to the grain.

Maximum load on the post while the bolster was in, was 120000 lbs. = 1744 lbs. per square inch. Under this load a crack at end of post was enlarged, giving evidence that failure of the post was imminent. The bolster in the mean time had become thoroughly cracked, owing to the unequal distribution of the load on the bolster, from imperfect workmanship. Slight cracks followed the first snapping sounds heard at 20000 lbs. compression. The cracks gradually developed as the loads were increased, the side nearest the heart of the bolster sliding off.

The post was taken from machine and bolster removed. The post was cut off $1\frac{1}{2}$ inches at the end, and squared; the total length, after cutting, being 149.35 inches. Ultimate strength, 375000 lbs. = 5451 lbs. per square inch.

A few of the conclusions in regard to these sizes of posts will now be quoted from the report of these tests, in which some general recommendations are made, and others having special reference to mill columns: —

1. I should recommend that the longitudinal holes in wooden mill columns be bored from one end only, and that all posts be rejected in

which the eccentricity at the other end is greater than a given small amount, as three-quarters of an inch. This recommendation is made in view of the facts that holes bored from the two ends are very liable not to meet in the middle, and hence not to allow a circulation of air; that, if the hole becomes very eccentric, the column is liable to be weakened; and also by the presence of two holes at the same section.

2. I should recommend that mill columns be not tapered, as the tapering is a source of weakness; the loss of strength in one of the cases tested amounting to about 120000 lbs.

3. I should also recommend that square columns be used in mills, instead of round ones, for the reason that the timber comes to the wharf in the form of square logs, and, when the columns are made round, they are cut from the square form; and this cutting-away of the wood is so much loss of strength.

4. The strength of a column of hard pine or oak, with "flat ends," the load being uniformly distributed over the ends, and of the diameters tested, is practically independent of the length up to a length of twelve feet (how much farther can only be decided by further experiment), such columns giving way practically by direct crushing; the deflection, if any, being as a rule very small, and exerting no appreciable influence on the breaking-strength.

5. The only exceptions to the above are found in cases where there is good reason for departure from the rule, as in the case of very imperfect wood or of very eccentric holes; but even there the influence of the deflection in reducing the strength is not nearly so great as has been generally supposed.

6. No formulæ founded on the generally received hypothesis, that the deflection exerts a very considerable influence on the breaking-strength of such columns as those referred to, represent correctly their breaking-strength for all lengths and diameters.

7. For such columns as those referred to, the most correct rule for determining the breaking-strength is to multiply the number of square inches in the section (the smaller section being used in the case of tapering columns) by the crushing-strength per square inch of the wood.

8. The crushing-strength per square inch varies considerably in

specimens of different degrees of seasoning, also in large and small specimens.

9. The average crushing-strength of wood is much less than has been supposed by many. That of some very highly seasoned hard pine was found at the arsenal to be 7386 lbs. For some hard pine of very slow growth and very highly seasoned, an average crushing-strength was found of 9339 lbs. For some very wet and green, they found a crushing-strength of 3015 lbs. For some yellow pine which had been seasoning about three months, I found 5400 lbs. For average crushing-strength of such posts as I tested, not thoroughly seasoned, and not very green, I found about 4400 lbs.; whereas in none of these cases did I obtain a greater result than about 4700 lbs. Hence it would be entirely unfair to assume a crushing-strength of 8000 lbs. for yellow pine. For two specimens of white oak tried at the arsenal, and very thoroughly seasoned, an average was obtained of about 7150 lbs.; whereas for such oak as was furnished me, which was green and knotty, but no more so than is usual for use in building, I obtained an average of about 3200 lbs.

10. I would recommend the use of iron caps and pintles, instead of wooden bolsters, as wood is very weak to resist crushing across the grain, and the wooden bolster will fail at a pressure far below that which the column is capable of resisting; and the unevenness of the pressure brought about by the bolster is so great as to sometimes crack the column at a pressure far below what it would otherwise sustain.

11. Any cause which operates to distribute the pressure on the ends unevenly, or to force its resultant out of centre, is a source of weakness, and brings about a very considerable deflection, which exerts an important influence in reducing the breaking-strength.

12. As far as these experiments have gone, it appears that such pintles as were used in these tests, when the fitting is perfect, exert no influence upon the breaking-strength of perfect and straight-grained columns, but that they probably are a source of weakness in the case of imperfect and knotty wood, and especially in cases where there is an incipient deflection. Further experiment is needed, however, to answer these questions fully.

13. I would also recommend that the horizontal holes connecting

the longitudinal holes with the outside air, be made in the iron cap, and not in the wood: this will prevent weakening of the post by the hole, and will prevent the closing of the hole by change in moisture and other causes.

14. Another conclusion which I think is very evident, is, that the crushing-strength of full-size columns cannot be fairly inferred from tests made on columns no larger than five feet long and two inches on a side.

The table of results of the tests on old and seasoned oak columns were made upon columns that had been in use for a number of years in different mills, from which they were removed, and replaced by new ones. Ten of them had been in use about twenty-five years, and the remainder for shorter periods. An inspection of this table will, I think, convince the reader that it would not be safe to calculate upon a higher breaking-strength per square inch in these than in the green ones.

TESTS MADE ON THE GOVERNMENT MACHINE.

In Executive Document 12, 47th Congress, first session, will be found a series of tests of white and yellow pine posts made at the Watertown Arsenal; and these tests probably furnish us the best information that we possess in regard to the strength of wooden columns.

The summary of results is appended: —

COMPRESSION OF WHITE-PINE POSTS.

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Width.	Depth.	Sectional Area.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
511	7.0	–	0	15.00	5.50	5.50	30.25	108000	3570	Fibres crushed.
491	26.5	–	4	11.95	5.48	5.48	30.00	111000	3700	Failed at knot 14 in. from end.
492	31.0	–	5	0.05	5.48	5.48	30.00	99300	3100	" " " 15 " " "
493	43.5	–	7	6.10	5.50	5.52	30.40	67400	2217	" " knots 28 " " "
494	37.5	–	7	6.10	5.47	5.47	29.90	70800	2368	" " " 16 " " "
495	45.0	–	7	6.10	5.50	5.45	29.80	74100	2487	" " " 20 and 25 in. from end.
512	62.5	–	10	0.10	5.48	5.49	30.10	82000	2724	" " " 40 in. from end.
519	49.5	–	10	0.10	5.48	5.42	29.70	75400	2539	" " " near middle.
520	59.5	–	10	0.10	5.45	5.45	29.70	61700	2077	" " " 30 in. from end.
516	60.0	–	12	6.15	5.46	5.46	29.80	68000	2282	" " " 64 " " "
517	70.0	–	12	6.16	5.50	5.50	30.25	70800	2340	" " " 58 " " "
518	67.5	–	12	6.10	5.48	5.52	30.25	100000	3306	" " " 31 " " "
513	84.5	–	15	0.27	5.47	5.47	29.90	54000	1806	" " " 66 " " "
514	90.0	–	15	0.31	5.48	5.48	30.00	93500	3117	Defl. diagonally, and failed at knots.
515	91.0	–	15	0.28	5.33	5.33	28.40	94000	3310	" " " " " "
521	96.5	–	17	6.24	5.35	5.35	28.60	53000	1853	" " " " " "
522	101.0	–	17	6.25	5.40	5.38	29.00	40500	1396	" downward, wind-shake on concave side.
523	95.0	–	17	6.22	5.33	5.35	28.50	64800	2274	" horizontally.
524	109.5	–	20	0.25	5.28	5.26	27.80	45000	1619	" diagonally.
525	106.5	–	20	0.27	5.29	5.27	27.90	39500	1416	" "

COMPRESSION OF WHITE-PINE POSTS.—*Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. in.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
526	95.0	–	20	0.28	5.25	5.29	27.80	37000	1331	Defl. horizontally.
505	110.0	–	22	6.26	5.16	5.18	26.70	37000	1386	" "
506	113.0	–	22	6.23	5.20	5.22	27.10	37200	1373	" "
507	118.0	–	22	6.26	5.18	5.18	26.80	46700	1743	" "
502	127.0	–	25	0.31	5.27	5.25	27.70	30900	1116	" diagonally.
503	125.0	–	25	0.34	5.25	5.25	27.60	25800	935	" sidewise.
504	127.0	–	25	0.37	5.25	5.25	27.60	22200	804	" diagonally.
496	169.0	–	27	6.35	5.32	5.34	28.40	32500	1144	" "
497	173.0	–	27	6.40	5.35	5.35	28.60	32000	1119	" "
498	157.0	–	27	6.30	5.35	5.35	28.60	28000	979	" "
569	94.0	–	6	8.07	7.77	9.64	74.90	204000	2724	Failed at knots 10 in. from middle.
570	93.0	7	6	8.10	7.70	9.70	74.70	177000	2369	" " " near middle.
571	120.5	12	6	8.13	7.73	9.63	74.40	185000	2487	" " " 9 in. from end.
575	166.0	15	10	0.13	7.75	9.62	74.60	186000	2493	" " " 21 " " "
576	166.0	11	10	0.16	7.72	9.76	75.30	204600	2717	" " " 13 and 27 in. from end.
577	160.0	10	10	0.13	7.73	9.74	75.30	135000	1793	" " " 8 " 11 " " "
581	221.0	7	13	4.12	7.75	9.75	75.60	168000	2222	" " " 14 " 44 " " "
582	182.0	3	13	4.14	7.75	9.75	75.60	127500	1687	" " " 13 " 18 " " "
583	196.5	13	13	4.14	7.48	9.23	69.00	208500	3022	" " " 50 in. from end.
590	242.0	7	16	8.20	7.72	9.63	74.30	161000	2167	" " " 20 " " "

COMPRESSION OF WHITE-PINE POSTS. — *Concluded.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Width.	Depth.	Sectional Area.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
591	210.5	9	16	8.22	7.78	9.58	74.50	163000	2188	Failed at knots near middle.
592	274.0	8	16	8.21	7.75	9.75	75.60	175300	2319	" " " 75 in. from end.
533	278.0	–	20	0.24	7.59	9.62	73.00	140000	1918	Deflected horizontally.
534	285.0	–	20	0.26	7.38	9.29	68.60	166000	2420	Failed at knot near middle.
535	295.0	–	20	0.25	7.39	9.27	68.50	206000	3007	" " " 24 in. from middle.
539	352.0	–	23	4.20	7.72	9.75	75.40	140000	1857	Defl. horizontally.
540	357.0	–	23	4.20	7.72	9.61	74.20	170000	2291	" "
541	340.0	–	23	4.18	7.63	9.50	72.50	150000	2069	" diagonally.
536	427.0	–	26	8.20	7.46	9.36	69.80	169000	2421	" horizontally.
537	397.0	–	26	8.39	7.49	9.34	70.00	140000	2000	" "
538	384.0	–	26	8.27	7.46	9.37	69.90	134000	1917	" " knots on concave.
702	300.0	10	15	0.02	5.60	15.58	87.20	159000	1823	Failed at knots 23 in. from end.
703	267.0	5	15	0.08	5.61	15.60	87.50	168000	1926	Defl. horizontally.
704	267.0	12	15	0.00	5.65	15.60	88.10	165000	1873	" "
699	356.0	7	15	0.00	6.60	15.64	103.20	240000	2326	Failed at knots 40 in. from end.
700	395.0	10	14	11.94	6.60	15.64	103.20	239000	2316	" " " 10 and 60 in. from end.
701	376.0	7	15	0.00	6.60	15.60	103.00	203000	1971	" " " 15 in. from middle.
705	657.0	6	15	0.03	8.47	16.43	139.20	279400	2007	" " " at middle.
706	437.0	5	15	0.00	8.48	16.46	139.60	292000	2092	" " " " "
707	447.0	5	14	11.93	8.48	16.48	139.80	359000	2568	Defl. horizontally, fibres crushed near end.

COMPRESSION OF WHITE PINE. — SINGLE STICKS AND BUILT POSTS.

In the multiple ones, dimensions of each stick are given.

No. of Test.	Weight.	Average Rate of Growth. Rings per Inch.	Dimensions of Post.			Sectional Area.	Compression in 150 In. Load = 500 lbs. per Sq. In.	Ultimate Strength.	
			Length.	Width.	Depth.			Actual.	Lbs. per Sq. In.
	lbs.		in.	in.	in.	sq. in.	in.	lbs.	
664	153	11	177.50	4.48	11.65	52.2	0.0545	110000	2107
665	143	10	180.00	4.48	11.64	52.1	0.1010	81500	1564
666	163	5	179.97	4.47	11.63	52.0	0.0895	70000	1346
667	228	13	180.00	5.40	11.30	61.0	0.0505	160000	2623
668	193	5	179.93	5.61	11.73	65.8	0.0622	156300	2375
669	253	5	180.00	5.64	11.76	66.3	0.0608	152300	2297
638	181 134 } 315	6 8	180.00 180.00	4.50 4.50	11.60 11.59	52.2 52.2 } 104.4	0.0670 0.0750	200000	1916
639	182 167 } 349	7 5	180.00 180.00	4.52 4.49	11.66 11.62	52.7 52.2 } 104.9	0.0645 0.0670	212000	2021
640	131 158 } 289	7 5	180.00 180.00	4.53 4.52	11.59 11.59	52.5 52.5 } 105.0	0.1060 0.0955	149000	1419
642	192 306 } 498	6 7	179.98 179.98	5.57 5.58	11.61 11.61	64.7 64.7 } 129.4	0.0770 0.0390	215000	1661
643	203 189 } 392	17 5	179.92 179.92	5.65 5.61	11.61 11.62	65.6 65.2 } 130.8	0.0440 0.0600	261000	1995
644	227 236 } 463	7 7	179.96 179.96	5.60 5.60	11.63 11.62	65.1 65.1 } 130.2	0.0596 0.0690	257800	1980
648	206 203 } 409	12 10	180.00 180.00	5.60 5.61	11.72 11.72	65.6 65.6 } 131.2	0.0590 0.0700	268000	2042
649	187 184 } 371	9 4	180.00 180.00	5.60 5.61	11.71 11.74	65.6 65.9 } 131.5	0.0600 0.0705	277000	2107
650	194 190 } 384	12 9	180 00 180.00	5.61 5.61	11.75 11.71	65.9 65.7 } 131.6	0.0530 0.0885	240000	1824
645	215 266 } 481	5 7	179.97 179.97	5.59 5.60	11.59 11.60	64.8 65.0 } 129.8	0.0560 0.0540	263200	2028
646	206 204 } 410	6 12	180.00 180.00	5.59 5.60	11.61 11.62	64.9 65.1 } 130.0	0.0493 0.0620	249000	1915
647	192 192 } 384	11 13	180.00 180.00	5.62 5.62	11.62 11.62	65.3 65.3 } 130.6	0.0630 0.0700	248000	1899
678	251 215 } 466	16 8	179.94 179.94	5.58 5.57	11.47 11.45	64.0 63.8 } 127.8	0.0529 0.0642	245500	1921
679	209 190 } 399	12 7	180.00 180.00	5.62 5.62	11.76 11.72	66.1 65.9 } 132.0	0.0664 0.0705	249000	1886
680	268 278 } 546	8 8	180.00 180.00	5.60 5.61	11.72 11.73	65.6 65.8 } 131.4	0.0650 0.0495	278000	2116
663	230 208 } 438	14 6	180.00 180.00	5.60 5.63	11.75 11.75	65.8 66.2 } 132.0	0.0621 0.0657	300000	2273
676	201 184 } 385	12 6	179.94 179.94	5.60 5.61	11.71 11.73	65.6 65.8 } 131.4	0.0530 0.0593	274500	2089
677	185 193 } 378	6 6	180.00 180.00	5.61 5.68	11.72 11.72	65.7 65.4 } 131.1	0.0551 0.0625	255000	1945

COMPRESSION OF WHITE PINE.— *Concluded.*

Single Sticks and Built Posts.

No. of Test.	Weight.		Average Rate of Growth. Rings per Inch.	Dimensions of Post. Length.	Width.	Depth.	Sectional Area.		Compression in 150 In. Load = 500 lbs. per Sq. In.	Ultimate Strength. Actual.	Lbs. per Sq. In.
	lbs.			in.	in.	in.	sq. in.		in.	lbs.	
690	175 226 199	600	18 11 18	180.00 180.00 180.00	4.52 5.56 4.46	11.62 11.70 11.62	52.5 65.0 51.8	169.3	0.0460 0.0580 0.0480	310000	1831
691	164 197 159	520	9 14 12	179.98 179.98 179.98	4.48 5.56 4.45	11.60 11.60 11.61	52.0 64.5 51.7	168.2	0.0526 0.0430 0.0390	372500	2215
692	248 153 213	614	13 12 5	177.25 177.25 177.25	5.62 4.50 5.60	11.60 11.60 11.57	65.2 52.2 64.8	182.2	0.0580 0.0641 0.0768	363000	1992
687	151 218 167	536	11 8 9	180.00 180.00 180.00	4.50 5.58 4.52	11.60 11.62 11.59	52.2 64.8 52.4	169.4	0.0460 0.0587 0.0533	325500	1919
688	176 236 163	575	10 10 11	180.00 180.00 180.00	4.50 4.62 4.50	11.60 11.60 11.60	52.2 65.2 52.2	169.6	0.0565 0.0645 0.0703	306000	1804
689	188 203 159	550	7 11 9	180.05 180.05 180.05	4.48 5.60 4.46	11.60 11.62 11.57	52.0 65.1 51.6	168.7	0.0510 0.0660 0.0789	340000	2015
681	193 192 146 157	688	5 5 9 9	179.95 179.95 179.95 179.95	4.47 4.52 4.48 4.51	11.60 11.65 11.65 11.65	51.9 52.7 52.2 52.5	209.3	0.0700 0.0714 0.0531 0.0762	362000	1734
682	161 156 163 151	631	7 14 7 9	180.00 180.00 180.00 180.00	4.50 4.50 4.49 4.50	11.63 11.64 11.60 11.63	52.3 52.4 52.1 52.3	209.1	0.0612 0.0546 0.0542 0.0916	414000	1980
683	169 219 218 181	787	10 11 9 7	180.03 180.03 180.03 180.03	4.46 5.62 5.60 4.46	11.63 11.69 11.70 11.61	51.9 65.7 65.5 51.8	234.9	0.0570 0.0530 0.0494 0.0590	501000	2133
684	146 210 234 166	756	10 12 7 10	180.00 180.00 180.00 180.00	4.50 5.64 5.60 4.48	11.64 11.59 11.58 11.61	52.4 65.4 64.8 52.0	234.6	0.0664 0.0610 0.0548 0.0514	529000	2255
685	145 220 209 166	740	10 10 12 9	180.00 180.00 180.00 180.00	4.50 5.63 5.61 4.48	11.60 11.62 11.62 11.62	52.2 65.4 65.2 52.0	234.8	0.0645 0.0650 0.0506 0.0546	430000	1831
686	145 183 159 150	637	11 10 12 8	180.00 180.00 180.00 180.00	4.48 4.50 4.50 4.52	11.61 11.56 11.36 11.61	52.0 52.0 51.1 52.5	207.6	0.0500 0.0315 0.0543 0.0680	395000	1903

COMPRESSION OF YELLOW-PINE POSTS.

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. In.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
565	47.0	20	5	0.06	5.50	5.50	30.3	172000	5677	Fibres crushed 12 in. from end.
566	44.0	12	5	0.10	5.50	5.49	30.2	141000	4669	Failed at knots 15 " " "
567	45.0	25	5	0.10	5.48	5.54	30.4	168000	5526	" " " 9 and 17 in. from end.
568	47.5	20	5	0.10	5.46	5.50	30.0	108000	3600	" " " 8 in. from end.
562	72.5	–	7	5.98	5.45	5.52	30.1	140000	4651	" " season cracks, crushed.
563	59.5	–	7	6.14	5.46	5.46	29.8	124000	4161	" " knots 30 in. from end.
564	57.0	–	7	6.15	5.47	5.47	29.9	143500	4799	" " " 27 " " "
559	86.0	–	10	0.17	5.49	5.48	30.1	146200	4857	Defl. upward.
560	88.5	–	10	0.16	5.50	5.47	30.1	139000	4618	" downward.
561	110.0	–	10	0.15	5.51	5.50	30.3	143600	4739	Failed at knots in middle.
556	126.0	–	12	6.12	5.54	5.51	30.5	156000	5114	Defl. horizontally.
557	128.0	–	12	6.17	5.57	5.52	30.2	156000	5166	" diagonally.
558	141.5	–	12	6.20	5.50	5.50	30.3	150000	4950	" "
553	140.0	–	15	0.22	5.50	5.48	30.1	99200	3296	" downward.
554	148.0	–	15	0.24	5.49	5.47	30.0	130000	4333	" horizontally.
555	148.0	–	15	0.25	5.50	5.50	30.3	129000	4257	" diagonally.
550	172.0	–	17	6.24	5.48	5.47	30.0	90000	3000	" horizontally.
551	180.5	–	17	6.25	5.46	5.46	29.8	109000	3658	" "
552	167.5	–	17	6.26	5.50	5.50	30.3	93000	3069	" "
548	185.0	–	20	0.30	5.40	5.45	29.4	82000	2789	" "

COMPRESSION OF YELLOW-PINE POSTS.—*Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Size of Post. Width.	Size of Post. Depth.	Sectional Area.	Ultimate Strength. Actual.	Ultimate Strength. Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
549	185.0	–	20	0.28	5.45	5.46	29.8	87000	2946	Defl. horizontally.
545	212.0	–	22	6.28	5.64	5.59	31.5	73000	2340	" "
546	189.0	–	22	6.37	5.42	5.49	29.8	56000	1879	" upward.
547	266.0	–	22	6.30	5.65	5.61	31.7	62500	1972	" diagonally.
508	210.0	–	25	0.36	5.46	5.48	29.9	52000	1739	" horizontally.
509	235.0	–	25	0.30	5.46	5.49	30.0	59000	1967	" "
510	210.0	–	25	0.33	5.45	5.46	29.8	55500	1862	" "
499	258.0	–	27	6.44	5.31	5.29	28.1	60800	2163	" "
500	236.0	–	27	6.43	5.32	5.30	28.2	39000	1383	" diagonally.
501	256.0	–	27	6.40	5.31	5.30	28.1	44400	1580	" horizontally.
572	124.0	6	6	8.10	7.78	9.75	75.8	260000	3430	Failed at knot 30 in. from end.
573	131.0	14	6	8.08	7.76	9.77	75.8	286500	3780	" " " 26 " " "
574	161.5	21	6	8.10	7.74	9.81	75.9	383000	5046	" " knots 8 " " "
578	233.5	23	10	0.13	7.74	9.73	75.3	304000	4037	" " " 18 " " "
579	240.0	19	10	0.14	7.78	9.75	75.9	399500	5264	" 36 in. from end, grain wavy.
580	240.0	28	10	0.14	7.75	9.75	75.6	350000	4629	" at knot 17 in. from end.
584	266.0	18	13	4.18	7.68	9.72	74.6	307500	4122	" " " 15 and 29 in. from end.
585	258.0	6	13	4.15	7.75	9.74	75.5	284000	3762	" " " 45 in. from end.
586	336.0	23	13	4.20	7.75	9.75	75.6	296000	3921	" " " near middle.
587	318.0	24	16	8.17	7.70	9.74	75.0	274500	3660	" " " 50 in. from end.

COMPRESSION OF YELLOW-PINE POSTS.—*Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. In.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
588	381.0	12	16	8.24	7.74	9.73	75.3	382000	5073	Defl. horizontally.
589	339.5	21	16	8.26	7.25	9.49	68.8	304000	4419	" "
530	461.0	–	20	0.25	7.71	9.80	75.6	287000	3796	" "
531	410.0	–	20	0.28	7.76	9.58	74.3	275000	3701	Failed at knots 70 in. from end.
532	391.0	–	20	0.25	7.61	9.68	73.7	220000	2985	Defl. horizontally.
542	501.0	–	23	4.20	7.69	9.77	75.1	225000	2996	" "
543	501.0	–	23	4.20	7.69	9.73	74.8	220500	2948	" "
544	560.0	–	23	4.22	7.70	9.76	75.1	298000	3968	" "
527	531.0	–	26	8.30	7.40	9.47	70.0	210000	3000	" "
528	627.0	–	26	8.29	7.45	9.45	70.4	205000	2912	" "
529	535.0	–	26	8.30	7.46	8.94	66.7	184500	2766	" diagonally.
711	375.0	12	15	0.08	5.60	15.60	87.4	349000	3993	" horizontally.
712	397.0	21	15	0.05	5.63	15.57	87.7	339000	3865	" "
713	382.0	15	15	0.00	5.65	15.50	87.6	272500	3111	" "
484	432.0	–	15	0.25	6.90	15.85	109.4	270000	2468	" " knot on concave side.
708	464.0	17	15	0.00	6.61	15.51	102.5	340000	3317	Failed at knots at middle.
709	557.0	18	15	0.00	6.60	15.64	103.2	455000	4400	Defl. horizontally.
710	491.0	18	14	11.92	6.61	15.61	103.2	432000	4186	" "
641	561.0	5	15	0.10	8.25	16.25	134.0	428000	3194	Fibres crushed at middle, in vicinity of knots.
671	585.0	12	15	0.07	8.25	16.20	133.6	425000	3181	Defl. horizontally.

COMPRESSION OF YELLOW-PINE POSTS.—*Concluded.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.			Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.	Width.	Depth.		Actual.	Lbs. per Sq. In.	
	lbs.	rings per in.	ft. in.	in.	in	sq. in.	lbs.		
672	578.0	13	14 11.93	8.20	16.20	132.8	526000	3961	Defl. horizontally.
485	434.0	–	15 0.27	6.75	15.95	107.7	175000	1625	" " knots on concave side. Second test of these posts.
487	450.0	–	15 0.25	6.80	15.72	106.9	310000	2900	Defl. horizontally, initial bend in post. Second test of these posts.
486	705.0	–	15 0.24	8.63	16.83	145.2	380000	2617	Opened shakes and seasoned cracks. Second test of these posts.

COMPRESSION OF YELLOW PINE.

Single Sticks and Built Posts.

No. of Test.	Weight, in lbs.	Average Rate of Growth. Rings per Inch.	Dimensions of Post. Length.	Width.	Depth.	Sectional Area.	Compression in 150 In. Load = 500 lbs. per Sq. Inch.	Ultimate Strength. Actual.	Lbs. per Sq. In.
			in.	in.	in.	sq. in.			
673 673*a* 673*b*	260 13 7¼	11	180.05 20.00 20.00	4.09 4.01 3.95	11.35 4.01 3.97	46.04 16.08 15.68	0.0370	142200 94000 99600	3065 5846 6352
674	233	17	180.00	4.51	11.60	52.30	0.0492	131500	2515
675	194	22	180.00	4.34	11.60	50.30	0.0490	121200	2410
490	269	–	180.00	5.05	12.10	61.10	–	230000	3764
670	309	12	180.00	5.65	11.74	66.30	0.0455	205900	3106
489	287	–	180.08	5.85	12.05	70.50	–	250000	3546
654	326 290 } 616	11 6	180.00 180.00	5.63 5.62	11.71 11.71	65.9 65.9 } 131.8	0.0418 0.0540	470000	3566
655	292 329 } 621	34 24	179.93 179.93	5.64 5.63	11.72 11.72	66.1 66.1 } 132.2	0.0292 0.0315	580000	4387
656	282 287 } 569	13 11	180.00 180.00	5.61 5.61	11.71 11.71	65.7 65.7 } 131.4	0.0368 0.0466	480000	3653
651	262 275 } 537	11 7	180.00 180.00	5.58 5.58	11.71 11.71	65.3 65.3 } 130.6	0.0620 0.0514	360000	2756
652	310 370 } 680	16 16	180.00 180.00	5.58 5.58	11.70 11.71	65.3 65.3 } 130.6	0.0395 0.0360	588500	4506
653	310 311 } 621	16 12	180.00 180.00	5.63 5.59	11.71 11.68	65.9 65.3 } 131.2	0.0559 0.0550	436600	3328
657	340 319 } 659	9 13	179.96 179.96	5.63 5.64	11.72 11.71	66.0 66.0 } 132.0	0.0375 0.0305	580000	4394
658	319 332 } 651	12 9	179.98 179.98	5.59 5.59	11.71 11.72	65.5 65.5 } 131.0	0.0320 0.0436	448000	3420
659	320 293 } 613	12 8	180.00 180.00	5.61 5.61	11.73 11.73	65.8 65.8 } 131.6	0.0312 0.0372	600000	4559
660	283 294 } 577	25 18	180.03 180.03	5.61 5.63	11.22 11.24	62.9 63.3 } 126.2	0.0325 0.0400	510000	4041
661	274 310 } 584	12 9	180.00 180.00	5.66 5.60	11.70 11.72	66.2 65.6 } 131.8	0.0410 0.0365	410000	3111
662	292 280 } 572	14 8	180.00 180.00	5.61 5.61	11.75 11.75	65.9 65.9 } 131.8	0.0540 0.0500	388000	2944
693	242 276 } 774 256	11 15 23	179.97 179.97 179.97	4.50 5.50 4.49	11.61 11.56 11.62	52.2 63.6 } 168.0 52.2	0.0320 0 0325 0.0148	564000	3357
694	193 290 } 691 208	17 24 10	180.00 180.00 180.00	4.50 5.59 4.46	11.35 11.36 11.35	51.1 63.5 } 165.2 50.6	0.0500 0.0650 0.0610	500000	3027
695	207 290 } 743 246	8 16 16	180.00 180.00 180.00	4.49 5.20 4.50	11.35 11.34 11.35	51.0 59.0 } 161.1 51.1	0.0429 0.0290 0.0410	474000	2942

COMPRESSION OF YELLOW PINE.—*Concluded.*

SINGLE STICKS AND BUILT POSTS.

No. of Test.	Weight, in lbs.		Average Rate of Growth. Rings per Inch.	Dimensions of Post.			Sectional Area.		Compression in 150 In. Load = 500 lbs. per Sq. Inch.	Ultimate Strength.	
				Length.	Width.	Depth.				Actual.	Lbs. per Sq. In.
				in.	in.	in.	sq. in.				
696	217 253 190	660	25 11 15	180.00 180.00 180.00	4.51 5.50 4.48	11.24 11.23 11.23	50.7 61.8 50.3	162.8	0.0337 0.0567 0.0715	480000	2948
697	242 249 215	706	15 12 15	180.00 180.00 180.00	4.52 5.43 4.47	11.60 11.60 11.58	52.4 63.0 51.8	167.2	0.0240 0.0330 0.0336	540000	3230
698	224 255 296	775	15 8 15	179.94 179.94 179.94	4.46 5.53 4.50	11.60 11.70 11.59	51.7 64.7 52.2	168.6	0.0290 0.0385 0.0341	544000	3227
488	911		-	180.20	6.88 6.72	15.75 15.75	108.36 105.84	214.2	-	700000	3268

These sets of tests furnish us practically the only reliable information we have in regard to the strength of full-size columns of white pine, yellow pine, and oak. In regard to white and yellow pine, Mr. Edward F. Ely, instructor in architecture at the Massachusetts Institute of Technology, has plotted the average, and also the lowest, results of the tables; and, from an inspection of the diagrams, he gives the following rules for determining the breaking-strength of a column with flat ends, the load being evenly distributed over the ends:—

Let A = area of section in square inches.

f = constant whose value is given in the tables following.

$\frac{l}{r}$ = ratio of length to least side of rectangle, all the tests having been made on rectangular sections.

Then

$$\text{Breaking-strength} = fA,$$

where f has the following values: —

White Pine.		Yellow Pine.	
$\frac{l}{r}$	f	$\frac{l}{r}$	f
0 to 10	2500	0 to 15	4000
10 to 35	2000	15 to 30	3500
35 to 45	1500	30 to 40	3000
45 to 60	1000	40 to 45	2500
		45 to 50	2000
		50 to 60	1500

In the case of oak, if it is desired to apply the results to greater ratios of length to diameter than those tested, a similar reduction can be made in the value of f to that which takes place here in the case of white and yellow pine.

For the ratios of length to diameter tested, $f = 3000$ would seem to be a fair value to use.

§ 238. **Factor of Safety.** — Whereas we are constantly told, that, in the case of iron bridge-work, we should use a factor of safety 4, but that for a timber construction we should use at least 8, or even 10 and 12, — the reason evidently is, that the figures that have generally been given us for breaking-strength have been really from two to four times the actual breaking-strength, — it would seem to the writer, that, when we use correct values for breaking-strength, a factor of safety 4 will be sufficient for all ordinary timber constructions; i.e., that we should use for working-strength per square inch, one-fourth the breaking-strength per square inch. This same reasoning will also apply to the case of beams bearing a transverse load when they are designed with reference to their breaking-weight.

§ 239. **Transverse Strength of Timber.** — In this regard, the common theory of beams has already been explained in § 185 *et seq.*

The tables of Rankine and Rodman, already given, represent the values of modulus of rupture that have been in common use. Other values, not differing essentially from these, are given by Hatfield, Laslett, Thurston, Trautwine, and others, all based upon tests of small pieces. No tables of these values will be given; as those above referred to furnish practically the same information. Confining ourselves to tests of full-size pieces, we find an account of a set of tests attributed by D. K. Clark, in his "Rules and Tables," to Edwin Clark and C. Graham Smith. The results are given below, and it will be seen that they are very much below those given by experimenters on small pieces. Two tests by R. Baker are also mentioned by D. K. Clark.

Kind of Timber.	Breadth and Depth.	Span.	How Loaded.	Breaking-Weight.	Modulus of Rupture.
	in.	ft.			
American red pine	12.0 × 12.0	15.00	Centre	33497	5238
" " "	12.0 × 12.0	15.00	"	29908	4680
" " "	6.0 × 6.0	7.50	"	7370	4608
Memel fir . . .	13.5 × 13.5	10.50	Distributed	68560	5274
" " . . .	13.5 × 13.5	10.50	"	68560	5274
Baltic fir	6.0 × 12.0	12.25	Centre	19145	4878
" "	6.0 × 12.0	12.25	"	23625	6020
Pitch pine . . .	6.0 × 12.0	12.25	"	23030	5868
" " . . .	6.0 × 12.0	12.25	"	23700	6048
" " . . .	14.0 × 15.0	10.50	"	134400	8064
" " . . .	14.0 × 15.0	10.50	"	132610	7956
Red pine	6.0 × 12.0	12.25	"	16800	4284
" "	6.0 × 12.0	12.25	"	19040	4860
Quebec yellow pine	14.0 × 15.0	10.50	Distributed	68600	4122
" " "	14.0 × 15.0	10.50	"	68600	4122
" " "	14.0 × 15.0	10.50	Centre	85792	5148
" " "	14.0 × 15.0	10.50	"	76160	4572

During the last three years tests of the strength and stiffness of full-size beams of spruce, yellow pine, oak, and white pine, both under centre loads and distributed loads, have been carried on in the Laboratory of Applied Mechanics of the Massachusetts Institute of Technology. Tests have also been made upon the effect of time on the stiffness of such beams, also on the strength of built-up beams, and of floors and framing-joints, all full size. A summary of the results obtained will be given, and conclusions drawn as to the proper values of the modulus of rupture and modulus of elasticity, etc., to be used in practice.

Before giving this summary, the following explanation and formulæ will be appended for the convenience of those who may not have read the former part of this book. The different tables of results given in different handbooks differ in the form of the constant. Thus, the constant given by Trautwine and Hatfield is one-eighteenth the modulus of rupture, or the hypothetical breaking centre load of a beam one inch square and one foot long supported at the ends; while Rodman gives one-sixth of the modulus of rupture, or the hypothetical breaking-load at the end of a cantilever one inch square and one inch long.

The formulæ for breaking-load in terms of the modulus of rupture, and for modulus of rupture in terms of the breaking-load, for some of the most usually occurring cases of rectangular beams, are appended.

Let $W =$ breaking-load in pounds.
$f =$ modulus of rupture in pounds per square inch.
$b =$ breadth of beam in inches.
$h =$ depth of beam in inches.
$l =$ length of beam in inches.

Then we shall have: —

(*a*) *Beam fixed at one end and free at the other.*

1°. Single load at free end,

$$W = f\frac{bh^2}{6l}, \qquad f = \frac{6Wl}{bh^2}.$$

2°. Load uniformly distributed,

$$W = f\frac{bh^2}{3l}, \qquad f = \frac{3Wl}{bh^2}.$$

(*b*) *Beam supported at both ends.*

1°. Single load at the middle,

$$W = \tfrac{2}{3}f\frac{bh^2}{l}, \qquad f = \frac{3}{2}\frac{Wl}{bh^2}.$$

2°. Load uniformly distributed,

$$W = \tfrac{4}{3}f\frac{bh^2}{l}, \qquad f = \frac{3}{4}\frac{Wl}{bh^2}.$$

3°. Single load at a distance a from the origin,

$$W = f\frac{lbh^2}{6a(l-a)}, \quad f = \frac{6Wa(l-a)}{lbh^2}.$$

DEFLECTION OF BEAMS.

While the preceding formulæ refer to the breaking-strength of beams, it is better engineering to determine, as the safe load of a timber beam, the load that will not deflect it more than a certain small fraction ($\frac{1}{300}$ or $\frac{1}{400}$) of the span.

Let W = given load in pounds.
b = breadth in inches.
h = depth in inches.
l = length in inches.
v = greatest deflection in inches.
E = modulus of elasticity of the material in pounds per square inch.

(*a*) *Beam fixed at one end and free at the other.*

1°. Single load at free end,

$$v = \frac{4Wl^3}{Ebh^3}, \qquad E = \frac{4Wl^3}{vbh^3}.$$

2°. Load uniformly distributed,

$$v = \frac{3}{2}\frac{Wl^3}{Ebh^3}, \qquad E = \frac{3}{2}\frac{Wl^3}{vbh^3}.$$

(*b*) *Beam supported at both ends.*

1°. Single load at the middle,

$$v = \frac{1}{4}\frac{Wl^3}{Ebh^3}, \qquad E = \frac{1}{4}\frac{Wl^3}{vbh^3}.$$

2°. Load uniformly distributed,

$$v = \frac{5}{32}\frac{Wl^3}{Ebh^3}, \qquad E = \frac{5}{32}\frac{Wl^3}{vbh^3}.$$

The above formulæ enable us to determine the deflection of a beam under a given load when the modulus of elasticity of the material is known, or to determine the modulus of elasticity of the material from the observed deflection.

LONGITUDINAL SHEARING.

In any rectangular beam, the greatest intensity of the longitudinal shearing-force at any section (i.e., its intensity at the neutral axis of the section) is, if we let

F = shearing-force at the section, technically so-called, in pounds,

b = breadth of beam in inches,

h = depth of beam in inches,

$$\frac{3}{2}\frac{F}{bh}.$$

In the case of a beam supported at the ends, and loaded at the middle with a single load, we have, for all sections except the middle, $F = \frac{W}{2}$; and hence, if we denote by i the greatest intensity of the longitudinal shearing-force at the neutral layer, we shall have

$$i = \frac{3}{4}\frac{W}{bh}: \quad (1)$$

and this is the intensity of the shearing-force at all points along the neutral layer, except at the middle section, where it is zero.

In the case of a beam supported at the ends, with the load uniformly distributed, the greatest intensity is that at the support, and is also given by equation (1), but decreases gradually to the middle.

SUMMARY OF THE TESTS.

The tests recorded may be divided into six classes: —

1°. Spruce beams.
2°. Yellow-pine beams.
3°. Time tests.
4°. Oak beams.
5°. White-pine beams.
6°. Framing-joints.

1°. *Spruce Beams.* — Before giving a summary of the tests made in this laboratory, I will insert some of the moduli of rupture and moduli of elasticity given by different authorities.

Moduli of rupture are given as follows: —

	Maximum.	Minimum.	Mean.
Hatfield	12996	7506	9900
Rankine	12300	9900	11100
Laslett	9707	7506	9045
Trautwine	–	–	8100
Rodman	–	–	6168

Hatfield's, Laslett's, Trautwine's, and Rodman's figures are from their own experiments. Trautwine advises, for practical use, to deduct one-third on account of knots and defects, hence to use 5400. The tables show the values obtained in these tests, and I will add a recommendation as to the values of modulus of rupture and modulus of elasticity suitable to use in practice.

As a result of the tests thus far made in my laboratory, it seems to me safe to say, if our Boston lumber-yards are to be taken as a fair sample of the lumber-yards in the case of spruce, — if such lumber is ordered from a dealer of good repute, no selection being made except to discard that which is rotten or has holes in it, — that 3000 lbs. per square inch is all that could with any safety be used for a modulus of rupture, and even this might err in some cases in being too large; (2°) that, if the lumber is carefully selected at any one lumber-yard, so as to take only the best of their stock, it would not be safe to use for modulus of rupture a number greater than 4000; and if we required a lot of spruce which should have a modulus of rupture of 5000, it would be necessary to select a very few pieces from each lumber-yard in the city. With a factor of safety four, we should have for greatest allowable outside fibre stress in the three cases respectively, 750, 1000, and 1250.

The modulus of elasticity (i.e., that determined from the immediate deflections) was: maximum, 1588548; minimum, 897961; mean, 1329479.

Two time tests were made on yellow pine; and, if we should consider the effect of time on spruce the same as for yellow pine, we should obtain for use, for spruce, about 886319.

SPRUCE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.	Manner of Loading.	Breaking-Load.	Modulus of Rupture, in lbs., per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	Remarks.
	inches.	ft. in.		lbs.			
3	2 × 12	15 0	Load at middle	5894	5526	1237215	
4	2 × 9	6 $7\frac{1}{2}$	" "	7322	5389	1067893	
5	2 × 12	15 0	" "	5586	5237	–	
5*a*	2 × 12	7 0	" "	8982	–	–	
6	$2\frac{3}{4}$ × 9	6 8	" "	7586	4082	938453	
7	3 × 9	4 0	" "	11086	3285	–	
8	3 × 9	10 0	" "	6086	4508	–	
9	3 × 9	15 0	" "	8086	5651	–	
10	$3\frac{7}{8}$ × 12	20 0	" "	6586	4253	–	
11	$2\frac{1}{2}$ × $13\frac{1}{2}$	10 0	" "	9585	3787	–	
12	$3\frac{3}{4}$ × 12	16 0	Load $4\frac{1}{2}$ ft. from end	7585	3271	–	
14	7 × 2	7 0	Load at middle	1944	8748	–	Seasoned on wharf 4 years.
15	$1\frac{3}{4}$ × $6\frac{3}{4}$	7 0	" "	4785	7562	–	Seasoned on wharf 4 years.
16	3 × 9	6 8	" "	9985	4931	–	
17	3 × 9	6 8	Load at four points 16 inches apart	16744	4961	–	
18	3.9 × 12	16 0	Load $4\frac{1}{2}$ ft. from end	12585	5218	–	
19	2 × 12	14 0	Load at middle	4404	3854	1482645	
20	2 × 12	14 0	" "	5108	4469	1588548	

SPRUCE BEAMS. — *Continued.*

No. of Test.	Width and Depth.	Distance between Supports.	Manner of Loading.	Breaking-Load.	Modulus of Rupture, in lbs., per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	Remarks.
	inches.	ft. in.		lbs.			
21	$3\frac{15}{16} \times 12$	14 0	Load at middle	8627	3834	1187073	
22	$3\frac{7}{8} \times 12$	14 0	" "	12545	5666	1332715	
23	$3\frac{7}{8} \times 12\frac{1}{4}$	14 0	" "	6917	2995	897961	
24	$3 \times 11\frac{7}{8}$	14 0	" "	8927	5442	1572470	
25	$2 \times 9\frac{3}{4}$	14 0	" "	3198	4139	1460620	
26	$2\frac{3}{4} \times 12$	14 0	" "	6819	4339	1396667	
27	$1\frac{15}{16} \times 10$	14 0	" "	4306	5601	1355860	
28	$4\frac{1}{8} \times 12$	18 0	" "	8829	4816	1397136	
29	$4 \times 12\frac{1}{4}$	18 0	" "	8324	4586	1259224	
31	$3\frac{1}{8} \times 12$	18 0	" "	7721	5559	1231498	
35	6×12	18 0	" "	11188	4196	1347910	
36	$2 \times 11\frac{7}{8}$	7 2	" "	7870	3599	–	
37	$4 \times 11\frac{3}{4}$	12 0	" "	10572	4135	–	
45	$3\frac{1}{8} \times 11\frac{3}{4}$	16 4	" "	8072	4436	–	
46	$3\frac{11}{16} \times 12$	10 2	" "	13772	4746	–	
49	$3\frac{3}{4} \times 11\frac{3}{4}$	14 0	" "	12076	5878	–	
60	4×12	17 4	Distributed equally at 12 points	26000	7448	1461039	Seasoned in steam-heat about 6 months without load.

SPRUCE BEAMS. — *Concluded.*

No. of Test.	Width and Depth.	Distance between Supports.	Manner of Loading.	Breaking-Load.	Modulus of Rupture, in lbs., per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	Remarks.
	inches.	ft. in.		lbs.			
66	4 × 12	15 8	Load at middle	14576	7211	1336162	Seasoned in steam-heat about 6 months without load.
70	$4\frac{1}{4}$ × $12\frac{1}{2}$	17 4	Distributed equally at 12 points	14633	3748	1551769	
72	4 × $12\frac{3}{16}$	17 4	Distributed equally at 12 points	11333	3252	1228563	
90	6 × 12	17 4	Distributed equally at 12 points	26100	5102	1587646	
					18)185560		
	Average	modulus	of rupture	. . . .	4884		
						21)27919067	
	Average	modulus	of elasticity	. . . .	. . . .	1329479	

Yellow-Pine Beams. — The moduli of rupture in common use are given as follows by different authorities; viz., —

	Maximum.	Minimum.	Mean.
Hatfield	21168	9000	15300
Laslett	14162	10044	12254
Trautwine	–	Yellow pine	9000
		Pitch pine	9900
Rodman	9876	8796	9293

A summary of the figures obtained from these tests will be given in a table at the end of these remarks.

It will be observed that we have for

	Maximum.	Minimum.	Mean.
Modulus of rupture . .	9380	4764	6984
Modulus of elasticity .	2386096	1256286	1779517

We also have a considerable reduction of the modulus of elasticity with time, as shown by the time tests; the proper values for use being, in the case of spruce and hard pine, from two-thirds to three-fourths the immediate moduli of elasticity. (See p. 536.)

YELLOW-PINE BEAMS.

No. of Test.	Width and Depth.	Span.		Manner of Loading.	Breaking-Weight, in lbs.	Modulus of Rupture.	Modulus of Elasticity.
	inches.	ft.	in.				
30	3 × $13\frac{7}{8}$	14	0	Load at centre	15158	6614	1937025
32	$4\frac{1}{16}$ × $12\frac{3}{16}$	18	0	" "	13751	7383	1733976
33	$3\frac{15}{16}$ × $12\frac{1}{4}$	18	0	" "	9832	5386	1793923
47	3 × $13\frac{3}{4}$	14	0	" "	19574	8696	2386096
50	4 × $14\frac{1}{16}$	21	0	" "	12875	5914	1256286
53	$3\frac{1}{8}$ × 14	24	6	" "	10076	7206	1784426
54	3 × $12\frac{1}{8}$	24	0	" "	9576	9380	2116821
56	$3\frac{1}{8}$ × 14	15	4	" "	10572	4764	1490396
57	$2\frac{15}{16}$ × 12	19	2	" "	8472	6950	1444521
59	9 × $13\frac{3}{4}$	24	0	" "	21083	5352	1417793
62	$4\frac{1}{8}$ × $12\frac{1}{8}$	19	10	" "	15461	9102	2037939
63	$4\frac{3}{16}$ × $12\frac{3}{16}$	20	0	" "	14073	8145	1599339
64	$4\frac{1}{8}$ × $12\frac{1}{4}$	19	10	" "	10573	6098	1917976
65	4 × $12\frac{1}{4}$	19	8	" "	11573	6782	1966717
67	$4\frac{1}{4}$ × 12	18	6	" "	13374	7277	1787610
68	4 × $12\frac{1}{8}$	19	9	" "	17676	10872	2381685
69	$3\frac{3}{16}$ × 14	20	0	" "	6675	3963	1169298
71	$4\frac{1}{4}$ × 12	18	2	" "	16074	8248	1512192
74	4 × 12	20	0	" "	11071	7004	1628134
75	4 × $11\frac{3}{4}$	19	9	" "	13771	9391	1850667
76	$4\frac{1}{8}$ × $12\frac{7}{16}$	17	4	Load equally distributed at 12 points	15825	4207	1344083
77	$4\frac{1}{4}$ × 12	17	4	Load equally distributed at 12 points	37325	10286	2123154
78	4 × $12\frac{1}{2}$	22	10	Load at centre	7172	4845	1455308
79	4 × 12	19	8	" "	–	–	2087583
81	$4\frac{1}{8}$ × $12\frac{1}{4}$	17	4	Load equally distributed at 12 points	16025	4349	1162467
82	4 × 12	19	8	Load at centre	15571	9671	1607336
84	$4\frac{1}{4}$ × $12\frac{1}{4}$	21	4	" "	11374	6985	1501854
85	4 × $11\frac{3}{4}$	20	6	" "	16874	11360	2246154

YELLOW-PINE BEAMS.—*Concluded.*

No. of Test.	Width and Depth.	Span.	Manner of Loading.	Breaking-Weight, in lbs.	Modulus of Rupture.	Modulus of Elasticity.
	inches.	ft. in.				
87	4 × $12\frac{1}{4}$	21 4	Load at centre	11272	7335	1535647
88	6 × $12\frac{1}{4}$	20 4	" "	15283	6112	1613012
91	4 × 12	19 10	" "	18074	11303	2223795
92	6 × 12	6 5	" "	38090	5092	–
					31)226072	
	Average	modulus	of rupture . .	. . .	7292	
						31)54113213
	Average	modulus	of elasticity . .	. . .	. . . .	1745587

In regard to the modulus of rupture to be used in practice for yellow pine, I should say, that, for the *modulus of rupture* of *yellow pine* of fair quality, I should not feel justified in using a number greater than 5000 lbs. per square inch, especially for large sizes, such as 9 × 14 inches, 12 × 16 inches, etc. My reason for this conclusion is, that, although the average modulus of rupture derived from the tests already enumerated is 7292, nevertheless, we have, in the case of beam No. 59, a modulus of rupture of 5300, notwithstanding the fact that this beam was quite free from knots, cracks, crooked grain, and other defects, and had been selected by a builder as one of exceptionally good quality. With a factor of safety four, we should have about 1200 as our greatest allowable outside fibre stress.

3°. *Time Tests.* — Two time tests have been made on yellow-pine beams.

In the first the beam was 4 × 12.5 inches, 20 feet span: it remained under load from Nov. 15 to Jan. 8. It was loaded at the centre, at first with 485 lbs., which was increased on

Nov. 21 to 1289, on Nov. 25 to 2093, etc.; finally breaking with 11741 lbs., giving a modulus of rupture of 6742 lbs. per square inch. The modulus of elasticity determined from the respective deflections was as follows: —

From immediate deflection,	1721608	At 8525 lbs.	1113684
At 4103 lbs.	1377166	" 9329 "	1101809
" 6113 "	1293046	" 10133 "	1108016
" 7118 "	1157931	" 10334 "	1103987
" 8123 "	1169451		

In the second time test, the beam was 4 × 12 inches, 21 feet 6 inches span, loaded at the centre.

It was loaded on Dec. 28 with 2070 lbs., which was increased to 3070 on Jan. 1; this load remaining on the beam till Jan. 30, when the load was increased until the beam broke at 11770 lbs., giving a modulus of rupture of 8019 lbs. per square inch.

The modulus of elasticity determined from the immediate deflection was 1715880 lbs. per square inch; that determined from the final deflection at 3070 lbs. was 1343918 lbs. per square inch, or about two-thirds the immediate modulus of elasticity.

A third time test, made with a distributed load on a spruce beam, has shown a similar result.

PROPER VALUE OF MODULUS OF ELASTICITY FOR USE IN COMPUTING DEFLECTIONS.

The fact that the strength of a structure is the strength of its weakest part, should lead us to select for use in ordinary cases, for immediate *modulus of elasticity*, a number less than the average.

Whether it is best, in any particular case, to use a number any greater than the minimum, I leave the reader to determine from the circumstances of the case and a perusal of the tests. Moreover, it should be distinctly understood, that, when the immediate modulus of elasticity is used in computing deflections, the deflections are those that will be assumed by the

beam immediately after the application of the load; while the deflections of the beam after the load has remained on it for a certain length of time will be greater.

MODULUS OF ELASTICITY TO BE USED IN COMPUTING THE DEFLECTIONS OF BEAMS AFTER THE LOAD HAS BEEN ON FOR SOME TIME.

A perusal of the time tests tends to show, that, in computing the final deflection of a beam under a given load, we ought to use a modulus of elasticity no greater than three-fourths (and two-thirds would be safer) of the immediate modulus of elasticity; and it will be noticed that the values of the immediate moduli of elasticity deduced from these tests are a little, though not very much, smaller than those given by Rankine, Trautwine, Hatfield, etc.

4°. *Oak Beams.*

SUMMARY.

No. of Test.	Span.	Width and Depth.	Description.	Breaking-Weight, in lbs.	Modulus of Rupture.	Modulus of Elasticity.
	ft. in.	inches.				
48	19 6	6 × 12	Load at middle	13776	5596	1766839
51	15 6	$4\frac{1}{4}$ × $14\frac{3}{8}$	" " "	19076	6060	1240728
55	13 8	3 × $13\frac{1}{4}$	" " "	10671	4984	853098
80	18 0	4 × 12	" " "	13371	7659	1307180
					4)24299	
	Aver	age modulus	of rupture . . .	. . .	6075	
						4)5169836
	Aver	age modulus	of elasticity . .	. . .	. . . .	1292459

While the average modulus of rupture is 6075, this is evidently too high a value to use in practice. I leave the reader to judge, but I should not feel safe with more than 4000 lbs. per square inch.

5°. *White-Pine Beams.*

No. of Test.	Width and Depth.	Span.	Breaking Centre Load, in lbs.	Modulus of Rupture.	Modulus of Elasticity.	
	inches.	ft. in.				
94	3 × 11½	15 8	5088	3613	924252	
95	3 × 13	14 0	12588	7251	1280832	Pattern stock. Clear piece. Seasoned 3 yrs.
96	3 × 13	16 6	9088	5324	1072889	
97	3 × 11	15 8	6088	4729	978256	
98	2⅞ × 9¾	16 0	6088	6415	1234880	
99	2⅞ × 13	15 6	5988	3438	1020390	
100	3 × 9¾	16 0	4288	4330	1165937	
102	3 × 10¾	15 6	4790	3855	990190	
103	3 × 11	16 6	6588	5390	1242649	
104	3 × 11¼	15 6	5088	3739	930760	
				48084	10842035	
	Average	modulus	rupture .	. 4808		
	Average	modulus	elasticity	. . .	1084200	

These averages become respectively, if 95 be omitted, 1062356 and 4537.

It would seem to the writer, therefore, that the rule already laid down in the case of spruce would apply also to white pine.

LONGITUDINAL SHEARING.

Below are given tables showing the greatest intensity of the shear at the neutral axis of each beam at the time of fracture. I will give tables showing these results: and we must observe, that, in the case of those beams which gave way by shearing, the figures given represent the shearing-strength of the wood along the grain; while in the case of those that did not give way by shearing, it is fair to assume that these numbers are less than the shearing-strength of the wood.

TABLE OF BEAMS THAT GAVE WAY BY LONGITUDINAL SHEARING.

Spruce.					Yellow Pine.				
No.	Width and Depth.	Span.		Intensity of Shear.	No.	Width and Depth.	Span.		Intensity of Shear.
		ft.	in.				ft.	in.	
22	$3\frac{7}{8} \times 12$	14	0	202	30	$3 \times 13\frac{7}{8}$	14	0	273
24	3×12	14	0	190	32	$4\frac{1}{16} \times 12\frac{3}{16}$	18	0	242
31	$3\frac{1}{8} \times 12$	18	0	154	33	$3\frac{15}{16} \times 12\frac{1}{4}$	18	0	153
35	6×12	18	0	117	50	$4 \times 14\frac{1}{16}$	21	0	172
36	$2 \times 11\frac{7}{8}$	7	2	248	92	6×12	6	8	397
46	$3\frac{3}{4} \times 12$	10	2	233					
				6)1144					5)1237
	Average .	. .	. .	191		Average .	. .	. .	248

TABLE OF BEAMS WHICH DID NOT FAIL BY SHEARING.

Spruce.		Yellow Pine.	
No. of Test.	Maximum Intensity of Shear at Fracture.	No. of Test.	Maximum Intensity of Shear at Fracture.
3	181	47	359
4	301	53	179
5	174	54	203
6	230	56	185
7	308	57	182
8	170	59	133
9	141	62	231
10	106	63	211
11	208	64	161
12	126	65	183
14	105	67	196
15	304	68	273
16	277	69	112
17	465	71	223
18	202	74	173

TABLE OF BEAMS WHICH DID NOT FAIL BY SHEARING. — *Concluded.*

Spruce.		Yellow Pine.	
No. of Test.	Maximum Intensity of Shear at Fracture.	No. of Test.	Maximum Intensity of Shear at Fracture.
19	138	75	230
20	160	76	231
21	137	77	549
23	108	78	108
25	123	81	238
26	155	82	343
27	167	84	164
28	134	85	270
29	129	87	172
37	169	88	156
45	133	91	282
49	205		
60	406		
66	228		
70	206		
72	174		
90	272		
Average . .	199	Average . .	221

One would naturally expect to find the intensity of the shearing-stress at fracture less in the case of the beams that did not fail by shearing than in the case of those that did; and this is seen to be generally true (making allowance for different qualities) both in the case of spruce and hard pine.

The notable exceptions seem to be, in the case of spruce, beams Nos. 4, 7, 15, 16, 17, 60, 90, all of which have this intensity very large. If these be omitted from the list, the average for those that did not give way by shearing would be 160 pounds per square inch, which is less than 191, the average for those that did.

In the case of yellow pine, the notable exceptions are beams Nos. 47, 77, and 92; and, if these be omitted, the average for those yellow-pine beams that did not fail by shearing would be 197 pounds, which is less than 248.

Moreover, it is to be observed, that, in the case of the spruce, Nos. 4, 7, 15, 16, and 17 were all of smaller dimensions than those used in practice.

In the face of these apparent exceptions, which I am unable to explain, I prefer not to state at present any definite rule for the guidance of one who wishes to take the shearing-force into account in his calculations, but rather to leave him to use his judgment, in connection with these results, for any particular case.

It will also be observed that these shearing-forces are less than those obtained from the experiments on direct shearing along the grain, made at the Watertown Arsenal; and this is naturally to be expected, for the shearing in their case took place along a section that was perfectly sound, while in these cases it took place at the weakest point.

6°. *Framing-Joints.* — Another matter intimately connected with the strength of timber beams is the strength of the beam after it has been cut in some of the various ways commonly employed in framing. We are often told that a notch cut on top of a beam, or at the middle of its depth, or near the support, does but little injury; but the tests made, show the injury to be very large, amounting to a reduction of the strength of the beam to one-fourth or one-fifth of its original strength, with some of the most approved framing. The fact is, that, with a material where the shearing-strength along the grain is so small as it is in the case of timber, almost any cutting does a great deal of injury; and it is much better to avoid framing whenever it is possible, and use stirrup irons instead. In these tests, only two of the most approved framing-joints have been tested; viz., the joint known as the "tusk-and-tenon," shown

in Fig. 243, and used for framing the tail-beams of a floor into the headers, and the "double tenon and joint bolt," shown in Fig. 244, and used for framing the headers into the trimmers.

Fig. 243.

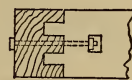


Fig. 244.

The arrangement is shown in plan in Fig. 245, where 1 and 2 are the trimmers, 3 is the header, and 4, 5, and 6 are the tail-beams; the latter being supported at one end on the header, and at the other on the wall, the header being supported by the trimmers, and the trimmers being supported on the walls at both ends.

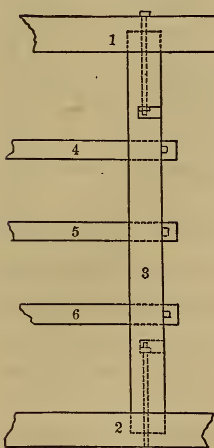


Fig. 245.

It is sometimes the practice to hang the header in stirrup irons, and this is an improvement; but it is very seldom that the tail-beams are hung in stirrup irons, and these tests have shown the weakening already referred to, from the mortises cut in the header to admit the tail-beams.

In our earlier tests, these joints were tested by loading the header at the top, distributing the load over the mortises; and the headers then tested were made for four tail-beams. The results are given below, these headers being all of spruce:—

No. of Test.	Span.	Width and Depth.	Description.	Total Breaking-Weights.
	ft. in.	in.		
2	6 8	4 × 12	Header. Framed at ends. Mortised for four tail-beams. Loads applied above mortises.	10338
38	6 8	6 × 12	Header. Framed at ends. Mortised for four tail-beams. Loads applied above mortises.	10798
39	6 8	6 × 12	Header. Hung in stirrup irons. Mortised for four tail-beams. Loads applied above mortises.	21298
41	6 8	$3\frac{1}{2}$ × 12	Header. Framed at ends. Mortised for four tail-beams. Loads applied above mortises.	10757

It is evident that the breaking-weights found here are greater than those that would actually break the header when used in a floor; for in the latter case the load is not applied at the top of the header, but lower down, and brings about an additional tendency to split the header.

A spruce floor was next built and tested, the following being a partial account of the test: —

No. 52. — Section of a floor between the trimmers. Spruce: three tail-beams, 2 inches by 12 inches each, framed into a $3\frac{3}{4}$-inch by $11\frac{3}{4}$-inch header; header in turn framed into sections of the trimmers by double tenon and joint-bolt, cross-bridged in two places; tail-beams framed by tusk-and-tenon joint, pinned, floored over and furred below; load at centre, distributed between the three tail-beams by bridging.

Span = 16 feet; weight of joist, flooring, etc., = 331 lbs.

11238 lbs. = breaking-load.

Joist on east side broke by splitting off at the tenon, bore 7988 lbs. after. The load was then increased. Centre tail-beam

broke by tension at 9988 lbs., on account of cross-grain in the lower fibres. A split also started at the lower tenon of the header, which at the time of breaking was rapidly increasing.

Average modulus of rupture of the tail-beams, including their own weight, etc., = 3801 lbs. per square inch.

Average modulus of elasticity of tail-beams = 1399141 lbs. per square inch.

It is to be noticed, that the header already began to crack when the tail-beams broke, and hence that the floor could have borne but little more, even if the load had been uniformly distributed: hence that, in this case, the breaking-strength of the floor would be determined by calculating the loads at the centre of the tail-beams, instead of accounting it as distributed; in other words, the breaking-weight would be about one-half what we should get by considering the load as distributed on the tail-beams. Since that time we have had six tests on yellow-pine headers, which will be given here.

It will be seen from these tests, that the first of these headers had for its breaking-weight 10916 lbs., and the second 13163, or in each case one-half the load on the floor. To institute a comparison, we may observe, that, if a 6-inch by 12-inch yellow-pine header 6 feet 8 inches long, with four tail-beams 18 feet long, were to support a floor, the floor surface would be 96 square feet, giving 48 square feet to be supported by the header. This, if the floor were loaded with 100 lbs. per square foot, would bring upon the header 4800 lbs., or about one-half the breaking-weight of a header only 5 feet 4 inches long; whereas, it would commonly be supposed, that, with such a construction for 100 lbs. per square foot of floor, we should have provided an unnecessarily large margin of safety.

As to the fact that the header supported in stirrup irons bore less than that which was framed, this must be due to a difference in the quality of the timber; and it would be unfair to conclude from only two tests that the second was a stronger

mode of construction than the first, even as far as the header itself is concerned.

The fact, also, that a 6-inch by 12-inch yellow-pine beam 5 feet 4 inches long bore 48000 lbs. centre load, equivalent to 96000 distributed, without breaking, while the header broke at 10916, shows what an enormous weakening is caused by cutting mortises, and how much strength would be gained by avoiding all framing, and using stirrup irons to support the tail-beams in all cases where they cannot be supported on top of the header bearing the latter.

No. 83. — Test of yellow-pine headers.

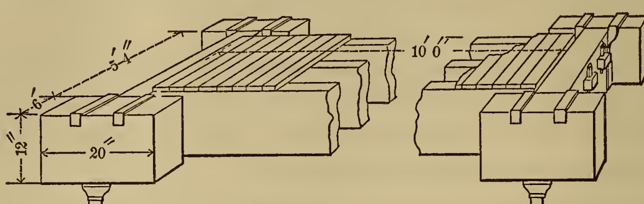


FIG. 246.

The headers, 6 inches by 12 inches, span 5 feet 4 inches, were hung at either end in iron stirrups, from trimmers 6 inches by 12 inches by 20 inches, which in turn were supported on jack-screws. The headers were mortised in three places (16 inches on centres) for three 3-inch by 12-inch yellow-pine tail-beams 10 feet in length.

The load was applied at the centre of the tail-beams, and divided equally among the three by iron bridging.

The tail-beams were cross-bridged in two places, 6 feet apart, by 2-inch by 3-inch spruce bridging, and also floored over with yellow-pine flooring 1 inch thick.

The weight of the tail-beams, bridging, flooring, etc., which was supported by header, was 833 lbs.

Weight of north header, 106 lbs.

Weight of south header, 122 lbs.

The headers were heart pieces, coarse grain, and sappy. The north one had a few season cracks extending from mortise to mortise on the outside. The tail-beams were of medium quality, two of them having sapwood on the edges; the third was much coarser, but contained more pitch, being heavy.

Weight of yoke and iron bridging, 366 lbs.

Details of the Test.

15366 lbs.	Cracks heard in north header.
19866 "	Loud cracks; season cracks opening in north header.
21366 "	North header giving way, load dropped 500 lbs.; cracks $\frac{3}{8}$ inch wide.
26366 "	South header began to show cracks.
26366 "	Centre tail-beam broke off below tenon at south end; load dropped 1500 lbs.

North header virtually broken at 21833 lbs., so each header bore $\frac{21833}{2} = 10916$ lbs.

The tail-beam which did not break was heavy and full of pitch, and of a coarser grain than the other.

No. 86. — Test of yellow-pine headers by means of tail-beams and floor.

The headers, 6 inches by 12 inches, span 5 feet 4 inches, were mortised at each end into the trimmers with a double tenon and joint-bolt. The trimmers were supported on jack-screws, as before. Three tail-beams, 3 inches by 12 inches, span 10 feet, were mortised into the headers with tusk and tenon, and pinned.

The load was applied at the centre of tail-beams, and distributed equally over the three by means of bridging. (See No. 83.) Tail-beams cross-bridged in two places with 2-inch by 3-inch spruce, and floored over with 1-inch yellow-pine flooring.

Weight of tail-beams, bridging, flooring, etc., which was supported by headers = 763 lbs.

No deflections taken.

Weight of yoke and iron bridging, 366 lbs.

Details of the Test.

14366 lbs. North header heard to crack internally.

24366 " Two tail-pieces began to crack under lower tenon (north end).

24866 " A few minutes later one of them broke under tenon (north end). The headers, as far as could be seen, were uninjured.

No. 89. — Test of headers in floor (yellow pine). The headers and trimmers were the ones used in No. 86, and were framed in the same way. Three tail-beams, 3 inches by 12 inches, 6 feet 6 inches span (inside measurement), were framed into headers with a tusk-and-tenon joint, and then pinned.

The experiment was precisely the same as No. 86, with the exception, that, instead of 10-foot tail-beams being used, the length of these was 6 feet 6 inches.

No deflections were taken.

Details of the Test.

* 20150 lbs. Season cracks in north header began to open wider.

* 26325 " North header broke through the middle, following the line of mortises, then held 18825 lbs.

South header cracked but little.

No. 105. — Tests of yellow-pine header in floors.

The headers and tail-beams were framed as in No. 86; and the experiment was exactly the same, with the exception that the tail-beams used were 6 feet long.

Details of the Test.

20762 lbs. Season crack in both headers opened.

23262 " North header failed, after three minutes, through the line of mortises; while south header was but little cracked.

* This weight includes half the weight of bridging, floor, and tail beams (325 lbs).

No. 106. — Test of yellow-pine header.

In this experiment the headers were hung in stirrup irons, exactly as in No. 83.

Details of the Test.

24262 lbs.	Slight cracking.
26662 "	East tail-beam split below line of tenon.
30262 "	Held for five minutes, when west stirrup on north tail-beam broke. The header was virtually broken.

No. 107. — Test of yellow-pine header.

The unbroken headers of Nos. 105 and 106 were used.

Details of the Test.

22262 lbs.	Crack in north header.
25162 "	North header (framed) suddenly failed.

§ 240. **Shearing of Timber along the Grain.** — The shearing of timber always takes place along, and not across, the grain; for it can be shown, that, wherever we have a tendency to shear on a certain plane, there is an equal tendency to shear on a plane at right angles to it. Hence if there is, at any point in a piece of wood, a tendency to shear it across the grain, there must necessarily accompany it an equal tendency to shear it along the grain; and, the resistance to the latter being very slight, the timber will give way in this manner, instead of across the grain.

As to the shearing-strength per square inch, some values have been given in Rankine's table; and the following table contains results obtained at the Watertown Arsenal, and recorded in Executive Document No. 12, 47th Congress, first session.

Kind of Wood.	Arsenal No.	Shearing-Strength per Square Inch.	Kind of Wood.	Arsenal No.	Shearing-Strength per Square Inch.
Ash	620	600	Oak (white) . .	631	752
	621	592	Pine (white) . .	752	324
	622	458		753	267
	623	700		754	352
Birch (yellow) .	623	563		755	366
	633	815	Pine (yellow) . .	607	399
	634	672		608	317
	635	612		614	409
Maple (white) . .	636	647		615	415
	637	537		616	409
	638	367		617	364
	639	431		618	286
Oak (red) . . .	624	775		619	330
	625	743	Spruce	748	253
	626	999		749	374
	627	726		750	347
Oak (white) . .	628	966		751	316
	629	803	Whitewood . .	609	406
	630	846		610	382

§ 241. **General Remarks.** — A perusal of the tests on columns and on beams will show that one of the principal sources of weakness in timber is the presence of knots, and it will be noticed that the position of the fracture is in most cases determined by the knots.

Sap-wood, season cracks, and decay are doubtless other sources of weakness. The tests, however, do not present such striking evidence of the deleterious effects of the first two as is the case with knots. In general, it may be said, however, that timber used in construction should be free, or nearly free, from sap-wood; as an excessive amount of sap-wood renders it weak.

It will often be found to be a common opinion among lumber-dealers, that a piece of timber which contains the heart is not as good as one which is cut from the wood on one side of the heart. This is very often true; as the timber which is sold in the market is very liable to have cracks at the heart, and also, if the tree has passed maturity, the heart is the place where decay is likely to begin. Nevertheless, the tests of beams would not, it seems to the author, bear out the conclusion that such pieces as contain the heart are always weaker than those that do not.

Another matter that claims serious consideration is the effect of seasoning upon the strength of timber. This question can only be decided by tests on full-size pieces, as the small pieces season much more rapidly and uniformly than full-size pieces.

In this regard, the observation should be made, that practically our buildings and other constructions are built with green lumber; i.e., lumber which has been cut from three months to a year. Unless it can be shown that the seasoning which the lumber receives while in use imparts to it a greater strength, it will only be proper to consider its strength the same as that of green lumber. Not very much evidence has thus far been obtained upon this point; but, such as it is, it will be noted here.

1°. We have, on p. 505, the results of the tests of a lot of old-mill columns; and, while some of them did exhibit a greater strength than green ones, a perusal of this set of tests will convince the reader that it would not be safe to rely upon any greater strength in these columns than in green ones. Moreover, these columns had been in a building heated by steam for a number of years, and during the seasoning process they had been subjected to the load they had to support. The writer has also observed some evidence of the same kind in connection with one of his time tests.

2°. In the case of beams, we have, in Nos. 60 and 66, examples of beams which had been seasoning, *unloaded*, in a building heated by steam; and in these cases there was a great gain in strength. Some yellow-pine beams exhibited a similar action. On the other hand, beams Nos. 18 and 19 had been seasoning on the wharf, in the open air, for about one year; and while some yellow-pine beams which had seasoned without load, in the building, showed great strength, in other cases the increase was not so marked.

In view of the fact that the above is practically all the evidence we have in the matter, it would seem to the writer, unless future experiments shall prove the contrary to be true, that we cannot rely, in our constructions, upon having any greater strength than that of the green lumber, and that the figures to be used should be those obtained by testing green lumber.

§ 242. **Strength of Building-Stones.** — Inasmuch as it is not intended to enter into a discussion of the work that has thus far been done in testing the strength of the ordinary building-stones, and as it will be a convenience to the reader to have some figures on which he can depend with reasonable certainty, the following table will be given, taken from Gen. Gillmore's report.

The specimens tested were cubes, 2 inches on a side: their faces were smoothed and polished, so as to obtain an even bearing.

In the following table, B means "on bed," E means "on edge."

Kind.	Locality.	Position.	Strength per Square Inch.	Weight of One Cubic Foot.
Granites:—			lbs.	lbs.
Blue	Staten Island, N.Y.	B.	22250	178.80
. . . .	Fox Island, Me.		14875	164.10
. . . .	Dix Island, Me.		15000	166.50
Dark	Quincy, Mass.		17750	166.20
Light	" "		14750	168.70
. . . .	Tarrytown, West Chester Co., N.Y.,	B.	18250	162.20
Flagging	North River, N.Y.		13425	168 10
Old Quarry . . .	Westerly, Washington County, R.I.,		17750	165.60
" " . . .	" " " "		17250	165.60
. . . .	Millstone Point, Conn.		16187	168.70
. . . .	" " "		18750	168.70
. . . .	Sprucehead, Me.		13500	171.90
. . . .	" "		17500	171.90
. . . .	Hewitt's Island, Me.		14375	164 60
. . . .	" " "		15062	164.60
Up-river	Richmond, Va.		21250	
"	" "		20000	
. . . .	Greenwich, Conn.		11300	177.20
. . . .	" "		11700	177.20
Niantic River . .	New London, Conn.		12500	166.25
" " . .	" " "	E.	14175	166.25
. . . .	Fox Island, Me.	B.	15062	166.30
. . . .	" " "		11700	166.30
. . . .	Vinalhaven, Me.		13150	170.00
. . . .	" "		16750	170.00
Harlem stone . . .	Morrisania, West Chester Co., N.Y.,		15800	170.00
Tombstone . . .	Sharkey's Quarry, Me.		22125	170.00
" . . .	" " "		20875	170.00
. . . .	Richmond, Va.		16063	170.50
. . . .	Cape Ann, Mass.	B.	12423	
. . . .	" " "	B.	19500	
Porter's rock . . .	Mystic River, Conn.	B.	18125	164.40
" " . . .	" " "	E.	22250	164.40
Gray	Westerly, R.I.	B.	14687	166.90
"	" "	E	14937	166.90
"	Richmond, Va.	B.	14100	164.40
"	" "	B.	13875	164.40
"	New Haven, Conn.	E.	7750	162.50
"	" " "	B.	9500	162.50
"	Stony Creek, Conn.	B.	15000	165.40
"	" " "	E.	16750	165.40
"	" " "	B.	15750	165.40
"	Fall River, Mass.	B.	15937	165.00
"	" " "	E.	9250	165.00

Kind.	Locality.	Position.	Strength per Square Inch.	Weight of One Cubic Foot.
Granites: —			lbs.	lbs.
Rose	Niantic, Conn.	B.	9550	162.50
Gray	" "	B.	9450	161.20
Gneiss	Sachem's-head Quarry, Conn. . .	E.	15937	163.70
"	" " " " . .	B.	14000	163.70
Gray	Hurricane Island, Me.	B.	14425	166.90
"	" " "	E.	14937	166.90
"	" " "	B.	11000	166.90
Dark, soft-looking .	Madison Avenue, N.Y.	B.	11250	182 50
" " " .	" " "	E.	12500	182.50
Gray	Port Deposit, Md..	B.	19750	170.00
"	" " "	E.	13100	170.00
"	" " "	B.	16500	170.00
"	" " "	B.	15000	170.00
"	Garrison's, North River, N.Y. . .	E.	12250	161.30
"	" " " " . . .	B.	13370	161.30
"	Palmer Quarry, Me.	B.	11500	161.90
"	" " "	E.	16500	161.90
"	Rockport, Mass.	B.	16300	163.20
"	" "	E.	19750	163.20
Limestones: —				
Glen's Falls . . .	Glen's Falls, Warren County, N.Y.,	B.	11475	168.80
" " . . .	" " " " "	E.	10750	168.80
Lake	Lake Champlain, N.Y.	B.	25000	171.90
"	" " "	E.	21500	171.90
"	Canajoharie, Montgomery Co., N.Y.,	B.	20700	169.80
"	" " " "	E.	19250	169.80
North River . . .	Kingston, N.Y.	B.	13900	168.20
" " . . .	" "	E.	11050	168.20
" " . . .	Garrison's Station, N.Y.	B.	18500	164.70
" " . . .	" " "	E.	18275	164.70
" " . . .	" " "	B.	17750	163.70
" " . . .	" " "	B.	17750	163.50
" " . . .	" " "	B.	18775	163.50
White	Marblehead, O.	B.	11250	150.00
"	" "	B.	10700	150.00
"	" "	B.	12600	155.00
Sandstones: —				
Brown	Little Falls, N.Y.	B.	9850	140.60
"	" " "	E.	9150	140.60
Gray	Belleville, N.J.	B.	11700	141.00
"	" "	E.	10250	141.00
Brown	Middletown, Conn.	B.	6950	148.50

Kind.	Locality.	Position.	Strength per Square Inch.	Weight of One Cubic Foot.
Sandstones: —			lbs.	lbs.
Brown	Middletown, Conn.	E.	5550	148.50
Red	Haverstraw, N.Y.	B.	4350	133.10
"	" "	E.	4025	133.10
Pink	Medina, N.Y.	B.	17250	150.60
"	" "	E.	14812	149.30
Drab	" "	B.	17725	151.10
"	Berea, O.	B.	10250	131.90
"	" "	B.	8300	133.10
"	" "	B.	7250	137.50

§ 243. **Cement Mortar.** — The reader who wishes to investigate this subject is referred, for the beginning of his research, to the treatise of Gen. Q. A. Gillmore on "Limes, Hydraulic Cements, and Mortars."

He made a large number of tests (and of these, only the following abridgment of one of his tables will be given) showing the resistance which Croton bricks cemented together crosswise in pairs, face to face, offer to a force of traction applied at right angles to the surfaces of contact.

Kind of Cement.	Composition of Mortar, by Volume.	Ultimate Tensile Strength per Sq. In.
Delafield & Baxter	Stiff paste of pure cement . .	1097
" " "	" " " " " . .	1101
" " "	" " " " " . .	less than 420
" " "	" " " " " . .	843
Lawrence Cement Company .	" " " " " . .	898
" " " .	" " " " " . .	1258
" " " .	" " " " " . .	1242

Kind of Cement.	Composition of Mortar, by Volume.	Ultimate Tensile Strength per Sq. In.
Lawrence Cement Company .	Stiff paste of pure cement . .	1284
" " " .	" " " " " . .	1398
Kingston & Rosendale . . .	" " " " " . .	1227
" " " . . .	" " " " " . .	969
" " " . . .	" " " " " . .	836
" " " . . .	" " " " " . .	1284
" " " . . .	" " " " " . .	1055
Hancock, Maryland	" " " " " . .	648
" "	" " " " " . .	777
" "	" " " " " . .	1023
" "	" " " " " . .	617
Newark & Rosendale	" " " " " . .	1213
James River	" " " " " . .	859
Delafield & Baxter	Cement in powder, 4; sand, 1 .	1023
" " "	" " " " .	1420
" " "	" " " " .	763
" " "	" " " " .	1023
" " "	" 2 " " .	1030
" " "	" " " " .	1113
" " "	" " " " .	1070
" " "	" " " " .	843
" " "	" 1 " " .	420
" " "	" " " " .	732
" " "	" " " " .	812
" " "	" " " " .	523
" " "	" " " 2 .	367
" " "	" " " " .	260
" " "	" " " " .	491
James River	" 8 " 1 .	978
" "	" " " " .	812
" "	" 4 " " .	992
" "	" " " " .	300
" "	" 1 " " .	740
" "	" " " 2 .	392

CHAPTER VIII.

CONTINUOUS GIRDERS.

§ 244. **Fundamental Principles.** — A continuous girder is one that is continuous over one or more supports; i.e., one that has at least one support in addition to those at the ends. The principle of continuity is, that the neutral line is throughout a continuous curve over the supports, the tangent to one branch of the curve at the support being a prolongation of the tangent to the other branch.

Whereas, in the girder supported at the ends, the bending-moment at the support is zero, in the continuous girder there is a bending-moment at the support, where the girder is continuous. There is also a shearing-force at each side of the support, the sum of the shearing-forces on the two sides of any one support forming the supporting-force.

In this chapter will be given the general methods of determining the bending-moments, slopes, and deflections of continuous girders.

1°. When the loads are distributed.

2°. When the loads are all concentrated.

3°. When there are both distributed and concentrated loads.

It is believed that the reader will thus have the means of solving all cases of continuous girders, and that, whenever it is desirable to have a set of simplified formulæ for a small but

definite number of spans, or for some special proportions or distribution of the load, he will be able to deduce such simplified formulæ from the more general ones.

§ 245. **Distributed Loads.** — In this case we assume that all the loads are distributed, whether they are uniformly distributed or not. The first step to be taken is, to find the bending-moment over each support: this is done by using what is known as the "*three-moment equation*," which we shall now proceed to deduce; and, in the course of the reasoning by which we deduce it, we shall derive a number of useful equations, expressing bending-moment, shearing-force, slope, deflection, etc., at various points.

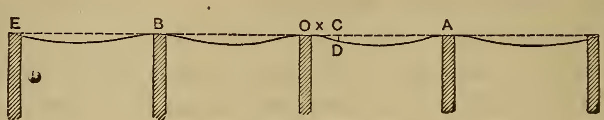


FIG. 247.

For the purpose in view, let us assume our origin at O (Fig. 247), and let

M_1 = bending-moment at B.
M_2 = bending-moment at O.
M_3 = bending-moment at A.
l_1 = OA.
l_{-1} = OB.
F_0 = shearing-force just to the right of O.
F_{-0} = shearing-force just to the left of O.
F_1 = shearing-force at distance x to the right of origin.
F_{-1} = shearing-force at distance x to the left of origin.

Shear is taken as positive when the tendency is to slide the part remote from the origin upwards.

If S_0 = supporting-force at O,

$$S_0 = F_0 + F_{-0}.$$

Beginning, now, by taking O as origin, and x positive to the right, —

Let $OC = x$.

$CD = v =$ deflection at distance x from origin.

$w \;=$ load per unit of length (either constant, or variable with x).

We shall then have, from the principles of the common theory of beams,

$$F_1 = F_0 - \int_0^x w dx; \qquad (1)$$

i.e., the shearing-force at a distance x to the right of O is found by subtracting from the shearing-force just to the right of O the sum of the loads between the section at x and the support; and this sum is

$$\int_0^x w dx.$$

In a similar manner, if we were to take origin at O, and x positive to the left, we should have

$$F_{-1} = F_{-0} - \int_0^x w dx. \qquad (2)$$

In § 204 we found the equation

$$\frac{dM}{dx} = F_1,$$

$$\therefore \quad \frac{dM}{dx} = F_0 - \int_0^x w dx.$$

Hence, integrating between $x = 0$ and $x = x$, and observing, that, when $x = 0$, $M = M_2$, we have

$$M - M_2 = F_0 x - \int_0^x \int_0^x w dx^2,$$

which reduces to

$$M = M_2 + F_0 x - \int_0^x \int_0^x w dx^2 ; \qquad (3)$$

or, in words, —

The bending-moment at a distance x to the right of O is equal to the bending-moment over the support at the origin, plus the product of the shearing-force just to the right of the origin by the distance of the section from the origin, minus the sum of the moments of the loads between the section and the support about the section.

Observe that this sum of the moments of the loads between the section and the support about the section has, for its mathematical equivalent, the expression

$$\int_0^x \int_0^x w dx^2 ;$$

and, as a particular instance, it may be noted, that when the load is *uniformly* distributed, and hence w is constant, this will reduce to

$$\frac{wx^2}{2} = (wx) \frac{x}{2},$$

wx being the load between the section and the support, and $\frac{x}{2}$ being the leverage of its resultant.

Now write, for brevity,

$$\int_0^x \int_0^x w dx^2 = m ;$$

then

$$M = M_2 + F_0 x - m. \qquad (4)$$

Now, from § 194, we have

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}.$$

Let α_1 $=$ slope at distance x to the right of the origin.
α_{-1} $=$ slope at distance x to the left of the origin.
α_0 $=$ value of α_1 when $x = 0$.
α_{-0} $=$ value of α_{-1} when $x = 0$.

Then

$$\tan \alpha_1 = \frac{dv}{dx} = \int_0^x \frac{M}{EI} dx + c,$$

where c is an arbitrary constant, to be determined from the conditions of the problem.

If, now, we substitute for M its value $M_2 + F_0 x - m$, we shall have

$$\tan \alpha_1 = \frac{dv}{dx} = M_2 \int_0^x \frac{dx}{EI} + F_0 \int_0^x \frac{xdx}{EI} - \int_0^x \frac{mdx}{EI} + c.$$

To determine c, observe, that, when $x = 0$, $\alpha_1 = \alpha_0$;

$$\therefore \quad c = \tan \alpha_0$$

$$\therefore \quad \tan \alpha_1 = \frac{dv}{dx} = \tan \alpha_0 + M_2 \int_0^x \frac{dx}{EI}$$

$$+ F_0 \int_0^x \frac{xdx}{EI} - \int_0^x \frac{mdx}{EI}. \quad (5)$$

Integrate again, and observe, that, when $x = 0$, $v = 0$, and we obtain

$$v = x \tan \alpha_0 + M_2 \int_0^x \int_0^x \frac{dx^2}{EI}$$

$$+ F_0 \int_0^x \int_0^x \frac{xdx^2}{EI} - \int_0^x \int_0^x \frac{mdx^2}{EI}. \quad (6)$$

Now write, for the sake of brevity,

$$m = \int_0^x \int_0^x w dx^2, \quad m_1 = \int_0^{l_1} \int_0^x w dx^2, \quad m_{-1} = \int_0^{l-1} \int_0^x w dx^2,$$

$$n = \int_0^x \int_0^x \frac{dx^2}{EI}, \quad n_1 = \int_0^{l_1} \int_0^x \frac{dx^2}{EI}, \quad n_{-1} = \int_0^{l-1} \int_0^x \frac{dx^2}{EI},$$

$$q = \int_0^x \int_0^x \frac{x dx^2}{EI}, \quad q_1 = \int_0^{l_1} \int_0^x \frac{x dx^2}{EI}, \quad q_{-1} = \int_0^{l-1} \int_0^x \frac{x dx^2}{EI},$$

$$V = \int_0^x \int_0^x \frac{m dx^2}{EI}, \quad V_1 = \int_0^{l_1} \int_0^x \frac{m dx^2}{EI}, \quad V_{-1} = \int_0^{l-1} \int_0^x \frac{m dx^2}{EI},$$

the last four being derived by taking x positive to the left. We shall have

$$v = x \tan \alpha_0 + M_2 n + F_0 q - V; \tag{7}$$

and, if $v_1 =$ deflection at $A =$ vertical height of A above O, we shall have, by substituting l_1 for x in (7),

$$v_1 = l_1 \tan \alpha_0 + M_2 n_1 + F_0 q_1 - V_1.$$

Now, if we assume any horizontal datum line entirely below all the points of support, and let the height of B above this line be y_b, that of A, y_a, and that of O, y_o, etc., we shall have

$$y_a - y_o = l_1 \tan \alpha_0 + M_2 n_1 + F_0 q_1 - V_1. \tag{8}$$

And, if we put $x = l_1$ in (4), we shall have

$$M_3 = M_2 + F_0 l_1 - m_1$$

$$\therefore \quad F_0 = \frac{M_3 - M_2 + m_1}{l_1}: \tag{9}$$

and, if we substitute this value of F_0 in (8), we obtain, by reducing,

$$y_a - y_o = l_1 \tan \alpha_0 + M_2\left(n_1 - \frac{q_1}{l_1}\right) + \frac{M_3 q_1}{l_1} + \frac{m_1 q_1}{l_1} - V_1;$$

and, solving for $\tan \alpha_0$, we obtain

$$\tan \alpha_0 = \frac{y_a - y_0}{l_1} + M_2\left(\frac{q_1}{l_1^2} - \frac{n_1}{l_1}\right) - M_3\frac{q_1}{l_1^2} - \frac{m_1 q_1}{l_1^2} + \frac{V_1}{l_1}. \quad (10)$$

This expression gives us the tangent of the slope at O in span OA; and equation (9) gives us the shearing-force just to the right of O in span OA, in terms of M_2, M_3, and known quantities.

If we were to take the origin at O, as before, and x positive to the left instead of the right, we should have, in place of (4),

$$M = M_2 + F_{-0}x - m; \quad (11)$$

in place of (9),

$$F_{-0} = \frac{M_1 - M_2 + m_{-1}}{l_{-1}}; \quad (12)$$

and in place of (10),

$$\tan \alpha_{-0} = \frac{y_b - y_0}{l_{-1}} + M_2\left(\frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right) - M_1\frac{q_{-1}}{l_{-1}^2} - \frac{m_{-1}q_{-1}}{l_{-1}^2} + \frac{V_{-1}}{l_{-1}}. \quad (13)$$

But, since the girder is continuous, we must have the tangent at O to the left-hand part, a prolongation of the tangent at O to the right-hand part, as shown in Fig. 248.

Hence we must have

$$\alpha_{-0} = -\alpha_0$$

$$\therefore \quad \tan \alpha_{-0} + \tan \alpha_0 = 0.$$

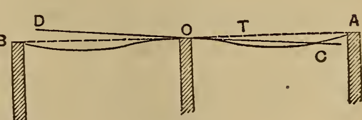


FIG. 248.

Hence, adding (10) and (13), we have

$$\frac{y_a - y_0}{l_1} + \frac{y_b - y_0}{l_{-1}} + M_2\left\{\frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right\} - M_3\frac{q_1}{l_1^2} - M_1\frac{q_{-1}}{l_{-1}^2} - \frac{m_1 q_1}{l_1^2} - \frac{m_{-1}q_{-1}}{l_{-1}^2} + \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} = 0; \quad (14)$$

and this is the "*three-moment equation*" for the case of a distributed load, whether it be *uniformly* distributed or otherwise.

CASE WHEN SUPPORTS ARE ON THE SAME LEVEL.

When the supports are all on the same level, then $y_a = y_b = y_o$, and the three-moment equation becomes

$$M_2\left\{\frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right\} - \frac{M_3 q_1}{l_1^2} - \frac{M_1 q_{-1}}{l_{-1}^2}$$
$$- \frac{m_1 q_1}{l_1^2} - \frac{m_{-1} q_{-1}}{l_{-1}^2} + \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} = 0. \quad (15)$$

MANNER OF USING THE THREE-MOMENT EQUATION.

When the dimensions and load of the girder are known, all the quantities in the three-moment equation, whether we use (14) or (15), are known, except the three bending-moments, M_1, M_2, and M_3.

Suppose, now, the girder to have any number of (say, seven) points of support; then, by taking the origin at B (Fig. 247), we obtain one equation between the bending-moments at E, B, and O, the first of which, if E is an end support, is zero. Next take the origin at O, and we obtain one equation between the three bending-moments at B, O, and A; and so, continuing, we obtain five equations between five unknown quantities.

Solving these, we obtain the bending-moments over the supports; and from these bending-moments, after they are found, we can obtain the shearing-forces, bending-moments, slopes, and deflections, by using the equations deduced in the course of the reasoning for the three-moment equation, as equations (4), (5), (7), (9), and (10).

SPECIAL CASE,

when, the supports being all on the same level, the load on any one span is uniformly distributed over that span, and when the girder is of uniform section throughout.

Let w_1 = load per unit of length on span OA, origin at O.
w_{-1} = load per unit of length on span OB, origin at O.
I = the constant moment of inertia of the section $y_a = y_b = y_o$.

Then

$$m_1 = \frac{w_1 l_1^2}{2}, \qquad m_{-1} = \frac{w_{-1} l_{-1}^2}{2};$$

$$n_1 = \frac{l_1^2}{2EI}, \qquad n_{-1} = \frac{l_{-1}^2}{2EI};$$

$$q_1 = \frac{l_1^3}{6EI}, \qquad q_{-1} = \frac{l_{-1}^3}{6EI};$$

$$V_1 = \frac{w_1 l_1^4}{24EI}, \qquad V_{-1} = \frac{w_{-1} l_{-1}^4}{24EI}.$$

With these substitutions, the three-moment equation, either (14) or (15), becomes

$$M_1 l_{-1} + 2M_2(l_{-1} + l_1) + M_3 l_1 + \tfrac{1}{4}(w_1 l_1^3 + w_{-1} l_{-1}^3) = 0. \quad (16).$$

This is a simpler form of the three-moment equation, applicable to this particular case only.

EXAMPLE I. — Suppose we have a continuous girder of uniform section, uniformly loaded, and of three equal spans, to find M_B and M_C, also the supporting-forces, shearing-forces, bending-moments, slopes, and deflections throughout.

FIG. 249.

Solution. — Take the origin at B, and we have

$$M_1 = 0, \qquad M_2 = M_3 = M_B = M_C;$$

since

$$l_1 = l_{-1} = l,$$

equation (16) gives

$$5M_B l + \tfrac{1}{2} w l^3 = 0 \qquad \therefore \quad M_B = M_C = -\frac{wl^2}{10}.$$

Next, to find the shearing-forces, we have, from (9),

$$F_B = F_{-C} = \frac{-\frac{wl^2}{10} + \frac{wl^2}{10} + \frac{wl^2}{2}}{l} = \frac{wl}{2},$$

equals shearing-force just to the right of B or left of C.

Shearing-force just to the right of C or left of B =

$$\frac{+\frac{wl^2}{10} + \frac{wl^2}{2}}{l} = \tfrac{3}{5}wl.$$

Hence supporting-forces are

$$S_B = S_C = (\tfrac{3}{5} + \tfrac{1}{2})wl = \tfrac{11}{10}wl,$$

$$S_A = S_D = \tfrac{1}{2}(3wl - \tfrac{22}{10}wl) = \tfrac{2}{5}wl.$$

Bending-moment in span AB at distance x from A, or in span CD at distance x from D,

$$M = \tfrac{2}{5}wlx - \frac{wx^2}{2}.$$

Bending-moment in middle span at a distance x from B or from C,

$$M = -\frac{wl^2}{10} + \frac{wlx}{2} - \frac{wx^2}{2}.$$

Shearing-force in span AB or CD at a distance x from A or D,

$$F = \tfrac{2}{5}wl - wx.$$

Shearing-force in middle span at distance x from B or C,

$$F = \frac{wl}{2} - wx.$$

Maximum bending-moment in span BC (when $x = \frac{l}{2}$),

$$M_0 = -\frac{wl^2}{10} + \frac{wl^2}{4} - \frac{wl^2}{8} = \frac{wl^2}{40}.$$

Maximum bending-moment in span AB or CD,

$$x = \tfrac{2}{5}l,$$

$$M_0 = \frac{4}{25}wl^2 - \frac{4}{50}wl^2 = \frac{2wl^2}{25}.$$

Hence the greatest bending-moment to which the girder is subjected is that at B or C, and its amount is $\dfrac{wl^2}{10}$.

Slope at B in middle span, from equation (10),

$$\tan \alpha_B = -\frac{wl^2}{10}\left(-\frac{l}{3EI}\right) + \frac{wl^2}{10}\left(\frac{l}{6EI}\right) - \frac{wl^3}{12EI} + \frac{wl^3}{24EI}$$

$$= \frac{wl^3}{EI}\left(\frac{1}{30} + \frac{1}{60} - \frac{1}{24}\right) = \frac{wl^3}{120EI},$$

which denotes an upward slope at B towards the right. In the same way, the girder slopes upwards at C towards the left. The slopes at B and C in the end spans are, of course, downwards.

Slope in the middle span at a distance x from B,

$$\tan \alpha = \frac{dv}{dx} = \frac{1}{EI}\left\{-\frac{wl^2}{10}x + \frac{wlx^2}{4} - \frac{wx^3}{6}\right\} + c.$$

When $x = 0$,

$$\tan \alpha = +\frac{wl^3}{120EI} \qquad \therefore \quad c = \frac{wl^3}{120EI}$$

$$\therefore \quad \tan \alpha = \frac{w}{EI}\left\{\frac{l^3}{120} - \frac{l^2x}{10} + \frac{lx^2}{4} - \frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}(l^3 - 12l^2x + 30lx^2 - 20x^3)$$

$$\therefore \quad \text{Deflection} = v = \frac{w}{120EI}(l^3x - 6l^2x^2 + 10lx^3 - 5x^4).$$

In order to make plain all methods of proceeding, the slope in the end spans will be found in two different ways, as follows: —

For bending-moment, slope, and deflection in left-hand span at a distance x from B (or in the right-hand span at distance x from C), we have

$$M = -\frac{wl^2}{10} + \frac{3}{5}wlx - \frac{wx^2}{2}.$$

$$\tan\alpha = \frac{dv}{dx} = \frac{1}{EI}\left\{-\frac{wl^2x}{10} + \frac{3wlx^2}{10} - \frac{wx^3}{6}\right\} + c.$$

When $x = 0$,

$$\tan\alpha = -\frac{wl^3}{120EI} \qquad \therefore \quad c = -\frac{wl^3}{120EI}$$

$$\therefore \quad \tan\alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{-\frac{l^3}{120} - \frac{l^2x}{10} + \frac{3}{10}lx^2 - \frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}(-l^3 - 12l^2x + 36lx^2 - 20x^3)$$

$$\therefore \quad \text{Deflection} = v = \frac{w}{120EI}(-l^3x - 6l^2x^2 + 12lx^3 - 5x^4).$$

We may, on the other hand, accomplish the same object by finding the slope and deflection in left-hand span at distance x from A, or in right-hand span at distance x from D, as follows: —

$$\tan\alpha = \frac{dv}{dx} = \frac{1}{EI}\int\left\{\frac{2}{5}wlx - \frac{wx^2}{2}\right\}dx = \frac{w}{EI}\left\{\frac{lx^2}{5} - \frac{x^3}{6}\right\} + c.$$

When $x = l$,

$$\tan\alpha = \frac{wl^3}{120EI}$$

$$\therefore \quad \frac{wl^3}{120EI} = \frac{wl^3}{30EI} + c \qquad \therefore \quad c = -\frac{wl^3}{40EI}$$

$$\therefore \quad \tan\alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{-\frac{l^3}{40} + \frac{lx^2}{5} - \frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}\left\{-3l^3 + 24lx^2 - 20x^3\right\}$$

$$\therefore \quad v = \frac{w}{120EI}\left\{-3l^3x + 8lx^3 - 5x^4\right\}.$$

The figure shows the mode of bending of the girder.

Fig. 250.

To find the greatest deflection in either span, put the expression for the slope equal to zero, and find x by the ordinary methods for solving an equation of the third degree, and then substitute this value in the expression for the deflection.

Example II. — Continuous girder of two equal spans, section uniform, and load uniformly distributed.

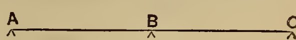


Fig. 251.

Solution. — Take origin at B.

$$M_1 = M_3 = M_A = M_C = 0, \quad M_2 = M_B, \quad l_1 = l_2 = l, \quad w_1 = w_2 = w;$$

therefore, from equation (16),

$$4M_Bl + \tfrac{1}{2}wl^3 = 0 \qquad \therefore \quad M_B = -\frac{wl^2}{8}.$$

Shearing-force either side of $B =$

$$F_B = F_{-B} = \frac{\dfrac{wl^2}{8} + \dfrac{wl^2}{2}}{l} = \tfrac{5}{8}wl.$$

Supporting-force at $B = \frac{5}{4}wl$.
Supporting-force at A and $C = \frac{3}{8}wl$.
Shear at distance x from A or C,

$$F = \tfrac{3}{8}wl - wx.$$

Bending-moment at distance x from A or C,

$$M = \frac{3}{8}wlx - \frac{wx^2}{2}.$$

Maximum bending-moment occurs when $x = \frac{3}{8}l$,

$$M_0 = \frac{9}{64}wl^2 - \frac{9}{128}wl^2 = \frac{9wl^2}{128}.$$

Hence greatest bending-moment to which the girder is subjected is that at B, and its magnitude is $\frac{wl^2}{8}$.

Slope at B, from equation (6),

$$\tan \alpha_B = \tan \alpha_{-B} = -\frac{wl^2}{8}\left(-\frac{l}{3EI}\right) - \frac{wl^3}{12EI} + \frac{wl^3}{24EI}$$

$$= \frac{wl^3}{EI}\left\{\frac{1}{24} - \frac{1}{12} + \frac{1}{24}\right\} = 0,$$

as was to be expected.

Slope at distance x from A in span AB,

$$\tan \alpha = \frac{1}{EI}\left(\frac{3}{16}wlx^2 - \frac{wx^3}{6}\right) + c.$$

When $x = l$, $\alpha = 0$;

$$\therefore \quad C = -\frac{wl^3}{48EI}$$

$$\therefore \quad \tan \alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{\frac{3}{16}lx^2 - \frac{x^3}{6} - \frac{l^3}{48}\right\}$$

$$= \frac{w}{48EI}(9lx^2 - 8x^3 - l^3).$$

Deflection,

$$v = \frac{w}{48EI}(-l^3x + 3lx^3 - 2x^4).$$

For maximum deflection, we have

$$\frac{3}{16}lx^2 - \frac{x^3}{6} - \frac{l^3}{48} = 0$$

$$\therefore \quad x = 0.44l.$$

$$\text{Maximum deflection} = \frac{wl^4}{48EI}\left\{-1 + 3(0.44)^2 - 2(0.44)^3\right\}(0.44)$$

$$= -0.0054\frac{wl^4}{EI}.$$

EXAMPLE III. — In order to solve a case where no simplifications enter, on account of symmetry or otherwise, we will take a continuous girder of five spans (as shown in the figure), the spans varying in length from $3l$ to $7l$; the loads being uniformly distributed, and varying in intensity from $3w$ on the longest span to $7w$ on the shortest; the beam being of uniform section.

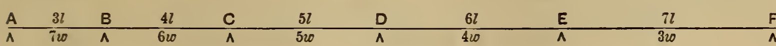


FIG. 252.

For this case we can use equation (9).

Origin at B,

$$0 + 14lM_B + 4lM_C + \tfrac{1}{4}[7w(27l^3) + 6w(64l^3)] = 0,$$

or

$$56M_B + 16M_C = -573wl^2.$$

Origin at C,

$$4lM_B + 18lM_C + 5lM_D + \tfrac{1}{4}wl^3[6(64) + 5(125)] = 0,$$

or

$$16M_B + 72M_C + 20M_D = -1009wl^2.$$

Origin at D,

$$5lM_C + 22lM_D + 6lM_E + \tfrac{1}{4}[5(125) + 4(216)]wl^3 = 0,$$

or

$$20M_C + 88M_D + 24M_E = -1489wl^2.$$

Origin at E,

$$6lM_D + 26lM_E + \frac{wl^3}{4}[4(216) + 3(343)] = 0,$$

or

$$24M_D + 104M_E = -1893wl^2.$$

The four equations are:

$$56M_B + 16M_C = \qquad - 573wl^2. \qquad (1)$$
$$16M_B + 72M_C + 20M_D = \qquad -1009wl^2. \qquad (2)$$
$$20M_C + 88M_D + 24M_E = -1489wl^2. \qquad (3)$$
$$24M_D + 104M_E = -1893wl^2. \qquad (4)$$

Eliminate M_E between (3) and (4), and we obtain

$$130M_C + 536M_D = -6839wl^2. \qquad (5)$$

Eliminate M_D between (2) and (5), and we obtain

$$2144M_B + 8998M_C = -101011wl^2. \qquad (6)$$

Eliminate M_C between (1) and (6), and we obtain

$$234792M_B = -1769839wl^2. \qquad (7)$$

$$\therefore \qquad M_B = - \ 7.5379wl^2,$$
$$\therefore \text{ from } (1), M_C = - \ 9.4299wl^2,$$
$$\text{from } (5), M_D = -10.4722wl^2,$$
$$\text{from } (4), M_E = -15.7853wl^2.$$

Shearing-force just to the right of

$$A = \frac{-0 - 7.3579 + 31.5}{3}wl \qquad = 7.9874wl,$$

$$B = \frac{-9.4299 + 7.5379 + 48}{4}wl \qquad = 11.5270wl,$$

$$C = \frac{-10.4722 + 9.4299 + 62.5}{5}wl = 12.2915wl,$$

$$D = \frac{-15.7853 + 10.4722 + 72}{6}wl = 11.1145wl,$$

$$E = \frac{15.7853 + 73.5}{7}wl \qquad = 12.7550wl.$$

Shearing-force to the left of

$$B = \frac{7.5379 + 31.5}{3}wl = 13.0126wl,$$

$$C = \frac{-7.5379 + 9.4299 + 48}{4}wl = 12.4730wl,$$

$$D = \frac{-9.4299 + 10.4722 + 62.5}{5}wl = 12.7084wl,$$

$$E = \frac{-10.4722 + 15.7853 + 72}{6}wl = 12.8855wl,$$

$$F = \frac{-15.7853 + 735}{7}wl = 8.2450wl.$$

Supporting-force at

$A = 7.9874wl$, $C = 24.7645wl$, $E = 25.6405wl$,
$B = 24.5396wl$, $D = 23.8229wl$, $F = 8.2450wl$.

Shearing-force at distance x to the right of

$$
\begin{aligned}
A \text{ in section } AB &= 7.9874wl - 7wx,\\
B \text{ in section } BC &= 11.5270wl - 6wx,\\
C \text{ in section } CD &= 12.2915wl - 5wx,\\
D \text{ in section } DE &= 11.1145wl - 4wx,\\
E \text{ in section } EF &= 12.7550wl - 3wx.
\end{aligned}
$$

Bending-moment at distance x from

$$
\begin{aligned}
A \text{ in section } AB &= + 7.9874wlx - \frac{7wx^2}{2},\\
B \text{ in section } BC &= - 7.5379wl^2 + 11.5270wlx - \frac{6wx^2}{2},\\
C \text{ in section } CD &= - 9.4299wl^2 + 12.2915wlx - \frac{5wx^2}{2},\\
D \text{ in section } DE &= -10.4722wl^2 + 11.1145wlx - \frac{4wx^2}{2},\\
E \text{ in section } EF &= -15.7853wl^2 + 12.7550wlx - \frac{3wx^2}{2}.
\end{aligned}
$$

For the sections of maximum bending-moments (put shearing-force = 0), —

$$\begin{aligned}
&\text{In } AB,\ x = 1.1410l;\\
&\text{In } BC,\ x = 1.9211l;\\
&\text{In } CD,\ x = 2.4583l;\\
&\text{In } DE,\ x = 2.7786l;\\
&\text{In } EF,\ x = 4.2517l.
\end{aligned}$$

Hence the maximum bending-moments are respectively, in —

Section AB,

$$wx\left(-\frac{7x}{2} + 7.9874l\right) = \qquad +4.5570wl^2.$$

Section BC,

$$-\ 7.5379wl^2 + wx(11.5270l - 3x) = 3.5347wl^2.$$

Section CD,

$$-\ 9.4299wl^2 + wx(12.2915l - \tfrac{5}{2}x) = 5.6781wl^2.$$

Section DE,

$$-10.4722wl^2 + wx(11.1145l - 2x) = 4.9693wl^2.$$

Section EF,

$$-15.7853wl^2 + wx(12.7550 - \tfrac{3}{2}x) = 11.3297wl^2.$$

Values of $\tan \alpha_0$ = slope in every case in the span, towards the right.

$$\tan \alpha_0 = M_2\left(\frac{q_1}{l_1^2} - \frac{n_1}{l_1}\right) - \frac{M_3 q_1}{l_1^2} - \frac{m_1 q_1}{l_1^2} + \frac{V_1}{l_1}.$$

Slope at B,

$$\tan \alpha_B = -7.5379\frac{wl^3}{EI}\left(\frac{2}{3} - 2\right) + 9.4299\left(\frac{2}{3}\right)\frac{wl^3}{EI} - \frac{32wl^3}{EI} + \frac{16wl^3}{EI}$$

$$= 0.3371\frac{wl^3}{EI}.$$

Slope at C,

$$\tan\alpha_C = -9.4299\frac{wl^3}{EI}\left(\frac{5}{6}-\frac{5}{2}\right) + 10.4722\left(\frac{5}{6}\right)\frac{wl^3}{EI} - \frac{625}{12}\frac{wl^3}{EI} + \frac{625}{24}\frac{wl^3}{EI} = -1.5983\frac{wl^3}{EI}.$$

Slope at D,

$$\tan\alpha_D = -10.4722\frac{wl^3}{EI}(1-3) + 15.7853\frac{wl^3}{EI} - \frac{72wl^3}{EI} + \frac{36wl^3}{EI} = 0.7297\frac{wl^3}{EI}.$$

Slope at E,

$$\tan\alpha_E = -15.7853\frac{wl^3}{EI}\left(\frac{7}{6}-\frac{7}{2}\right) - \frac{1029}{12}\frac{wl^3}{EI} + \frac{343}{8}\frac{wl^3}{EI} = -6.0426\frac{wl^3}{EI}.$$

The manner of bending, very much exaggerated, is shown in the accompanying figure.

FIG. 253.

Slope at $A = -4.096\frac{wl^3}{EI}$, slope at $F = +23.4578\frac{wl^3}{EI}$.

For the deduction, see what follows.

Slopes in General.

Span AB, origin at A,

$$\tan\alpha = \frac{w}{EI}\left\{3.9937lx^2 - \frac{7}{6}x^3\right\} + c.$$

When $x = 3l$, $\tan\alpha = 0.3371\frac{wl^3}{EI}$;

$$\therefore \quad \frac{wl^3}{EI}(35.9433 - 31.5) + c = 0.3371\frac{wl^3}{EI}$$

$$\therefore \quad c = -4.106\frac{wl^3}{EI}$$

$$\therefore \quad \tan\alpha = \frac{w}{EI}\left\{3.9937lx^2 - \frac{7}{6}x^3 - 4.106l^3\right\}.$$

Span BC, origin at B,

$$\tan\alpha = 0.3371\frac{wl^3}{EI} - 7.5379\frac{wl^2x}{EI} + 5.7635\frac{wlx^2}{EI} - \frac{wx^3}{EI}.$$

Span CD, origin at C,

$$\tan\alpha = \frac{w}{EI}\left\{-1.5983l^3 - 9.4299l^2x + 6.1458lx^2 - \frac{5}{6}x^3\right\}.$$

Span DE, origin at D,

$$\tan\alpha = \frac{w}{EI}\left\{0.7297l^3 - 10.4722l^2x + 5.55725lx^2 - \frac{2}{3}x^3\right\}.$$

Span EF, origin at E,

$$\tan\alpha = \frac{w}{EI}\left\{-6.0426l^3 - 15.7853l^2x + 6.3775lx^2 - \frac{wx^3}{2}\right\}.$$

When $x = 7l$,

$$\tan\alpha = \frac{wl^3}{EI}(-6.0426 - 110.4971 + 312.4975 - 172.5)$$

$$= +23.4578\frac{wl^3}{EI}.$$

Deflections.

Span AB,

$$v = \frac{w}{EI}\left\{1.3312lx^3 - \frac{7}{24}x^4 - 4.106l^3x\right\}.$$

Span BC,

$$v = \frac{w}{EI}\left\{0.3371l^3x - 3.7689l^2x^2 + 1.9211lx^3 - \frac{x^4}{4}\right\}.$$

Span CD,

$$v = \frac{w}{EI}\left\{-1.5983l^3x - 4.7149l^2x^2 + 2.0486lx^3 - \frac{5}{24}x^4\right\}.$$

Span DE,

$$v = \frac{w}{EI}\left\{0.7297l^3x - 5.2361l^2x^2 + 1.8524 + lx^3 - \frac{x^4}{6}\right\}.$$

Span EF,

$$v = \frac{w}{EI}\left\{-6.0426l^3x - 7.8927l^2x^2 + 2.1258lx^3 - \frac{x^4}{8}\right\}.$$

The maximum deflections can be obtained by putting the slopes equal to zero, as before.

§ 246. **Continuous Girder with Concentrated Loads.**—For our next general case, we will take that where there are no distributed loads, but where all the loads are concentrated at single points, and the section uniform throughout; and we will begin by assuming only one concentrated load on each span.

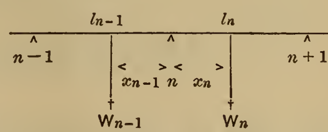


FIG. 254.

Let the support marked $n - 1$ be the $(n - 1)^{\text{th}}$ support, and the length of the $(n - 1)^{\text{th}}$ span be l_{n-1}; let the load on this span be W_{n-1}, and likewise for the other spans. Assume the origin at n, and let

F_n = shearing-force just to the right of n.
F_{-n} = shearing-force just to the left of n.
F_1 = shearing-force at distance x to the right of n.
F_{-1} = shearing-force at distance x to the left of n.

Shear is taken as positive when the tendency is to slide the part remote from the origin upwards.

If S_n = supporting-force at n,

$$S_n = F_n + F_{-n}. \qquad (1)$$

Let, also, x_n = distance from origin to point of application of load W_n, and let x_{n-1} = distance from origin to point of application of load W_{n-1}.

Take x positive to the right. Then, for

$$\left.\begin{array}{ll} x < x_n, & F_1 = F_n; \\ x > x_n, & F_1 = F_n - W_n. \end{array}\right\} \qquad (2)$$

Moreover, we have

$$\frac{dM}{dx} = F_1;$$

hence, by integration, for

$$x < x_n, \int_{M_n}^{M} dM = \int_0^x F_n dx;$$

$$x > x_n, \int_{M_n}^{M} dM = \int_0^x F_n dx - \int_0^x W_n dx + c;$$

the value of c being determined from the condition, that, when $x = x_n$ the two results must be identical. Hence we have, for

$$\left.\begin{array}{ll} x < x_n, & M = M_n + F_n x; \\ x > x_n, & M = M_n + F_n x - W_n(x - x_n). \end{array}\right\} \quad (3)$$

Make $x = l_n$ in the last equation, and we have

$$M_{n+1} = M_n + F_n l_n - W_n(l_n - x_n). \quad (4)$$

Now let $l_n - x_n = a_n$, and (4) becomes

$$M_{n+1} = M_n + F_n l_n - W_n a_n; \quad (5)$$

hence

$$F_n = \frac{M_{n+1} - M_n + W_n a_n}{l_n}. \quad (6)$$

Moreover, we have, as before,

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad \therefore \quad EI\frac{d^2v}{dx^2} = M,$$

I being a constant.

Let, as before, —

α_1 = slope at distance x to the right of origin.
α_{-1} = slope at distance x to the left of origin.
α_n = value of α_1 when $x = 0$.
α_{-n} = value of α_{-1} when $x = 0$.

Then by integration, determining the constant in the same way as in (3), we have, for

$$\left.\begin{aligned} x < x_n, \quad & EI(\tan\alpha_1 - \tan\alpha_n) = M_n x + F_n\frac{x^2}{2}; \\ x > x_n, \quad & EI(\tan\alpha_1 - \tan\alpha_n) = M_n x + F_n\frac{x^2}{2} - \frac{W_n(x - x_n)^2}{2}. \end{aligned}\right\} \quad (7)$$

Hence

$$x < x_n, \quad EI\frac{dv}{dx} = EI\tan\alpha_n + M_n x + F_n\frac{x^2}{2};$$

$$x > x_n, \quad EI\frac{dv}{dx} = EI\tan\alpha_n + M_n x + F_n\frac{x^2}{2} - \frac{W_n(x - x_n)^2}{2}.$$

Integrate again, and determine constants in the same way, and for

$$\left.\begin{aligned} x < x_n, \quad & EIv = EIx\tan\alpha_n + M_n\frac{x^2}{2} + F_n\frac{x^3}{6}; \\ x > x_n, \quad & EIv = EIx\tan\alpha_n + M_n\frac{x^2}{2} + F_n\frac{x^3}{6} - \frac{W_n(x - x_n)^3}{6}. \end{aligned}\right\} \quad (8)$$

Make $x = l_n$ in the last equation, and denote the heights of the supports above the datum line in the same way as in § 245, and we have

$$EI(y_{n+1} - y_n) = EIl_n\tan\alpha_n + M_n\frac{l_n^2}{2} + F_n\frac{l_n^3}{6} - \frac{W_n(l_n - x_n)^3}{6}. \quad (9)$$

Substitute for $l_n - x_n$, a_n, and for F_n, its value from (6), and we have

$$EI(y_{n+1} - y_n) = EIl_n\tan\alpha_n + M_n\frac{l_n^2}{3} + M_{n+1}\frac{l_n^2}{6} + \frac{W_n a_n}{6}(l_n^2 - a_n^2). \quad (10)$$

Hence

$$EI\tan\alpha_n + \frac{l_n}{6}(2M_n + M_{n+1}) + \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) = EI\frac{(y_{n+1} - y_n)}{l_n}. \quad (11)$$

Now, if we take origin at n and x positive to the left, we should obtain, instead of (11),

$$EI \tan \alpha_{-n} + \frac{l_{n-1}}{6}(2M_n + M_{n-1})$$
$$+ \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2) = EI\left(\frac{y_{n-1} - y_n}{l_{n-1}}\right). \quad (12)$$

Now add (11) and (12), and observe, that, since the girder is continuous,

$$\tan \alpha_n + \tan \alpha_{-n} = 0,$$

and we obtain

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + \frac{M_{n+1}l_n}{6}$$
$$+ \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) + \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2)$$
$$= EI\left\{\frac{y_{n+1} - y_n}{l_n} + \frac{y_{n-1} - y_n}{l_{n-1}}\right\}; \quad (13)$$

and this is the "three-moment equation" for the case of a single concentrated load on each span, and a uniform section.

When the supports are all on the same level, this becomes

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + \frac{M_{n+1}l_n}{6} + \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2)$$
$$+ \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2) = 0. \quad (14)$$

Either of these equations can be used (when it is applicable) just as the three-moment equation was used in the case of distributed loads.

CASE OF MORE THAN ONE LOAD ON EACH SPAN.

When there is more than one load on each span, the three-moment equation becomes as follows: —

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + M_{n+1}\frac{l_n}{6}$$
$$+ \Sigma\frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) + \Sigma\frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2)$$
$$= EI\left\{\frac{y_{n+1} - y_n}{l_n} + \frac{y_{n-1} - y_n}{l_{n-1}}\right\}. \quad (15)$$

In using these equations for concentrated loads, we can determine the moments over the supports; but we must observe, that, in getting slopes and deflections, bending-moments, etc., the algebraic expressions that represent them are different on the two sides of any one load, and hence we must deduce new values whenever we pass a load, determining the constants for our integration to correspond.

Example. — Given a continuous girder of three spans, the middle span = 20 feet, each end span = 15 feet; supports on same level. The only loads on the girder are two; viz., a load of 5000 lbs. at 5 feet from the left-hand end, and one of 4000 lbs. 5 feet from the right-hand end. The supports are lettered from left to right, A, B, C, D, respectively. Find the greatest bending-moment and greatest deflection.

Solution. — Origin at B,

$$\frac{M_B}{3}(20 + 15) + \frac{M_C}{6}(20) + \frac{5000 \times 5}{6 \times 15}(225 - 25) = 0. \quad (1)$$

Origin at C,

$$\frac{M_C}{3}(20 + 15) + \frac{M_B}{6}(20) + \frac{4000 \times 5}{6 \times 15}(225 - 25) = 0. \quad (2)$$

These reduce to

$$\left.\begin{aligned} 70M_B + 20M_C + \frac{1000000}{3} &= 0 \\ 20M_B + 70M_C + \frac{800000}{3} &= 0 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} M_B &= -4000 \text{ foot-lbs.} \\ M_C &= -2667 \text{ foot-lbs.} \end{aligned}$$

Shearing-forces.		Supporting-forces.	Slopes at supports.
$F_A = 3067,$	$F_{-C} = -67.$	$S_A = 3067.$	$\tan \alpha_A = -\dfrac{59444}{EI}.$
$F_{-B} = 1934,$	$F_C = 1511.$	$S_B = 2000.$	$\tan \alpha_B = +\dfrac{35557}{EI}.$
$F_B = 67,$	$F_{-D} = 2489.$	$S_C = 1444.$	$\tan \alpha_C = -\dfrac{31111}{EI}.$
		$S_D = 2489.$	$\tan \alpha_{-D} = -\dfrac{48888}{EI}.$

Span AB, origin at A,

$$x < 5, \quad M = 3067x.$$
$$x > 5, \quad M = 3067x - 5000(x - 5) = 25000 - 1933x.$$

Maximum bending-moment occurs when $x = 5$ and therefore $M_0 = 15333$.

$$x < 5, \quad EI \tan \alpha_1 = -59444 + 1533x^2;$$
$$x > 5, \quad EI \tan \alpha_1 = 25000x - 967x^2 + c.$$

Determine c by condition, that, when $x = 5$, these two become equal;

$$\therefore \quad c = -121944;$$
$$\therefore \quad x > 5, \quad EI \tan \alpha_1 = -121944 + 25000x - 967x^2.$$

For deflections,

$$x < 5, \quad EIv = -50444x + 511x^3;$$
$$x > 5, \quad EIv = -121944x + 12500x^2 - 322x^3 + c.$$

Determine c from condition, that, when $x = 15$, $v = 0$;

$$\therefore \quad c = 103410;$$
$$\therefore \quad x > 5, \quad EIv = 103410 - 121944x + 12500x^2 - 322x^3.$$

For maximum deflection, equate slope to zero, and find x.

$$\text{We find it at } x = 6.53.$$

$$\therefore \quad EIv_0 = -249531.$$

Span BC, origin at B,

$$M = -4000 + 67x,$$

$$EI \tan \alpha_1 = 35557 - 4000x + 33x^2,$$

$$EIv = 35557x - 2000x^2 + 11x^3.$$

For maximum deflection, equate slope to zero, and find x.

$$\text{We find it at } x = 9.78.$$

$$\therefore \quad EIv_0 = 166740.$$

Span CD, origin at C,

$x < 10, M = -2667 + 1511x;$

$x > 10, M = -2667 + 1511x - 4000(x - 10) = 37333 - 2489x;$

$x < 10, EI \tan \alpha_1 = -31111 - 2667x + 756x^2;$

$x > 10, EI \tan \alpha_1 = -131055 + 37333x - 1245x^2.$

For deflections,

$x < 10, EIv = -31111x - 1334x^2 + 252x^3;$

$x > 10, EIv = -131055x + 18667x^2 - 415x^3 + c.$

When $x = 15, v = 0$;

$$\therefore \quad x > 10, \quad EIv = -833625 - 131055x + 18667x^2 - 415x^3.$$

For maximum deflection, equate slope to zero, and find x.

$$\text{We find it at } x = 8.41.$$

$$\therefore \quad EIv_0 = -24506.$$

Hence greatest bending-moment and greatest deflection are both in span AB.

Observe, that, since we have used one foot as our unit of measure, all dimensions must be taken in feet, and the value of E is also 144 times that ordinarily given.

§ 247. **Continuous Girder, with both Distributed and Concentrated Loads.** — In this case we may either calculate the bending-moments, slopes, and deflections due to each separately, and then add the results with their proper signs, or we may modify the solution that was used for the case of a distributed load, so as to extend its applicability to this case.

Let W represent any one concentrated load, and let x_1 represent the distance of its point of application from the origin. Then, in the general formulæ deduced for the distributed load, make the following changes; viz., —

1°. Instead of

$$m = \int_0^x \int_0^x w dx^2,$$

put

$$m = \int_0^x \int_0^x w dx^2 + \Sigma_0^x W(x - x_1),$$

since, as was shown, m represents the sum of the moments of the loads, between the section and the support, about the section.

2°. Instead of

$$V = \int_0^x \int_0^x \left\{ \int_0^x \int_0^x w dx^2 \right\} \frac{dx^2}{EI},$$

put

$$V = \int_0^x \int_0^x \left\{ \int_0^x \int_0^x w dx^2 \right\} \frac{dx^2}{EI} + \Sigma W \int_{x_1}^x \int_{x_1}^x \frac{(x - x_1) dx^2}{EI},$$

and make the corresponding changes in the values of m_1, m_{-1}, V_1, and V_{-1}, leaving n and q just as before; then use the same three-moment equation as before, with these substitutions, i.e.,

$$M_2 \left\{ \frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}} \right\} - M_3 \frac{q_1}{l_1^2} - M_1 \frac{q_{-1}}{l_{-1}^2} - \frac{m_1 q_1}{l_1^2} - \frac{m_{-1} q_{-1}}{l_{-1}^2}$$
$$+ \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} + \frac{y_a - y_0}{l_1} + \frac{y_b - y_0}{l_{-1}} = 0.$$

SPECIAL CASE,

when the distributed load is uniformly distributed on each span, but may be different on the different spans, and when the girder is of uniform section.

Let w_1 = weight per unit of length on OA.

w_{-1} = weight per unit of length on OB.

Denote by W_1 any concentrated load on OA at distance x_1 from O.

Denote by W_{-1} any concentrated load on OB at distance x_{-1} from O.

Then we shall have

$$m_1 = \frac{w_1 l_1^2}{2} + \Sigma W_1(l_1 - x_1),$$

$$m_{-1} = \frac{w_{-1} l_{-1}^2}{2} + \Sigma W_{-1}(l_{-1} - x_{-1}),$$

$$V_1 = \frac{w_1 l_1^4}{24EI} + \Sigma \frac{W_1(l_1 - x_1)^3}{6EI},$$

$$V_{-1} = \frac{w_{-1} l_{-1}^4}{24EI} + \Sigma \frac{W_{-1}(l_{-1} - x_{-1})^3}{6EI},$$

and, as before,

$$n_1 = \frac{l_1^2}{2EI}, \qquad n_{-1} = \frac{l_{-1}^2}{2EI},$$

$$q_1 = \frac{l_1^3}{6EI}, \qquad q_{-1} = \frac{l_{-1}^3}{6EI}.$$

Making these substitutions in the three-moment equation, and clearing fractions, we obtain for the case, when the supports are all on the same level,

$$\begin{aligned} 0 = M_1 l_{-1} + 2M_2(l_1 + l_{-1}) + M_3 l_1 + \tfrac{1}{4}(w_1 l_1^3 + w_{-1} l_{-1}^3) \\ + \frac{1}{l_1}\Sigma\{W_1[l_1^2 - (l_1 - x_1)^2](l_1 - x_1)\} \\ + \frac{1}{l_{-1}}\Sigma\{W_{-1}[l_{-1}^2 - (l_{-1} - x_{-1})^2](l_{-1} - x_{-1})\}. \end{aligned}$$

CONCENTRATED AND DISTRIBUTED LOADS.

EXAMPLE. — Let the girder be of uniform section, of two equal spans, each being 10 feet; let the concentrated loads be as shown in the figure, the distributed load being 96 lbs. per foot. Find the

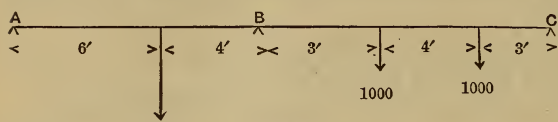


FIG. 255.

value of EI, so that the deflection may nowhere exceed $\frac{1}{400}$ of the span.

Solution. — Use equation (12); and, in deducing value of M_B, use dimensions in feet; afterwards use inches.

Origin at B, $\quad M_A = M_C = 0;$

$$40M_B + 2\frac{96}{4}(1000 + 1000) + \frac{1}{10}\{2000(64)(6) + 1000(51)(7) + 1000(91)(3)\} = 0,$$

$$40M_B + 48000 + 139800 = 0, \quad \text{or} \quad M_B = -4695 \text{ foot-lbs.},$$

or

$$M_B = -56340 \text{ inch-lbs.}, \quad M_A = M_C = 0.$$

$$m_1 = 177600, \qquad m_{-1} = 201600,$$

$$n_1 = \frac{7200}{EI}, \quad \frac{n_1}{l_1} = \frac{60}{EI}. \qquad n_{-1} = \frac{7200}{EI}, \quad \frac{n_{-1}}{l_{-1}} = \frac{60}{EI}.$$

$$q_1 = \frac{288000}{EI}, \quad \frac{q_1}{l_1^2} = \frac{20}{EI}. \qquad q_{-1} = \frac{288000}{EI}, \quad \frac{q_{-1}}{l_{-1}^2} = \frac{20}{EI}.$$

$$V_1 = \frac{175680000}{EI}, \quad \frac{V_1}{l_1} = \frac{1464000}{EI}. \qquad V_{-1} = \frac{193536000}{EI}, \quad \frac{V_{-1}}{l_{-1}} = \frac{1612800}{EI}.$$

$$\text{Shear right side of middle} = \frac{0 + 56340 + 177600}{120} = 1949.5,$$

$$\text{Shear left side of middle} = \frac{0 + 56340 + 201600}{120} = 2149.5;$$

$$\text{Shear left end} = \frac{-56340 + 153600}{120} = 810.5,$$

$$\text{Shear right end} = \frac{-56340 + 177600}{120} = 1010.5;$$

$$\text{Middle supporting-force} = 4099.$$

Bending-Moments in Each Span.

Span AB, origin at A,

$$810.5x - 4x^2$$

or

$$810.5x - 4x^2 - 2000(x - 72).$$

Span BC, origin at B,

$$-56340 + 1949.5x - 4x^2,$$
$$-56340 + 1949.5x - 4x^2 - 1000(x - 36),$$
$$-56340 + 1949.5x - 4x^2 - 1000(x - 36) - 1000(x - 84).$$

To ascertain position of the greatest bending-moment, differentiate each one.

$$\begin{array}{ll} 810.5 - 8x = 0, & x = 101.31; \\ 810.5 - 8x - 2000 = 0, & x = \text{a minus quantity}; \\ 1949.5 - 8x = 0, & x = 243.69; \\ 1949.5 - 8x - 1000 = 0, & x = 118.69; \\ 1949.5 - 8x - 1000 - 1000 = 0, & x = \text{a minus quantity}. \end{array}$$

Hence, in span AB, maximum bending is at the load, and its amount is

$$(810.5)(72) - 4(72)(72) = 37620.$$

Span BC, maximum is at right-hand load, and is

$$-56340 + 1949.5(84) - 4(84)(84) - 1000(48) = 31194.$$

SLOPES.

Slope at B,

$$T = \frac{-56340}{EI}(20 - 60) - \frac{(177600)(20)}{EI} + \frac{1464000}{EI} = \frac{165600}{EI}.$$

Slope and Deflection in Span AB.

First part,

$$\tan \alpha = \frac{1}{EI}\left\{405.25x^2 - \frac{4}{3}x^3\right\} + \tan \alpha_0,$$

α_0 being slope at A.

Second part,

$$\tan\alpha = \frac{1}{EI}\left\{405.25x^2 - \frac{4}{3}x^3 - 1000x^2 + 144000x\right\} + c.$$

When $x = 72$, α is the same in both cases;

$$\therefore \quad \frac{1}{EI}\left\{1000(72)^2 - (144000)(72)\right\} + \tan\alpha_0 - c = 0$$

$$\therefore \quad c = \tan\alpha_0 - \frac{5184000}{EI}.$$

When $x = 120$, the second value of $\tan\alpha$ becomes $\dfrac{165600}{EI}$;

$$\therefore \quad \frac{1}{EI}\left\{(405.25)(120)^2 - \frac{4}{3}(120)^3 - (1000)(120)^2 + 144000(120)\right\}$$
$$+ \tan\alpha_0 - \frac{5184000}{EI} = \frac{165600}{EI},$$

$$\tan\alpha_0 = \frac{1}{EI}\left\{-6411600 + 5184000 + 165600\right\} = -\frac{1062000}{EI},$$

$$\therefore \quad c = -\frac{6246000}{EI}.$$

Hence slope in first part (between A and the load),

$$\tan\alpha = \frac{1}{EI}\left\{-1062000 + 405.25x^2 - \frac{4}{3}x^3\right\}.$$

Second part (between B and the load),

$$\tan\alpha = \frac{1}{EI}\left\{-6246000 + 144000x - 594.75x^2 - \frac{4}{3}x^3\right\}.$$

Deflection.

First part,

$$v = \frac{1}{EI}\left\{-1062000x + 135.08x^3 - \frac{x^4}{3}\right\}.$$

Second part,

$$v = \frac{1}{EI}\left\{-6246000x + 72000x^2 - 198.25x^3 - \frac{1}{3}x^4\right\} + c.$$

When $x = 120$, $v = 0$;

$\therefore \quad c =$

$$\frac{1}{EI}\left\{(6246000)(120) - (72000)(120)^2 + (198.25)(120)^3 + \frac{(120)^4}{3}\right\}$$

$$= \frac{120}{EI}(1036800) = \frac{124416000}{EI}.$$

Point of greatest deflection is found by putting slope equal zero. Moreover, it is plain that the greatest deflection is in the first, and not the second, part.

Hence equation is

$$\tfrac{4}{3}x^3 - 405.25x^2 + 1062000 = 0$$

$$\therefore \quad x = 56''.77;$$

and, substituting this in the expression for the deflection, we obtain

$$v_0 = -\frac{39037720}{EI}.$$

Hence, putting $\dfrac{120}{400} = \dfrac{39037720}{EI}$, we obtain

$$EI = 130125733.$$

If $E = 1400000$, $I = 92.9$; therefore, if $b = 3$ inches, $h = 7$ inches.

Slope and Deflection in Span BC.

Portion nearest B,

$$\tan\alpha = \frac{1}{EI}\left\{165600 - 56340x + 974.8x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 36$ inches, we obtain

$$\tan\alpha = \frac{1}{EI}(165600 - 2028240 + 1263341 - 62208) = -\frac{661507}{EI}.$$

Middle portion,

$$\tan\alpha = \frac{1}{EI}\left\{-20340x + 474.75x^2 - \frac{4}{3}x^3\right\} + c.$$

When $x = 36$ inches, then $\tan\alpha = \dfrac{661507}{EI}$;

$$\therefore \quad -661507 = -732240 + 615276 - 62208 + EIc$$

$$\therefore \quad c = -\frac{482335}{EI}$$

$$\therefore \quad \tan\alpha = \frac{1}{EI}\left\{-482335 - 20340x + 474.75x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 84$ inches,

$$\tan\alpha = \frac{1}{EI}(-482335 - 1708560 + 3349836 - 790272)$$

$$= +\frac{368669}{EI}.$$

Portion nearest C,

$$i = \frac{1}{EI}\left\{63660x - 25x^2 - \frac{4}{3}x^3\right\} + c.$$

When $x = 84$ inches, then $\tan\alpha = \dfrac{368669}{EI}$;

$$\therefore \quad 368669 = 5347440 - 176400 - 790272 + EIc$$

$$\therefore \quad c = -\frac{4012099}{EI}$$

$$\therefore \quad \tan\alpha = \frac{1}{EI}\left\{-4012099 + 63660x - 25x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 120$ inches,

$$\tan\alpha = \frac{1}{EI}(-4012099 + 7639200 - 360000 - 2304000) = \frac{963101}{EI}.$$

Deflection.

Portion nearest B,

$$v = \frac{1}{EI}\left\{165600x - 28170x^2 + 324.9x^3 - \frac{1}{3}x^4\right\}.$$

When $x = 36$ inches,

$$v = \frac{1}{EI}(165600 - 1014120 + 421070 - 15552)(36)$$

$$= -\frac{(443002)36}{EI} = -\frac{15948072}{EI}.$$

Middle portion,

$$v = \frac{1}{EI}\left\{-482335x - 10170x^2 + 158.25x^3 - \frac{1}{3}x^4\right\} + c.$$

When $x = 36$ inches, then $v = -\dfrac{15948072}{EI}$;

$$\therefore \quad -15948072 = (-482335 - 366120 + 205092 - 15552) + EIc,$$

$$\therefore \quad c = -\frac{15289157}{EI}.$$

$$\therefore \; v = \frac{1}{EI}\left\{-15289157 - 482335x - 10170x^2 + 158.25x^3 - \frac{1}{3}x^4\right\}.$$

Greatest deflection occurs in the middle portion, and the point is given by the equation.

$$0 = -482335 - 20340x + 474.75x^2 - \tfrac{4}{3}x^3 = 0;$$

$$\therefore \quad x = 71.4.$$

Greatest deflection in span BC,

Fig. 256.

$$v = \frac{1}{EI}(-15289157 - 344387190 - 51846253 + 57602105 - 8662899)$$
$$= -\frac{52634923}{EI}.$$

Hence, putting $\dfrac{120}{400} = \dfrac{52634923}{EI}$, we obtain

$$EI = 175449743;$$

therefore, if $E = 1400000$, we have

$$I = 125.3.$$

If $b = 3$ inches, $h = 8$ inches.

EXAMPLES OF CONTINUOUS GIRDERS.

1°. Let I = uniform moment of inertia of girder.

w = load per unit of length uniformly distributed.

Find expressions for

1, the bending-moment over each support,

2, the supporting-forces,

3, the greatest bending-moment,

4, the slopes at the supports,

5, the greatest deflection,

in each of the following cases: —

(*a*) Two equal spans, length l.

(*b*) Three equal spans, length l.

(*c*) Four equal spans, length l.

(*d*) Two spans, lengths l_1 and l_2 respectively.

(*e*) Three spans, lengths l_1, l_2, and l_3 respectively.

(*f*) Four spans, lengths l_1, l_2, l_3, and l_4 respectively.

(*g*) Two equal spans; loads per unit of length on each span, w_1 and w_2 respectively.

(*h*) Three equal spans; loads per unit of length on each span, w_1, w_2, and w_3 respectively.

2°. Do the same in the case where each span is loaded with a centre load W, and has no distributed load.

3°. Find greatest bending-moment and greatest deflection for a continuous girder of two spans, uniformly loaded on these two spans with load w per unit of length, and which overhang the outer supports; the overhanging parts having lengths l_{-0} and l_0 respectively, and the same distributed load per unit of length on the overhanging parts.

CHAPTER IX.

EQUILIBRIUM CURVES.—ARCHES AND DOMES.

§ 248. **Loaded Chain or Cord.**—It has been already shown (§ 126), when the form of a polygonal frame is given, that the loads must be adapted, in direction and magnitude, to that form, or else the frame will not be stable. The same is true of a loaded chain or cord, which would be realized if the frame were inverted.

If a set of loads be applied which are not consistent with the equilibrium of the frame under that form, it will change its shape until it assumes a form which is in equilibrium under the applied loads.

As to the manner of finding (when a sufficient number of conditions are given) the stresses in the different members, etc., this was sufficiently explained under the head of "Frames," and will not be repeated here, as the figures speak for themselves.

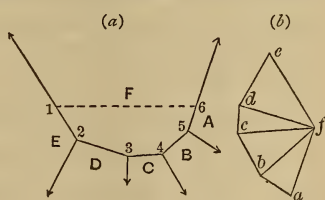


Fig. 257.

In Fig. 257 the polygon *fedcbaf* is the force polygon, while the equilibrium polygon is 123456, an open polygon. A straight line joining *e* and *a* would represent the resultant of the loads.

CHAIN WITH VERTICAL LOADS.

If all the loads are vertical, the broken line *edcba* becomes a straight line and vertical, as shown in Fig. 258*b*. Whenever the loads are concentrated at single points, as 2, 3, 4, 5, the form of the chain is polygonal; and when the load is distributed, it becomes a curve, as shown in Fig. 259.

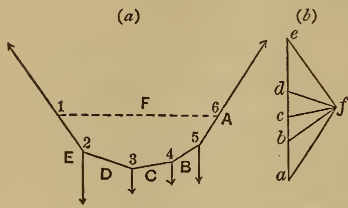


FIG. 258.

CURVED CHAIN WITH A VERTICAL DISTRIBUTED LOAD.

Given the form of the chain *AOE* supported at *A* and *E*, and the total load upon it (*bc*, Fig. 259*b*), to find the distribution of the load graphically. First lay off *bc* to scale, to represent the total load: this is balanced by the two supporting-forces at *A* and *E* respectively, as shown in the figure. Hence draw *ca* parallel to the tangent at *E*, and *ba* parallel to that at *A*, and we have the force polygon *abca;* the equilibrium curve being the chain *AOE* itself. Moreover, if the lowest point of the chain be *O*, then the load must be so distributed that the portion between *O* and *A* shall be balanced by the tension at *O* and that at *A*, and hence that its resultant shall pass through the intersection of the tangents at *O* and *A*. Its amount will be found by drawing from *a* a horizontal line; and then we shall have *ao* as the tension at *o*, *ab* as the tension at *A*, and *bo* as the load between *A* and *O*. Hence the load between *E* and *O* will be *oc*.

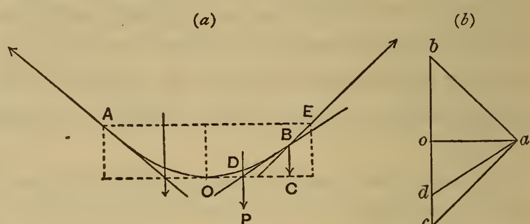


FIG. 259.

Moreover, the load between O and any point, as B, will be balanced by the tension at O, and the tension at B, and hence will be od, where ad is drawn parallel to the tangent BD, so that the load between b and c will be dc; and in this way we see that we can find the tension at any point of the chain by simply drawing a line from a, parallel to the tangent at that point, till it meets the load-line bc.

It is to be observed, that, if the tension at any point of the chain be resolved into horizontal and vertical components, the horizontal component will, when the loads are all vertical, be a constant, and the vertical component will be equal to the portion of the load between the lowest point and the point in question.

If we assume our origin at O, axis of x horizontal and axis of y vertical, and let the co-ordinates of B be x and y, and if w be the intensity of the load at the point (x, y), we shall have, for the load od between O and B,

$$P = \int_0^x w dx;$$

and, since the angle oad = angle BDC, we shall have

$$\frac{dy}{dx} = \frac{BC}{DC} = \frac{od}{oa} = \frac{P}{H}.$$

By differentiation, we shall have

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dP}{dx}\right)}{H}$$

or

$$\frac{d^2y}{dx^2} = \frac{w}{H}, \tag{1}$$

and this is the equation for all vertically loaded cords.

From it we can find the form of the cord to suit a given distribution of the load.

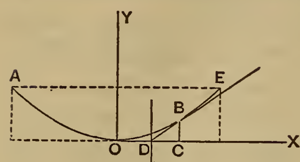


FIG. 260.

§ 249. **Chain with the Load Uniformly Distributed Horizontally.** — In this case w is a constant; and if we assume our origin at the lowest point of the chain, and use the same notation as before, we shall have

$$\frac{d^2y}{dx^2} = \frac{w}{H}.$$

Hence, integrating, and observing, that, when $x = 0, \frac{dy}{dx} = 0$, we have

$$\frac{dy}{dx} = \frac{wx}{H};$$

and by another integration, observing, that, when $x = o, y = o$, we obtain

$$y = \frac{w}{2H}x^2.$$

This is the equation of a parabola; hence a chain so loaded assumes a parabolic form.

EXAMPLE I. — Given the heights of the piers for supporting a chain so loaded, above the lowest point of the chain, as 8 and 18 feet respectively, the span being 100 feet, to find the distance of the lowest point from the foot of each pier, and the equation of the curve assumed by the chain.

Solution. — If (with the lowest point of the chain as origin) we call (x_1, y_1) the co-ordinates of the top of the first pier, and (x_2, y_2) those of the top of the second pier, we shall have, since $y_1 = 18$ and $y_2 = 8$, and since we must have

$$y = \frac{w}{2H}x^2,$$

$$18 = \frac{w}{2H}x_1^2 \qquad \text{and} \qquad 8 = \frac{w}{2H}x_2^2$$

$$\therefore \quad \frac{x_1}{x_2} = \sqrt{\frac{18}{8}} = \frac{3}{2} \qquad \therefore \quad x_1 = \frac{3}{2}x_2 \qquad \therefore \quad x_1 + x_2 = \frac{5}{2}x_2;$$

but

$$x_1 + x_2 = 100 \qquad \therefore \quad \tfrac{5}{2}x_2 = 100 \qquad \therefore \quad x_2 = 40, \qquad x_1 = 60.$$

Hence, since $18 = \dfrac{w}{2H}(60)^2$

$$\therefore \quad \frac{w}{2H} = \frac{18}{3600} = \frac{1}{200},$$

therefore equation of the curve is

$$y = \tfrac{1}{200}x^2.$$

EXAMPLE II. — Given the load on the above chain as 4000 lbs. per foot of horizontal length, to find the tension at the lowest point, also that at each end.

Solution.

$$\frac{w}{2H} = \frac{1}{200}, \qquad\qquad w = 4000,$$

$$\therefore \quad 2H = 800000 \qquad\qquad \therefore \quad H = 400000 \text{ lbs.}$$

Moreover, load between lowest point and highest pier $=$ $60 \times 4000 = 240000$ lbs.

Therefore tension at highest pier $=$

$$\sqrt{(240000)^2 + (400000)^2} = 10000\sqrt{(24)^2 + (40)^2}$$
$$= 10000\sqrt{2176} = 466480 \text{ lbs.}$$

Tension at lowest pier $=$

$$\sqrt{(160000)^2 + (400000)^2} = 10000\sqrt{256 + 1600}$$
$$= 10000\sqrt{1856} = 437919 \text{ lbs.}$$

Example III. — Given the span of the chain as 20 feet, and its length as 25 feet, the two points of support being on the same level, to find the position of the lowest point.

§ 250. **Catenary.** — The catenary is the form of the curve of a chain, which, being of uniform section, is loaded with its own weight only, i.e., with a load uniformly distributed along the length of the chain.

To deduce the equation of the catenary: if we assume the origin, as before, at the lowest point of the curve, we shall have still the general equation

$$\frac{d^2y}{dx^2} = \frac{w}{H};$$

but w in this case is not constant.

If we let $w_1 =$ the load per unit of length of chain, we shall have

$$w = w_1\frac{ds}{dx} = w_1\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

hence

$$\frac{d^2y}{dx^2} = \frac{w_1}{H}\frac{ds}{dx}.$$

Or, if we let

$$\frac{w_1}{H} = \frac{1}{m},$$

a constant,

$$\frac{d^2y}{dx^2} = \frac{1}{m}\frac{ds}{dx}, \qquad (1)$$

which is the differential equation of the catenary; and we only need to integrate it to obtain the equation itself.

To do this, we have

$$\frac{d^2y}{dx^2} = \frac{1}{m}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \therefore \quad \frac{\frac{d^2y}{dx^2}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{1}{m};$$

therefore, integrating, and observing, that, when $x = 0$, $\frac{dy}{dx} = 0$, we shall have

$$\log_e\left\{\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right\} = \frac{x}{m}$$

$$\therefore \quad \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{m}} \tag{2}$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = e^{\frac{2x}{m}} - 2e^{\frac{x}{m}}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) \qquad \therefore \quad y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) + c.$$

But, when $x = 0, y = 0 \quad \therefore \quad c = -m;$ hence the equation is

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) - m, \tag{3}$$

and this is the equation of the catenary when the origin is taken at O, the lowest point of the chain.

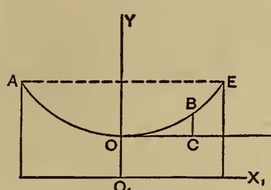


FIG. 261.

If it be transferred to O_1, where $OO_1 = m$, the equation becomes (by putting for y, $y - m$)

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \tag{4}$$

This is the most common form of the equation to the catenary, the origin being taken, at a distance below the lowest point of the curve equal to $m = \frac{H}{w_1}$, the horizontal tension divided by the weight per unit of length of chain.

To find x in terms of y, we have

$$e^{\frac{x}{m}} + \frac{1}{e^{\frac{x}{m}}} = \frac{2y}{m}$$

$$\therefore \quad e^{\frac{2x}{m}} + 1 = \frac{2y}{m} e^{\frac{x}{m}} \qquad \therefore \quad e^{\frac{2x}{m}} - \frac{2y}{m} e^{\frac{x}{m}} = -1.$$

Solving, we have

$$e^{\frac{x}{m}} = \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \qquad \therefore \quad \frac{x}{m} = \log_e \left\{ \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \right\}$$

$$\therefore \quad x = m \log_e \left\{ \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \right\}. \qquad (5)$$

To find the length of the rope: from the equation

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

we obtain

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4}\left(e^{\frac{2x}{m}} - 2 + e^{-\frac{2x}{m}}\right)}$$

$$= \sqrt{\frac{1}{4}\left(e^{\frac{2x}{m}} + 2 + e^{-\frac{2x}{m}}\right)} = \frac{1}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

$$\therefore \quad \frac{ds}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) \qquad (6)$$

$$\therefore \quad s = \frac{1}{2}\int_0^x \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) dx = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \qquad (7)$$

To find the area OO_1AB, we have

$$\text{Area} = \int_0^x y\,dx = \frac{m}{2}\int_0^x \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) dx = \frac{m^2}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \qquad (8)$$

But

$$\text{arc } OB = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = s;$$

hence area $OO_1AB = ms$.

This shows, that, if the load should be distributed in such a way as to be like a uniformly thick sheet of metal, having for one side the catenary and for the other the straight line O_1A, the equilibrium curve would be a catenary.

It may be convenient to have the development of $e^{\frac{x}{m}}$ and $e^{-\frac{x}{m}}$; hence they will be written here:—

$$e^{\frac{x}{m}} = 1 + \frac{x}{m} + \frac{x^2}{m^2\underline{|2}} + \frac{x^3}{m^3\underline{|3}} + \frac{x^4}{m^4\underline{|4}} + \text{etc.},$$

$$e^{-\frac{x}{m}} = 1 - \frac{x}{m} + \frac{x^2}{m^2\underline{|2}} - \frac{x^3}{m^3\underline{|3}} + \frac{x^4}{m^4\underline{|4}} + \text{etc.}$$

EXAMPLE I. — Given a rope 90 feet long, spanning a horizontal distance of 75 feet; find the equation of the catenary, the sag of the rope, and the inclination of the rope at each support, supposing these to be on the same level.

§ 251. **Transformed Catenary.** — We have just seen that the catenary is the form of chain suited to a load which may be represented by a uniformly thick sheet of metal, with a horizontal extrados, provided the distance OO_1 is equal to m, a definite quantity. A more general case, however, would be that of a chain loaded with a load which might be represented by a uniformly thick sheet of metal, where the length OO_1 is any given quantity whatever. A chain so loaded is called a *transformed catenary*, and the catenary itself becomes a particular case of the transformed catenary.

We may deduce its equation as follows :—

Let the chain be represented by ACB, and let it be so loaded that the load on CD is represented by w times area $OCDE$, so that $w =$ weight per unit of area; then we shall have, for this load,

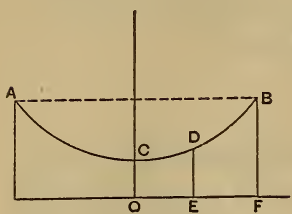


FIG. 262.

$$P = w\int_0^x ydx.$$

Hence, from what we have already seen,

$$\frac{dy}{dx} = \frac{P}{H} = \frac{w}{H}\int_0^x ydx$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{w}{H}y \qquad \therefore \quad \frac{dy}{dx}\frac{d^2y}{dx^2}dx = \frac{w}{H}y\frac{dy}{dx}dx.$$

Hence, integrating, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{w}{H}y^2 + c.$$

But, when $\dfrac{dy}{dx} = 0, y = a;$

$$\therefore \qquad c = -\frac{w}{H}a^2$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{w}{H}(y^2 - a^2).$$

Or, if we write, for brevity, $\dfrac{H}{w} = m^2$, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2 - a^2}{m^2} \qquad \therefore \quad \frac{dy}{dx} = \frac{1}{m}\sqrt{y^2 - a^2} \qquad \therefore \quad \frac{dy}{\sqrt{y^2 = a^2}} = \frac{dx}{m}$$

$$\therefore \quad \log(y + \sqrt{y^2 - a^2}) = \frac{x}{m} + c.$$

But, when $x = 0, y = a$;

$$\therefore \quad \log(a) = c \qquad \therefore \quad \log\left\{\frac{(y + \sqrt{y^2 - a^2})}{a}\right\} = \frac{x}{m}$$

$$\therefore \quad \frac{y + \sqrt{y^2 - a^2}}{a} = e^{\frac{x}{m}} \qquad \therefore \quad y^2 - a^2 = a^2 e^{\frac{2x}{m}} - 2aye^{\frac{x}{m}} + y^2$$

$$\therefore \quad \frac{2y}{a} = e^{\frac{x}{m}} + e^{-\frac{x}{m}} \qquad \therefore \quad y = \frac{a}{2}(e^{\frac{x}{m}} + e^{-\frac{x}{m}}),$$

which is the equation of the transformed catenary. This becomes the catenary itself whenever $a = m$.

EXAMPLE. — Given a chain loaded so that the load on OD is proportional to the area $OEDC$. Let $OC = 5$ feet, $BF = 8$ feet, $OF = 4$ feet; weight per unit of area $= 80$ lbs. Find the equation of the transformed catenary, also the tension at C and that at B.

§ 252. **Linear Arch.** — In all the preceding cases, the chain or cord is called upon to resist a tensile stress arising from a load that is hung upon it. If, now, the cord be inverted, we have the proper equilibrium curve for a load placed upon it, distributed in the same manner as before; only in this latter case the cord would be subjected to direct compression throughout its whole extent. The equilibrium curve is, then, sometimes called a *linear arch*. The general equation of the equilibrium curve remains just as before,

$$\frac{d^2y}{dx^2} = \frac{w}{H},$$

the axes being so chosen that OX is horizontal and OY vertical.

Thus, if it were required to find the form of the equilibrium curve or linear arch, with the upper boundary of the loading horizontal, we should obtain a *transformed catenary*.

§ 253. **Arches.** — In the case of arches composed of a series of blocks, as in stone or brick arches, the mathematical treatment generally used for determining the proper form and proportions of the arch has been quite different from that used for the determination of the proper form and proportions of the iron arch, whether made in one piece, or two pieces hinged together, or of a lattice.

In the case of the iron arch, the treatment involves necessarily a determination of the stresses acting in all its parts, and an adaptation of its form and dimensions to the load, so that at no point shall the stress exceed the working-strength of the material.

In the case of the stone arch, it is still a question under discussion whether it would not be best to adopt the same method, although it would lead to a great deal of complexity, on account of the joints.

Nevertheless, the question usually raised is one merely of stability ; i.e., as to the proper form and dimensions to prevent, not the crushing of the stone, though this must also be taken into account if there is any danger of exceeding it, but more especially the overturning about some of the joints.

The question of the stability of the stone arch may present itself in either of the two following ways : —

1°. Given the arch and its load, to determine whether it is stable or not.

2°. Given the distribution of the load, to determine the suitable equilibrium curve, and hence the form of arch, suited to bear the given load with the greatest economy of material.

§ 254. **Modes of giving Way of Stone Arches.** — An arch may yield, (1°) by the crushing of the stone, (2°) by sliding of the joints, (3°) by overturning around a joint. The following figures show the modes of giving way of an arch by the last two methods. The first two show the dislocation of the arch by the slipping of the voussoirs. In the former case the

haunches of the arch slide out, and the crown slips down; in the other case the reverse happens. The second two figures show the two methods by which an arch may give way by rotation of the voussoirs around the joints.

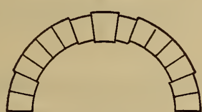


Fig. 263.

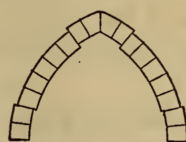


Fig. 264.

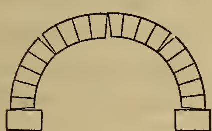


Fig. 265.

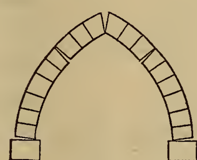


Fig. 266.

Before proceeding farther with the problem of the arch, two or three matters of a more general nature will be treated, which will be necessary in its discussion.

§ 255. **Friction.**—Let AB be a plane inclined to the horizon at an angle θ. Let D be a body resting on the plane, of weight $DG = W$. Resolve W into two components, DE and DF respectively, perpendicular and parallel to the plane. The component $DE = W \cos \theta$ is entirely neutralized by the re-action of the plane; while $DF = W \sin \theta$, on the other hand, is the only force tending to make the body slide down the plane. It is an experimental fact, that when the angle θ is less than a certain angle ϕ, called the angle of repose, the body does not slide; when $\theta = \phi$, the body is just on the point of sliding; and when θ is greater than ϕ, the body slides down the plane with an accelerated motion, showing that in this case an unbalanced force is acting. This

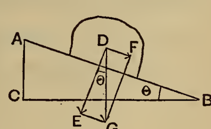


Fig. 267.

angle ϕ depends upon the nature of the material of the plane and of the body, and on the nature of the surfaces. Hence, in the first and second cases, the friction actually developed by the normal pressure DE just balances the tangential component DF; whereas, in the third case, when the angle of inclination of the plane to the horizon is greater than ϕ, the tangential component DF is only partially balanced by the friction.

Let ab be the plane when inclined to the horizon at an angle ϕ. The body is then just on the point of sliding, hence the component $df = W \sin \phi$ is just equal to the friction developed between the two surfaces. Moreover, if we represent by N the normal pressure $de = W \cos \phi$ on the plane, we shall have

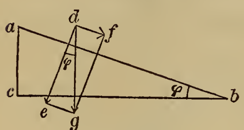


FIG. 268.

$$df = N \tan \phi.$$

Now, it is an experimental fact, that the friction developed between two given surfaces depends only on the normal pressure, i.e., that the friction bears a constant ratio to the normal pressure; and since, in this case, the friction just balances the tangential component $df = N \tan \phi$, the friction due to the normal pressure N is

$$N \tan \phi.$$

Now, it makes no difference what be the position of the plane surface: if a normal pressure N be exerted, the friction that is capable of being exerted to resist any force F tangential to the plane, tending to make the bodies slide upon each other, is $N \tan \phi$; and if the force F is greater than $N \tan \phi$, the bodies will slide, but if F is less than $N \tan \phi$, they will not slide. The quantity $\tan \phi$ is called the co-efficient of friction, and will be denoted by f.

From the preceding it is evident, that, if the resultant pressure on the body makes with the normal to the plane an angle less than the angle of repose, the sliding will not take place; whereas, if the resultant force makes with the normal to that plane an angle greater than the angle of repose, the body will slide.

§ 256. **Stability of Position.**—To determine under what conditions the stability of the block $DGHF$ is secure against turning around the edge D: if the resultant of the weight of the block and the pressure thereon pass outside the edge D, as OR_1, then the block will overturn; the moment of the couple tending to overturn it being $OR_1 \times DE$. If, on the other hand, it pass within the edge, as OR_2, the block will not overturn, since the force has a tendency to turn it the opposite way around D. Hence, in order that a block may not overturn around an edge at a plane joint, the resultant pressure must cut the joint within the joint itself.

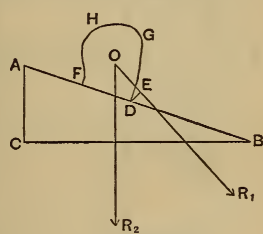


FIG. 269.

In any structure composed of blocks united at plane joints, we must have both stability of position ond stability of friction at each joint, in order that the structure may not give way.

§ 257. **Stability of a Buttress about a Plane Joint.**—Let $DCEF$ be a vertical section of a buttress, against which a strut rib or piece of framework abuts, exerting a thrust $P = ZG = OR$. In order that the buttress may not give way, it must fulfil the conditions of stability at each joint. Let AB be a joint. Should several pressures act against the buttress, the force P in the line ZO may be taken to represent the resultant of all the thrusts which act on the buttress above the joint AB. Let G be the centre of gravity of the part $ABEF$, and let $W = OL$ be the weight of that part of the buttress. Let O be the point of intersection of the line of

direction of the thrust, and of the weight W. Draw the parallelogram $ORNL$. Then will ON be the resultant pressure on the joint AB: and the conditions of stability require that the resultant pressure should cut the joint AB at some point between A and B, and that its line of direction should make with the normal to AB an angle less than the angle of repose, ϕ; and, in order that the buttress may not give way, these conditions must be fulfilled at each and every joint.

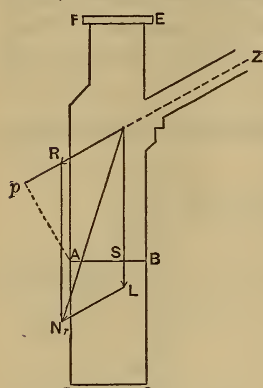


Fig. 270.

Another way of expressing this condition is as follows: The force tending to overturn the upper part of the buttress around A is the force $F = OR$; and its moment around A is $F(Ar) = Fp$ if we let $Ar = p$, whereas the moment of the weight which resists this is $W(AS) = Wq$ if we let $AS = q$. Now, when ON passes through A, we have $Fp = Wq$; when ON passes inside of A, we have $Wq > Fp$; when ON passes outside of A, we have $Wq < Fp$. Hence the conditions of stability require that

$$Wq \geq Fp \qquad \text{or} \qquad Fp \leq Wq.$$

Example. — Given a rectangular buttress 8 feet high, 1 foot wide, and 4 feet thick; the weight of the material being 100 lbs. per cubie foot, the buttress being composed of 8 rectangular blocks $1 \times 4 \times 1$ foot. On this buttress is a load of 500 lbs., whose weight acts through K, where $OK = 3$ feet. Find the greatest horizontal pressure P that can be applied along the line OK, consistent with stability, against overturning around each of the edges a, b, c, d, e, f, g, h.

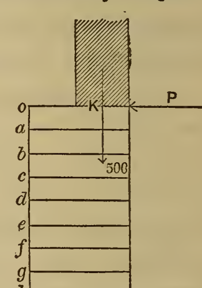


Fig. 271.

Solution. — The weight of each block will be 400 lbs. Hence we shall have the following equations: —

$$\text{Stability about } a, \text{ max } P = \frac{1500 + 400 \times 2}{1} = 2300.$$

$$\text{" " } b, \text{ " } = \frac{1500 + 800 \times 2}{2} = 1550.$$

$$\text{" " } c, \text{ " } = \frac{1500 + 1200 \times 2}{3} = 1300.$$

$$\text{" " } d, \text{ " } = \frac{1500 + 1600 \times 2}{4} = 1175.$$

$$\text{" " } e, \text{ " } = \frac{1500 + 2000 \times 2}{5} = 1100.$$

$$\text{" " } f, \text{ " } = \frac{1500 + 2400 \times 2}{6} = 1050.$$

$$\text{" " } g, \text{ " } = \frac{1500 + 2800 \times 2}{7} = 1014.$$

$$\text{" " } h, \text{ " } = \frac{1500 + 3200 \times 2}{8} = 987.$$

The least of these being 987 lbs., it follows that the greatest pressure consistent with stability is 987 lbs.

§ 258. **Line of Resistance in a Stone Arch.** — In order to solve any problem involving the stability of a stone arch, it is necessary that the student should be able to draw a line of resistance. To make plain the meaning of the term, the following solution of an example is given. The method of drawing the line of resistance employed in this solution is given purely for purposes of illustration, and is not recommended for use in practice, as a suitable method will be given later.

EXAMPLE. — Given three blocks of stone of the form shown in the figure (Fig. 272), their common thickness (perpendicular to the plane of the paper) being such that the weight per square inch of area (in the plane of the paper) is just one pound.

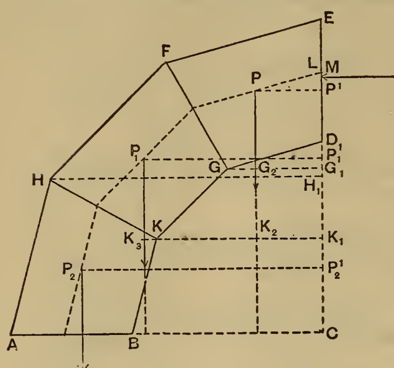


FIG. 272.

Given $AC = 13$ inches, $BC = 8$ inches. Suppose these three blocks to be kept from overturning by a horizontal force applied at the middle of DE. Find the least value of this horizontal force consistent with stability about the inner joints, also its greatest value consistent with stability about the outer joints.

Solution.

$$BK = 16 \sin 15° = 4.14112.$$

$$AH = 26 \sin 15° = 6.72932.$$

$$CR = \frac{2}{3}\left\{\frac{(13)^3 - (8)^3}{(13)^2 - (8)^2}\right\} = 10.7.$$

Altitude of each trapezoid $= 5 \cos 15°$ $= 4.8296.$

Area of each trapezoid $= \frac{105}{2} \sin 30°$ $= 26.25$ sq. in.

Weight of each stone $= 26.25$ lbs.

$GG_2 = 8 \sin 30° - 10.7 \cos 15° \sin 15° = 1.325.$

$KK_2 = 8 \cos 30° - 10.7 \cos 15° \sin 15° = 4.253.$

$KK_3 = 10.7 \cos 15° \cos 45° - 8 \cos 30° = 0.380.$

$BN_2 = 8 - 10.7 \cos 15° \sin 15° = 5.325.$

$BN_3 = 8 - 10.7 \cos 15° \cos 45° = 0.692.$

$BN_4 = 10.7 \cos 15° \sin 15° = 2.675.$

$HH_2 = 13 \cos 30° - 10.7 \cos 15° \sin 15° = 8.583.$

$HH_3 = 13 \cos 30° - 10.7 \cos 15° \cos 45° = 3.950.$

$$AN_2 = 13 - 10.7 \cos 15° \sin 15° = 10.325.$$
$$AN_3 = 13 - 10.7 \cos 15° \cos 45° = 5.692.$$
$$AN_4 = 13 - 10.7 \cos^2 15° = 3.017.$$
$$G_1M = 10.5 - 8 \cos 30° = 3.572.$$
$$K_1M = 10.5 - 8 \sin 30° = 6.500.$$
$$CM = 10.5 = 10.500.$$
$$H_1M = 10.5 - 13 \sin 30° = 4.000.$$

Let us represent the thrust at M by T. Then, to find what is the thrust required to produce equilibrium about G, we take moments about G, and likewise for the other joints. We may proceed as follows: —

INNER JOINTS.

Stability about G,

$$T(G_1M) = (26.25)(GG_2)$$

or

$$T(3.572) = (26.25)(1.325) \qquad \therefore \quad T = 9.74.$$

Stability about K,

$$T(K_1M) = (26.25)(KK_2 - KK_3)$$

or

$$T(6.500) = (26.25)(4.253 - 0.380) \qquad \therefore \quad T = 15.49.$$

Stability about B,

$$T(CM) = (26.25)(BN_2 + BN_3 - BN_4) \quad \therefore \quad T = 8.36.$$

OUTER JOINTS.

Stability about H,

$$T(H_1M) = (26.25)(HH_2 + HH_3) \qquad \therefore \quad T = 82.25.$$

Stability about A,

$$T(CM) = (26.25)(AN_2 + AN_3 + AN_4) \quad \therefore \quad T = 47.59$$

It is plain, therefore, that, in order to have equilibrium, the

thrust at M must be between 15.49 lbs. and 47.59 lbs.: for, if it is less than 15.49 lbs., the arch will turn about an inner joint; and if it is greater than 47.59 lbs., it will turn around an outer joint.

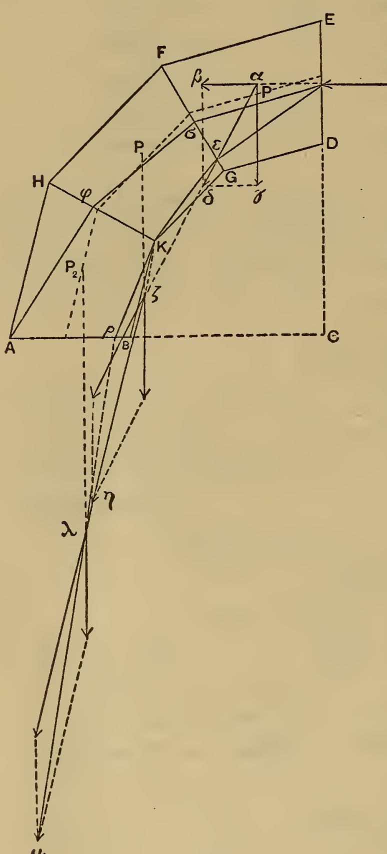


Fig. 273.

If, now, we draw through M a horizontal line to meet the vertical drawn through the centre of gravity of the first stone, and lay off $\alpha\beta = 15.49$, and $\alpha\gamma = 26.25$, then will the resultant of this thrust $\alpha\beta$ and the weight of the first stone $\alpha\gamma$ be $\alpha\delta$; this being the resultant pressure on the joint FG, its point of application being ϵ. Next, prolong this line $\alpha\delta$ to meet the vertical through the centre of gravity of the second stone, and combine $\alpha\delta$ with the weight of the second stone, thus obtaining, as resultant pressure on the joint KH, the force $\zeta\eta$, whose point of application is at K. Compounding, now, $\zeta\eta$ with the weight of the third stone, we obtain, as final resultant pressure on AB, the force $\lambda\mu$ applied at ρ. Now, joining $M\epsilon K\rho$ by a broken line, we have the *Line of Resistance* corresponding to the thrust 15.49, or the *minimum* horizontal thrust at M. If, now, we construct a line of resistance with 47.59 lbs., we obtain the line $M\sigma\phi A$, corresponding to maximum horizontal thrust at M.

If the arch is in equilibrium, and if the horizontal thrust is applied at M, it is plain that the actual thrust would either be one of these two or else somewhere between these two, and hence, that, if the requisite thrust is furnished at M to keep the arch in equilibrium, the true line of resistance cannot lie outside of these two; viz., the line corresponding to maximum and that corresponding to minimum horizontal thrust at M.

If the separate stones supported loads, it would be necessary to take into account these loads, in addition to the weights of the stones, in determining the horizontal thrust, and drawing the lines of resistance.

§ 259. **Arches with Symmetrical Distribution of the Load.** — Before considering the conditions of stability of an arch, we shall proceed to some propositions about lines of resistance corresponding to maximum and minimum horizontal thrust. If, in an arch, we draw a line of resistance AB through the point A of the crown, and then, by changing the horizontal thrust, we change the line of resistance continuously till it touches the extrados of the arch at C', we shall evidently have, in the line $AC'B'$, a line of resistance which has the greatest horizontal thrust of any line that passes through A, and lies wholly within the arch-ring. If, on the other hand, we decrease gradually the horizontal thrust until the line touches the intrados at D', then we have in this line the line of minimum horizontal thrust that passes through A. By lowering the point A, however, and keeping the point C the same, we should obtain new lines of resistance with greater and greater horizontal thrust; the greatest being attained when the line comes to have one point in common with the intrados. Hence a line of maximum horizontal thrust will have one point in common with the extrados and one point in common with the intrados, the latter being above the former.

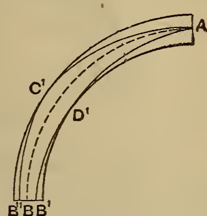


Fig. 274.

On the other hand, by retaining the point D' the same, and raising the point A, we should decrease the horizontal thrust, and thus obtain lines of resistance with less and less horizontal thrust; the least being attained when the line of resistance comes to have a point in common with the extrados. Hence the minimum line of resistance has a point in common with the extrados and one in common with the intrados, the latter being below the former.

These cases are exhibited in the following figures: —

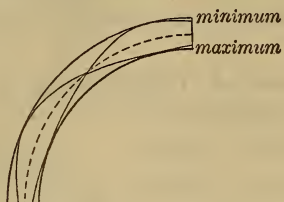


FIG. 275.

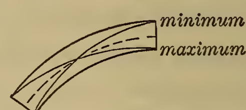


FIG. 276.

§ 260. **Conditions of Stability.** — The question of the stability of an arch must depend upon the position of its true line of resistance. If this true line of resistance lies within the arch-ring, the arch will be stable provided the material of which it is made is incompressible. If this is not the case, the stability of the arch will depend upon how near the true line of resistance approaches the edge of the joints; for the nearer it approaches the edge of a joint, the greater the intensity of the compressive stress at that joint, and the greater the danger that the crushing-strength of the stone will be exceeded at that joint. Thus, if the true line of resistance cuts any given joint at its centre of gravity, the stress upon that joint will be uniformly distributed over the joint. If, however, it cuts the joint to one side of its centre of gravity, the intensity of the stress will be greater on that side than on the opposite side; and, if it is carried far enough to one side, we may even have tension on the other side.

§ 261. **Criterion of Safety for an Arch.** — There are two criteria of safety for an arch, that have been used :—

1°. That the line of resistance should cut each joint within such limits that the crushing-strength of the stone should not be exceeded by the stress on any part of the joint.

2°. That, inasmuch as the joint is not suited to bear tension at any point, there should be no tension to resist.

The distribution of the stress is assumed to be uniformly varying from some line in the plane of the joint. The three following figures will, on this supposition, represent the three cases :—

1°. When the stress is wholly compression.

2°. When the stress becomes zero at the edge B.

3°. When the stress becomes negative or tensile at B.

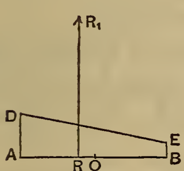


FIG. 277.

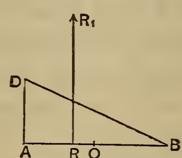


FIG. 278.

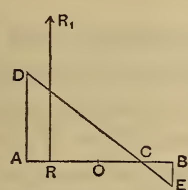


FIG. 279.

In all three figures, AB represents the joint which is assumed to be rectangular in section, AD represents the intensity of the stress at A, and BE that of the stress at B; while R represents the point of application of the resultant stress, RR_1 representing that resultant.

PROPOSITION. — If the stress on a rectangular joint vary uniformly from a line parallel to one edge, the condition that there shall be no tension on any part requires that the resultant of the compressive stress shall be limited to the middle third of the joint.

PROOF. — Let AB (Fig. 278) represent the projection of the joint on the plane of the paper. It is assumed that the

stress is uniformly varying; and, if there is to be no tension anywhere, the intensity at one edge must not have a value less than zero, hence at the limiting case the value must be zero; hence this limiting case is correctly represented by the figure, and the resultant of the compression will be for this case at the centre of stress. Thus, if AD represent the greatest intensity of the stress, then we shall have, if B be the origin and BA the axis of x, if the axis of y be perpendicular to AB at B, and if we let a = intensity of stress at a unit's distance from B, that $RR_1 = a\int\int xdxdy$, and $(BR)(RR_1) = a\int\int x^2dxdy$;

$$\therefore \quad BR = \frac{\int\int x^2dxdy}{\int\int xdxdy} = \frac{\frac{bh^3}{3}}{\frac{bh^2}{2}} = \tfrac{2}{3}h,$$

if b = breadth, and $h = BA$ = height of rectangle.

Hence, if the resultant of the compression be nearer A than R, there will be tension at B; and, on the other hand, if it be nearer B than $\frac{2}{3}h$, there will be tension at A. Hence follows the proposition as already stated.

While the above is probably the condition most generally used to determine the stability of an arch, at the same time, if there is any danger that the intensity of the stress at any part of any joint may exceed the working compressive strength of the stone, this ought to be examined, and hence a formula by which it may be done will be deduced.

Let AB (Fig. 279) be the joint, and let, as before, b be its breadth, and $h = AB$ = depth; then, suppose the pressure to be uniformly varying, $DA = f$ = the working-strength per unit of area = greatest allowable intensity of compression; then the entire stress on the joint will be represented by the triangle ACD, for the joint is incapable of resisting tension.

Hence

$$AR = \tfrac{1}{3}AC \qquad \therefore \quad AC = 3AR;$$

but

$$P = \frac{fb(AC)}{2} = \frac{3}{2}fb(AR) \qquad \therefore \quad AR = \frac{2}{3}\frac{P}{fb},$$

and this is the least distance from the outer edge at which the resultant should cut the joint.

We thus obtain, in terms of the pressure on any joint, and of the working-strength of the material, the limits within which the line of resistance should pass, in order that the working-strength of the stone may not be exceeded.

§ 262. **Position of the True Line of Resistance.** — The question of the most probable position of the true line of resistance involves the discussion of the properties of the elastic arch. This discussion will be given later; but, for the present, the statement only of the following proposition, due to Dr. Winkler, will be given: —

"*For an arch of constant section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring, as determined by the method of least squares.*"

From this it will follow: —

1°. That, if a line of resistance can be drawn in the arch-ring, then the true line of resistance will lie in the arch-ring; and

2°. That, if a line of resistance can be drawn within the middle third of the arch-ring, then the true line of resistance will lie in the middle third.

But, before proving this proposition, the proposition will be used, and the method explained, for determining whether a line of resistance can be drawn within the arch-ring: for, if it can, then the true line of resistance must lie within the arch-ring; and if no line of resistance can be drawn within the arch-ring, then the true line of resistance cannot pass within the arch-ring, and the arch would necessarily be unstable, even if the materials were incompressible.

By following the same method, we could determine whether

it was possible to draw a line of resistance within the middle third of the arch-ring; and, if this is found to be possible, we should know that the true line of resistance will pass within the middle third of the arch-ring.

Hence our most usual criterion of the stability of a stone arch is, whether a line of resistance can be passed within the middle third of the arch-ring.

If the condition be used, that the working-strength of the stone for compression be not exceeded, then, instead of the middle third, we shall have some other limits.

In what follows, an explanation will be given of Dr. Scheffler's method (that most commonly employed) of determining whether a line of resistance can be drawn within the arch-ring, inasmuch as the same method can be employed to determine whether such a line can be drawn within the middle third or within any other given limits.

§ 263. **Preliminary Proposition referring to Arches Symmetrical in Form and Loading.** — *An arch and its load being given, a line of resistance can always be made to pass through any two given points; hence, if any two points of a line of resistance are given, the line is determined.*

Proof. — Let the arch be that shown in Fig. 281; and let us consider first the special case when the two given points are A, the top of the crown-joint, and G_4, the foot of the springing-joint. In this case, the only quantity to be determined is the thrust at A. Let this thrust be denoted by T; let P be the total weight of the half-arch and its load; let a be the perpendicular distance of the point G_4 from a vertical line through the centre of gravity of the entire half-arch and its load; let h be the vertical depth of G_4 below A. Then, taking moments about G_4, we must have

$$Th = Pa$$

$$\therefore \quad T = \frac{Pa}{h}; \qquad (1)$$

and the line of resistance can then be drawn with this thrust, as has been done in the figure. Next take the general case, when the given points are not in these special positions. Let them be any two points, as A_2 and G_3.

In this case, the point of application of the thrust at the crown is not necessarily A, but may be some other point of the crown-joint: hence the quantities to be determined are two; viz., the thrust T at the crown, and the distance x of its point of application below A. Let the combined weight of the first two voussoirs and their load be P_1, and the horizontal distance of A_2 from a vertical line through the centre of gravity of P_1 be a_1.

Let P_2 be the combined weight of the first three voussoirs and their load, and let a_2 be the horizontal distance of G_3 from a vertical line through the centre of gravity of P_2.

Let the vertical depth of A_2 below A be h_1, and that of G_3 below A be h_2. Then, taking moments about A_2 and G_3 respectively, we shall have

$$T(h_1 - x) = P_1 a_1 \quad \text{and} \quad T(h_2 - x) = P_2 a_2,$$

two equations to determine the two unknown quantities T and x, which can easily be solved in any special case; and the resulting line of resistance can be drawn, which will pass through the two given points.

§ 264. **Dr. Scheffler's Method of Determining the Possibility of Passing a Line of Resistance within the Arch-Ring.** — In using Scheffler's method of determining whether it is possible to pass a line of resistance within the arch-ring or not, we should proceed as follows; viz., —

First pass a line of resistance through 1, the top of the crown-joint (Fig. 280), and e, the inside of the springing-joint. If this line lies wholly within the arch-ring, it proves that a line of resistance can be drawn within the arch-ring.

If this line of resistance does not pass entirely within the

arch-ring, proceed as follows: Suppose the line thus drawn to be 1*abcde*, passing without the arch-ring on both sides, as shown in the figure. Then from *a*, the point where it is farthest from the extrados of the arch-ring, draw a normal to the extrados, and find the point where this normal cuts the extrados: in this case, α is the point in question. In this way determine also the point 7, where the normal from *d* cuts the intrados; then pass a new line of resistance through the points α and 7, determining the thrust and its point of application. If this new line of resistance lies within the arch-ring, then it is plain that it is possible to draw a line of resistance within the arch-ring; if not, it is not at all probable that it is possible to draw such a line.

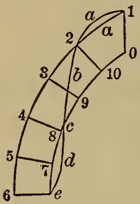


Fig. 280.

If the line of resistance drawn through 1 and *e* goes outside the arch-ring only beyond the extrados, as at *a*, we should draw our second line of resistance through α and *e*; if, on the other hand, it goes outside only below the intrados, as at *d*, we should draw our second line through 1 and 7.

In the construction, we make use of a slice of the arch included between two vertical planes a unit of distance apart; and we take for our unit of weight the weight of one cubic unit of the material of the voussoirs, so that the number of units of area in any portion of the face of the arch shall represent the weight of that portion of the arch.

We next draw, above the arch, a line (DD_4 in Fig. 281), straight or curved, such that the area included between any portion of it, as D_1D_2, the two verticals at the ends of that portion, and the extrados of the arch-ring, shall represent by its area the load upon the portion of the arch immediately below it. This line will limit the load itself whenever this is of the same material as the voussoirs; otherwise it will not. We shall always call it, however, the extrados of the load.

The mode of procedure will best be made plain by the solution of examples; and two will be taken, in the first of which only one trial is necessary to construct a line of resistance that shall lie wholly within the arch-ring, and, in the second, two trials are necessary.

EXAMPLE. — The half-arch under consideration is shown in Fig. 281, GG_4 being the intrados, AA_4 the extrados of the arch,

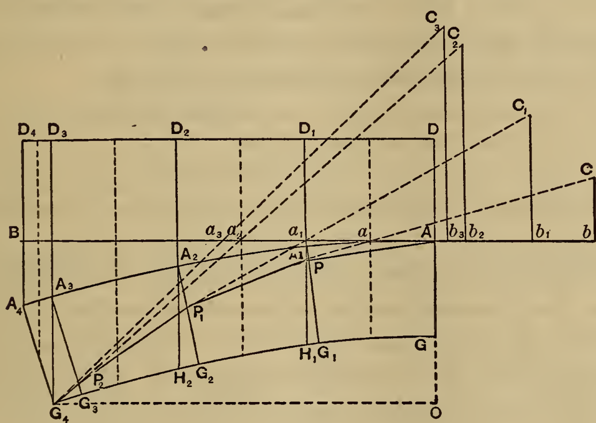


FIG. 281.

and DD_4 the extrados of the load. The arcs GG_4 and AA_4 are concentric circular arcs. The data are as follows: —

Span	$= 2(G_4O)$	$= 6.00$ feet,
Rise	$= GO$	$= 0.50$ foot,
Thickness of voussoirs $= AG$	$= A_4G_4$	$= 0.75$ foot,
Height of extrados of load above A	$= AD$	$= 1.60$ foot.

The position of the joints is not assumed to be located. We therefore draw through A a horizontal line AB, and divide this into lengths nearly equal, unless, as is usual near the springing, there is special reason to the contrary. Thus, we make the first three lengths each equal to 1 foot, and thus reach a vertical

through G_4; and then the last division has a length of 0.24 foot. We have thus divided the half-arch and its load into four parts; viz., GDD_1H_1, $H_1D_1D_2H_2$, $H_2D_2D_3G_4$, and $G_4D_3D_4A_4$, the loads on these respective portions being represented by their areas respectively. We assume the centre of gravity of each load to lie on its middle vertical; and we then proceed to determine the numerical values of the several loads, the distances of their centres of gravity from a vertical through the crown, also the amount and centre of gravity of the first and second loads together, then of the first, second, and third, etc.

The work for this purpose is arranged as follows:—

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Number of Voussoir.	Length.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.57	1.570	0.50	0.785	1	1.570	0.785	0.500
2	1.00	1.68	1.680	1.50	2.520	1+2	3.250	3.305	1.017
3	1.00	1.90	1.900	2.50	4.750	1+2+3	5.150	8.055	1.563
4	0.24	1.72	0.413	3.12	1.287	1+2+3+4	5.563	9.344	1.680
–	3.24	–	5.563	–	9.344	–	–	–	–

Column (1) shows the number of the voussoir.

" (2) gives the horizontal lengths of the several trapezoids.

" (3) gives the middle heights of the trapezoids.

" (4) gives the areas of the trapezoids, and is obtained by multiplying together the numbers in (2) and (3).

" (5) gives the distances from A to the middle lines of the trapezoids.

" (6) gives the products of (4) and (5), giving the moments of the respective loads about an axis through A perpendicular to the plane of the paper.

Column (7) merely indicates the successive combinations of voussoirs.

" (8) has for its numbers, —

1°. The area representing the first load.

2°. The area representing the first two loads.

3°. The area representing the first three.

4°. The area representing the first four.

" (9) has for its numbers, —

1°. The moment of the first load about A.

2°. The moment of the first and second loads about A.

3°. The moment of the first, second, and third loads about A.

4°. The moment of the first, second, third, and fourth loads about A.

" (10) is obtained by dividing column (9) by column (8); the quotients being respectively the distance from A to the centres of gravity of the first, of the first and second, of the first, second, and third, and of the first, second, third, and fourth loads.

The calculation thus far is purely mathematical, and merely furnishes us with the loads and their points of application; in other words, furnishes us the data with which to begin our calculation of the thrust. Before passing to this, it should be said, however, that we now assume the joints to be drawn through the points A_1, A_2, A_3, and A_4, and generally normal to the extrados of the arch.

In this proceeding, we, of course, make an error which is very small near the crown and increases near the springing of the arch; this error, in the case of voussoir $A_1A_2G_1G_2$, amounts to the difference of the two triangles $A_2G_2H_2$ and $A_1G_1H_1$. A manner of making a correction by moving the joint will be explained later; but now we will complete our example, as the

errors are not serious in this example. We now pass a line of resistance through A, the upper point of the crown-joint, and G_4, the lower point of the springing. For this purpose take moments about G_4; and we shall have, if T = thrust at the crown,

$$1.25T = (5.563)(3 - 1.68),$$

since 5.563 is the whole weight, and 3 — 1.68 is its leverage about G_4.

Hence

$$1.25T = (5.563)(1.32) = 7.343$$

$$\therefore \quad T = 5.87.$$

Hence we proceed to draw a line of resistance through A, assuming, as the horizontal thrust, 5.87. To do this we proceed as follows: From a, the point of intersection of AD with the vertical through the centre of gravity of the first trapezoid, we lay off ab to scale equal to 5.87, and then lay off bC vertically to scale equal to 1.57, the first load; then will Ca be the resultant pressure on joint A_1G_{11}, and its point of application will be P, which gives us one point in the line of resistance. To obtain the point P_{11}, we lay off Aa_1 = 1.017, the lever arm of the first two loads; then lay off a_1b_1 = 5.87, the thrust; then lay off b_1C_1 equal to 3.25, the weight of the first two loads. Then will C_1a_1 be the pressure on the second joint; and the point P_{11}, or its point of application, is at the intersection of C_1a_1 with A_2G_2.

Then lay off Aa_2 = 1.563, a_2b_2 = 5.87, b_2C_2 = 5.150; and P_2, the next point of the line of resistance, is the intersection of C_2a_2 with A_3G_3. Then lay off Aa_3 = 1.680, a_3b_3 = 5.87, b_3C_3 = 5.563; and C_3a_3 is the pressure on the springing, and this will intersect A_4G_4 at G_4 unless some mistake has been made in the work. Then is $APP_1P_2G_4$ the line of resistance through A and G_4, and this lies entirely within the arch-ring.

Hence we conclude that it is possible to draw a line of resistance within the arch-ring without having recourse to another trial.

§ 265. **Scheffler's Mode of Correcting the Joints.** — The following is the approximate construction given by Scheffler for correcting the joint: Let DCG be the side of the trapezoid, and CH the uncorrected joint. From b, the middle point of GH, draw Db; then draw Gc parallel to bD, and ch parallel to CH. Then will ch be the corrected joint.

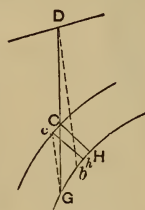


Fig. 282.

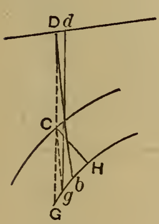


Fig. 283.

Conversely, having given the joint CH, to find the side of the trapezoid which limits the portion of the load upon it: through C draw DG vertical; join D with b, the middle point of GH; then draw Cg parallel to Db; then, from g, drawing dg vertical, we thus have the desired side of the trapezoid.

§ 266. **Another Example.** — Another example will now be solved, which necessitates two trials, and where some of the joints have to be corrected. It is practically one of Scheffler's. The dimensions of the arch are as follows: —

Half-span	32.97 feet.
Rise	24.74 feet.
Thickness of ring	5.15 feet.
Height of load at crown	8.24 feet.
Height of load at springing	33.50 feet.

The arch may be drawn by using, for the intrados, two circular arcs. Beginning at the springing, draw a 60° arc with a radius of one-fourth the span; then, with an arc tangent to this arc, continue to the crown, the proper rise having been previously laid off. The work for drawing a line of resistance

through the top of the crown-joint and the inside of the springing will be given without comment. It is as follows: —

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Number of the Voussoir.	Length.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	8.24	14.1	116.18	4.12	476	1	116.18	476	4.12
2	8.24	16.4	135.14	12.36	1675	1+2	251.32	2151	8.57
3	8.24	18.3	150.79	20.60	3106	1+...+3	402.11	5257	13.09
4	4.13	22.6	93.34	26.79	2600	1+...+4	495.45	7857	15.87
5	4.13	27.1	111.92	30.92	3448	1+...+5	607.37	11305	18.62
6	5.14	34.7	178.36	35.55	6351	1+...+6	785 73	17656	22.49
–	–	–	785.73	–	17656	–	–	–	–

$$29.89T = (785.73)(32.97 - 22.49),$$
$$29.89T = 8234.45$$
$$\therefore \quad T = 275.5.$$

Hence we construct the line of resistance passing through the top of the crown-joint and the inside of the springing, using the thrust 275.5.

The construction is shown in the figure, and is entirely similar to that previously used, with the single exception that the upper, instead of the lower, half of the rectangle is used, in each case, in constructing the parallelogram of forces, to determine the pressure on each joint: this is merely a matter of convenience. The student will readily identify this first line of resistance, and will see that it goes outside the arch-ring both above and below, being farthest above the extrados at the first joint from the crown, and farthest inside of the intrados opposite the first joint from the springing. Hence we

proceed to pass a new line of resistance through the top of the first joint from the crown, and the inside of the first joint from the springing.

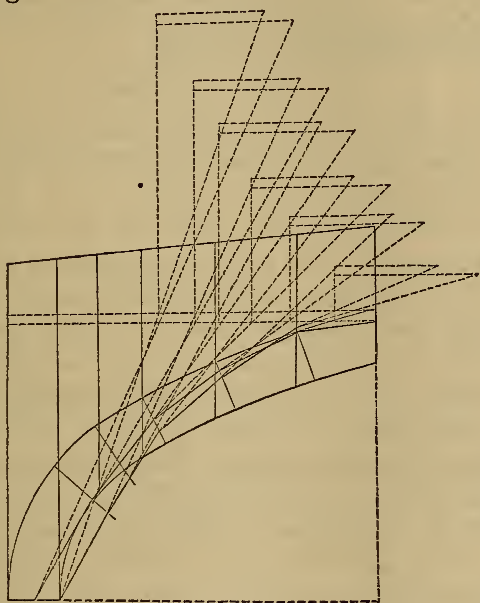


Fig. 284.

For this purpose we do not need to make out a new table, as it is not necessary to insert any new joints. We need only two more dimensions, i.e., the vertical depth of each of these points below the top of the crown: these depths are respectively 2.38 and 21.85.

Hence we proceed as follows:—

Let $T =$ thrust at the crown,

$x =$ distance of its point of application below the top of the crown-joint.

1°. Take moments about the upper one of the two points, and we have

$$T(2.38 - x) = (116.18)(8.24 - 4.12) = 478.66.$$

2°. Take moments about the lower one of the two points, and we have

$$T(21.85 - x) = (607.37)(30.90 - 18.62) = 7458.50.$$

Solving these equations, we obtain

$$T = 358.9, \quad x = 1.046.$$

Hence we lay off on the crown-joint a distance 1.046 below the top of the crown-joint, and through this point draw a horizontal line, this line being the line of action of the thrust. Then, making the construction for a new line of resistance just as before, only using this new point of application of the thrust, and using for thrust 358.9, we shall obtain a new line of resistance, which passes through the desired points; and, since this line lies within the arch-ring, we therefore conclude that it is possible to draw a line of resistance within the arch-ring.

§ 267. **Examples.** — Four more examples will now be given, to be worked out by the student. The dimensions are approximately those given in some of Scheffler's examples.

Example I. — Half-span $= CD = 65.16$ feet, rise $= FD = 13.85$ feet, $AF = 5.32$ feet, $AE = 6.40$ feet. The arcs CF and AG are concentric circular arcs. Given width of first five horizontal divisions of line AB, counting from A, each 10.66 feet; width of sixth division, 11.84 feet; of seventh, 3.68 feet. Determine the possibility of drawing a line of resistance in the arch-ring.

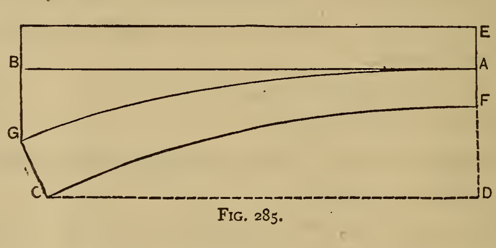


Fig. 285.

Example II. — Half-span $= 63.98$ feet, rise $= FD = 31.99$ feet, $AF = CG = 5.32$ feet, $AE = 2.13$ feet. The intrados and

extrados of this arch are seven centred ovals, both drawn from the same centres. Beginning at the springing, an arc with a radius of 21 feet is drawn, subtending 39°; the curve is continued by a curve subtending 24°, and having a radius of 35.55 feet. From F an arc subtending 10° is drawn from a centre on FD produced, and with a radius of 152 feet; the curve is completed by an arc connecting the second and last.

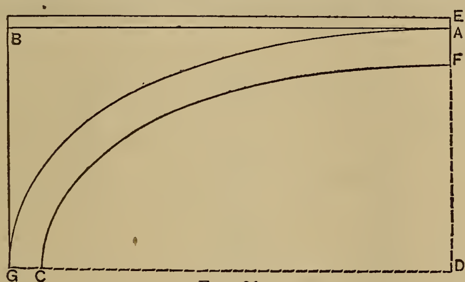


Fig. 286.

Given horizontal width of each of first six divisions, counting from A, 10.66 feet; horizontal width of seventh division, 5.32 feet. Determine the possibility of drawing a line of resistance in the arch-ring.

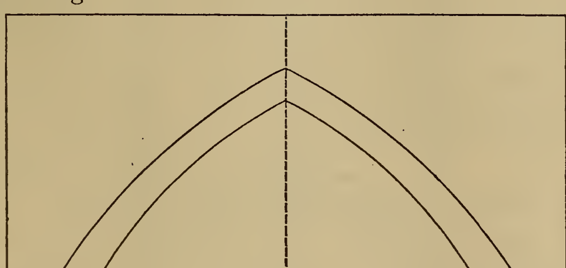


Fig. 287.

Example III. — Given span = 74.18 feet; rise = 45.83

feet; radius of intrados = 82.42 feet; radius of extrados = 91.18 feet; height of load at crown = 8.24 feet; width of each of five divisions nearest crown = 8.24 feet; width of sixth stone = 4.13 feet. Determine the possibility of drawing a line of resistance within the arch-ring.

EXAMPLE IV. — Given span = 37.07 feet; thickness of ring = AB = 3.08 feet; height of load = BC = 82.42 feet. Determine the possibility of drawing a line of resistance within the arch-ring.

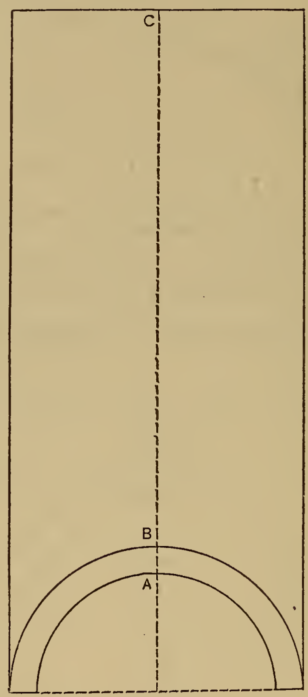


FIG. 288.

§ 268. **Criterion of Stability.** — It has already been stated, that, if a line of resistance can be drawn within the arch-ring, then the true line of resistance will lie within the arch-ring.

With those who, like Scheffler, consider the material of the voussoirs incompressible, the criterion of stability of an arch is, that it should be possible to draw a line of resistance within the arch-ring.

On the other hand, Rankine would decide upon the stability of an arch by determining whether a line of resistance can be drawn within the middle third of the arch-ring.

Other limits have been adopted instead of the middle third. In some cases the only reason for deciding upon what these limits should be has been custom or precedent.

They might also be determined so that there should be no danger of exceeding the crushing-strength of the stone.

It is needless to say that the first method is incorrect; for the material of the voussoirs is never incompressible, and an arch where the true line of resistance touches the intrados or extrados could not stand, as the stone would be crushed.

Nevertheless, no example will be solved here, where we determine the possibility of drawing a line of resistance within the middle third, or other limits than the entire arch-ring, as the method of procedure is entirely similar to what we have done, the computation of the entire table being the same in all cases, the only difference occurring in the computation of the thrust and its point of application, and the consequent construction of the line of resistance. The method to be pursued is, as before, by taking moments about the points through which it is desired that the line of resistance shall pass.

§ 269. **Unsymmetrical Arrangement.** — When the arch is unsymmetrical, either in form or loading, the same criterion as to being able to pass a line of resistance within the middle third or other limits of the arch-ring will serve to determine its stability. The method of procedure differs, however, from the fact, that whereas we have heretofore found it necessary to study only the half-arch and its load; and have had the advantage of knowing, from the symmetry of arch and load, that the thrust at

the crown is horizontal, we have not that advantage here, and hence we must study the entire arch, and we must assume that the thrust at the crown may be oblique, and hence have a vertical as well as a horizontal component.

In this case it will be necessary to have three instead of two points given, in order to determine a line of resistance.

If we assume (Fig. 289) a vertical joint at the crown, and let $P =$ vertical component of the thrust at the crown, $A =$ horizontal component of the thrust at the crown, $x =$ distance of point of application of thrust at the crown below upper point of crown-joint, we have thus three unknown quantities, and we shall therefore need three equations to determine them.

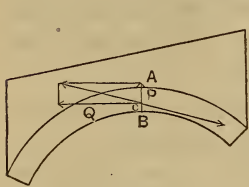


Fig. 289.

In this case, therefore, we must have three points of the line of resistance given, in order to determine it; and a reasoning similar to that pursued in § 263 would show that a line of resistance can always be passed through any three given points.

In performing the work, we should need to make out a table for the part of the arch on each side of the crown-joint, showing the loads, and centres of gravity of the loads, on each voussoir, and on combinations of the first two, first three, etc.; this portion of the work being entirely similar to that done in the case of arches of symmetrical form and loading, only that we require a separate table for the parts on each side of the crown-joint.

When these two tables have been worked out, we next proceed to impose the conditions of equilibrium by taking moments about each of the three points given.

Thus, suppose that (as is usually done first) we pass a line of resistance through the top of the crown-joint and the inside of each springing-joint, we then have only two unknown quantities to determine; viz., P and Q, inasmuch as x becomes zero.

Hence we take moments about the inner edge of each of the springing-joints.

In taking moments about the inner edge of the left-hand springing-joint, we impose the conditions of equilibrium upon the forces acting on that part of the arch that lies to the left of the crown-joint. These forces are, (1°) its load and weight, which tend to cause right-handed rotation; (2°) the horizontal component of the thrust exerted *by* the right-hand portion *upon* the left-hand portion; (3°) the vertical component P of the thrust exerted *by* the right-hand portion *upon* the left-hand portion.

It is necessary to adopt some convention, in regard to the sign of P, to avoid confusion: and it will be called positive when the vertical component of the thrust exerted *by* the right-hand portion *on* the left-hand portion is upwards; when the reverse is the case, it is negative.

We next take moments about the inner edge of the right-hand springing-joint, and impose the conditions of equilibrium upon the forces acting upon the right-hand portion of the arch. In doing this, we must observe that we have for these forces, (1°) the weight and load which tend to cause left-handed rotation; (2°) the horizontal component Q of the thrust exerted *by* the left-hand portion *upon* the right-hand portion, — this acts towards the right; (3°) the vertical component P of the thrust exerted by the left-hand portion upon the right-hand portion; and this, when positive, acts downwards.

Having determined the values of Q and P, we next proceed to draw the line of resistance; and this is done in a similar way to that pursued with symmetrical arches, only that the thrust, i.e., the resultant of P and Q, is now oblique, and that it acts in opposite directions on the two sides of the crown-joint.

Having drawn this line of resistance, if we find that it passes outside of the arch-ring, we draw normals through the points where it is farthest from the arch-ring, and thus obtain three points through which to draw a line of resistance: then, taking

moments about each of these three points, we determine, from the three resulting equations, values of Q, P, and x, and proceed to draw our new line of resistance; and, if this does not pass entirely within the arch-ring, it is not at all probable that a line of resistance can be drawn within the arch-ring. All the above will be made clearer by the following example: —

Example. — Given an unsymmetrical circular arch, shown in the figure, the intrados and extrados being concentric circles, $AB = 4'$, $AC = 1'.85$, $MH = 1'.5$, $AH = 0'.5$, $AK = 0'.8$, to

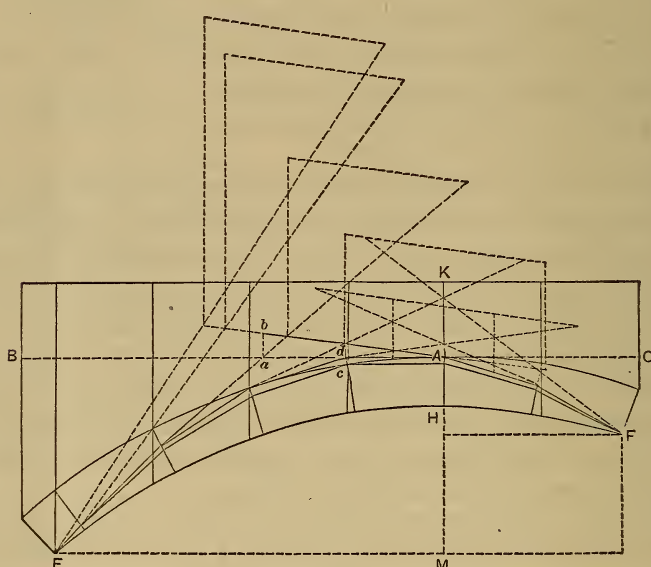


Fig. 290.

determine the possibility of drawing a line of resistance in the arch-ring. The tables following show the mode of dividing up the load, and getting the centres of gravity, also the mode of arranging the work for this purpose.

LEFT-HAND PORTION.

Number of Voussoir.	Width.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.32	1.32	0.50	0.660	1	1.32	0.660	0.50
2	1.00	1.48	1.48	1.50	2.220	1 + 2	2.80	2.880	1.03
3	1.00	1.84	1.84	2.50	4.600	1 + 2 + 3	4.64	7.480	1.61
4	1.00	2.42	2.42	3.50	8.470	1 + ... + 4	7.06	15.950	2.26
5	0.33	2.63	0.87	4.17	3.628	1 + ... + 5	7.93	19.578	2.47
–	–	–	7.93	–	19.578	–	–	–	–

RIGHT-HAND PORTION.

Number of Voussoir.	Width.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.32	1.32	0.50	0.660	1	1.32	0.660	0.50
2	1.00	1.48	1.48	1.50	2.220	1 + 2	2.80	2.880	1.03
–	–	–	2.80	–	2.880	–	–	–	–

Now take moments about the left-hand springing inner edge, and we have

$$2.02Q + 4P = 7.93(4 - 2.47) = 12.1239.$$

Then take moments about the right-hand springing inner edge, and we have

$$0.79Q + 1.86P = 2.80(1.86 - 1.33) = 2.3240.$$

Solving these two equations gives us

$$Q = 4.602,$$

$$P = 0.707.$$

If R represent the resultant of P and Q, we have

$$R = \sqrt{P^2 + Q^2} = 4.66;$$

hence we proceed, as follows, to pass a line of resistance through the top of the crown-joint and the inner edge of each springing: —

Through the top of the crown draw a horizontal line BAC. Lay off $Aa = 4.602$ and $ab = 0.707$, and draw Ab; then $Ab = 4.66$ represents, in direction and magnitude, the thrust at the crown. Using this thrust in the same way as we did the horizontal thrust in the case of symmetrical arches, we obtain the line of resistance $EdAF$, which is farthest outside of the arch at d; hence, drawing a normal to the arch from d, we obtain c, the upper edge of the first joint from the crown. Hence we proceed to pass a new line of resistance through E, c, and F.

To do this we must assume Q, P, and x all unknown.

1°. Take moments about E, and we have

$$(2.02 - x)Q + 4P = 12.1239.$$

2°. Take moments about F, and we have

$$(0.79 - x)Q - 1.86P = 2.3240.$$

3°. Take moments about c, and we have

$$(0.82 - x)Q + P = (1.32)(0.50) = 0.66.$$

Solving these three equations, we obtain

$$Q = 4.92,$$
$$P = 0.64,$$
$$x = 0.078.$$

Hence

$$R = \sqrt{P^2 + Q^2} = 4.96.$$

Hence, if we lay off a distance 0.078 below A, we shall have the point on the crown-joint at which the thrust is applied;

and making the same kind of construction as we just made, only using this point instead of A, and these new values of Q and P, we construct the second line of resistance. The construction is omitted in order not to confuse the figure; but the line of resistance is drawn, and the student can easily make the construction for himself. It will be seen, that, in this case, this new line of resistance lies entirely within the arch-ring.

§ 270. **General Remarks.** — Whenever there are also horizontal external forces acting upon the arch, these should be taken into account in imposing the conditions of equilibrium.

It will be noticed, that, in the preceding discussion, it has always been assumed that the load upon any one voussoir is the weight of the material directly over that voussoir. This is the assumption usually made in computing bridge arches: and it may be nearly true when the height of the load above the crown is not great; but even then it is not strictly true, and when this depth becomes great, as would be the case with an arch which supports the wall of a building, it is far from true, as the distribution of the load actually coming upon different parts of the arch must vary with, and depend upon, the bonding of the masonry, and also upon the co-efficient of friction of the material. Thus, in the case of an arch supporting a part of the wall of a building, it is probable that the only part of the load that comes upon the arch is a small triangular-shaped piece directly over the arch, and that above this the material of the wall is supported independently of the arch. This will be plain when we consider, that, were such an arch removed, the wall would remain standing, only a few of the bricks near the arch falling down; and though the number of bricks that would fall would be greater while the mortar is green, still even then only a few would drop out.

In regard to these matters, we need experiments; but thus far we have none that are reliable.

Then, again, we have arches supporting a mass of sand or gravel; and then the mutual friction of the particles on each other comes into play, and it is not true in this case that the load on any voussoir is the weight of the material directly above that voussoir. In some cases this has been accounted as a mass of water pressing normally upon the arch, but we cannot assert that such a course is correct.

On the other hand, there are cases where we know that an arch is subjected to horizontal as well as to vertical forces, and sometimes we cannot tell how great these horizontal forces are. Thus, the forms of sewers are an arch for the top and an inverted arch for the bottom; but in this case the sides of the ditch in which the sewer is laid when building it, are capable of furnishing whatever horizontal thrust is needed to force the line of resistance into the arch-ring, provided that a *horizontal thrust* is what is needed to force it in. Hence it is, that, were the attempt made to pass a line of resistance within the arch-ring of almost any successful sewer, accounting the load as the weight of the earth above it, the line would almost invariably go outside; but the earth on the sides is capable of furnishing the necessary horizontal thrust to force it inside, unless a careless workman has omitted to ram it tight, or unless some other cause has loosened it on the sides of the sewer.

If we know, in any case, the actual law of the distribution of the load, we can determine the proper form for the arch by the methods of the first part of this chapter, as was done in the case of the parabola and of the catenary. Scheffler's method is, however, the one almost always used for determining the stability of any stone arch against overturning around the joints.

Should there ever arise a case where there was danger that the resultant pressure on any joint made an angle with the joint greater than the angle of friction, this could be remedied by merely changing the inclination of the joint.

§ 271. **General Theory of the Elastic Arch.** — In the case of the iron arch, the loads upon the arch are all definitely known; and it is necessary to ascertain with certainty the stress in all parts of the structure, and to so proportion the different members as to bear with safety their respective stresses.

The general discussion of the method used in calculating such arches will now be given; the method used being practically that followed by Dr. Jacob J. Weyrauch, and explained more at length in his "Theorie der Elastigen Bogenträger."

This discussion is also necessary in order to prove the proposition already enunciated in § 262; viz., that "for an arch of constant cross-section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring as determined by the method of least squares."

In this discussion the following definitions are adopted: —

1°. The axis of the arch is a plane curved line passing through the centres of gravity of all its normal sections.

2°. The plane of this axis is called the plane of the arch.

3°. The axial layer of the arch is a cylindrical surface perpendicular to the plane of the arch, and containing its axis.

4°. A section normal to the axis is called a cross-section.

5°. The length of the axis between two sections is called the length of arch between the sections.

The loads may be single isolated loads, or they may be distributed loads.

We shall, in this discussion, assume in the plane of the arch a pair of rectangular axes, OX and OY, positive to the right and upwards respectively.

We will, then, assuming any point on the axis of the arch before the loads are applied, call x, y, the co-ordinates of that point, s the length of axis from some arbitrary fixed point, ϕ the angle made by the tangent line at that point with OX, r the radius of curvature of the axis at that point, $x + dx, y + dy$, $s + ds$, and $\phi + d\phi$, the corresponding quantities for a point

very near the first before the load is applied; also we will denote by η the perpendicular distance of any fibre from the axial layer, by S_η the length of arc measured to that point where this fibre cuts the cross-section through (x, y), and $S_\eta + dS_\eta$ the length of arc measured on this fibre to the next cross-section, so that ds will be the distance apart of the cross-sections measured on the axis, and dS_η on the other fibre.

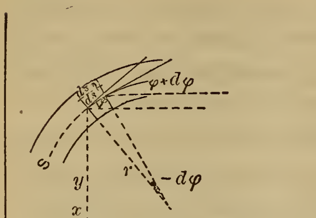


FIG 291.

All this is done before the load is applied, and is shown in Fig. 291; while the changes brought about by the application of the loads are denoted by Δ's, and shown in Fig. 292. Thus, x, y, s, and ϕ become respectively $x + \Delta x$, $y + \Delta y$, $s + \Delta s$, and $\phi + \Delta\phi$.

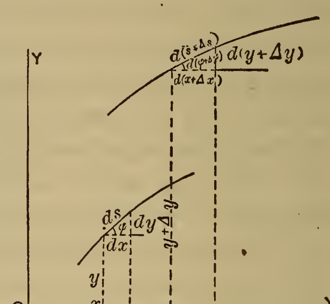


FIG. 292.

Now, the course we are to follow in the discussion is, to imagine a cross-section dividing the arch into two parts, and to impose the conditions of equilibrium between the external forces acting on the part to one side of the section, and the forces exerted by the other part upon this part at the section. These latter forces may be reduced to the three following: —

1°. A normal thrust T_x uniformly distributed over the section, the resultant acting at the centre of gravity of the section.

2°. A shearing-force S_x at the section.

3°. A bending-couple at the section; this comprising a stress varying uniformly from the axial layer, and amounting to a statical couple, tension below, and compressions above, the axial layer.

Moreover, (1) and (3) combined amount to a uniformly vary-

ing stress, the magnitude of whose resultant is T_x, its point of application not being at the centre of gravity of the section; this sort of composition having been already exhibited in the case of the short strut (§ 207).

Now, let r be the radius of curvature of the axial layer at the section; and we have, from Fig. 291, by similar sectors,

$$dS_\eta = dS + \eta(-d\phi) = dS - \eta d\phi. \qquad (1)$$

But

$$r(-d\phi) = ds \qquad \therefore \quad \frac{d\phi}{ds} = -\frac{1}{r}$$

$$\therefore \quad ds_\eta = ds\left(1 + \frac{\eta}{r}\right) = ds\left(\frac{r + \eta}{r}\right). \qquad (2)$$

Now, if the loads are applied, and the changes take place that are indicated in Fig. 292, we shall have, by suitable substitutions in (1),

$$d(S_\eta + \Delta S_\eta) = d(S + \Delta s) - \eta d(\phi + \Delta\phi); \qquad (3)$$

and, combining this with (1) and (2), we obtain

$$\frac{d\Delta S_\eta}{dS_\eta} = \left(\frac{d\Delta S}{ds} - \eta\frac{d\Delta\phi}{ds}\right)\frac{r}{r + \eta}. \qquad (4)$$

Now, the change of length of fibre from dS_η to $d(S_\eta + \Delta S_\eta)$ is due to two causes: (1) the change of temperature, (2) the stress acting on the fibre normal to the section.

Let ϵ = co-efficient of expansion per degree temperature.
τ = difference of temperature, in degrees.
p_η = intensity of stress along the fibre at section.
E = modulus of elasticity of the material.

Then

$$\epsilon\tau - \frac{p_\eta}{E} = \frac{d\Delta S_\eta}{dS_\eta} = \left(\frac{d\Delta s}{ds} - \eta\frac{d\Delta\phi}{ds}\right)\frac{r}{r + \eta}. \qquad (5)$$

Hence, solving for p_η, we have

$$p_\eta = E\left(\eta\frac{d\Delta\phi}{ds} - \frac{d\Delta s}{ds}\right)\frac{r}{r + \eta} + E\epsilon\tau, \qquad (6)$$

this being the expression for the stress per square inch on the fibre whose distance is η from the axial layer.

Hence we shall have, by summation, if elementary area $= dA$,

$$T_x = \Sigma p_\eta dA = E\left[\frac{d\Delta\phi}{ds}\Sigma\frac{r\eta dA}{r+\eta} - \frac{d\Delta s}{ds}\Sigma\frac{rdA}{r+\eta} + \epsilon\Sigma\tau dA\right]; \quad (7)$$

and for the moment M_x we have, by taking moments about the neutral axis of the section (i.e., horizontal line through its centre of gravity),

$$M_x = \Sigma p_\eta \eta dA = E\left[\frac{d\Delta\phi}{ds}\Sigma\frac{r\eta^2 dA}{r+\eta} - \frac{d\Delta s}{ds}\Sigma\frac{r\eta dA}{r+\eta} + \epsilon\Sigma\tau\eta dA\right]. \quad (8)$$

Let $\Sigma dA = A$, $\Sigma\dfrac{\eta^2 dA}{r+\eta} = \Omega$, and observe that $\Sigma\eta dA = 0$, since the axis passes through the centre of gravity of the section, and we have

$$\Sigma\frac{r\eta dA}{r+\eta} = \Sigma\eta dA - \Sigma\frac{\eta^2 dA}{r+\eta} = -\Omega,$$

$$\Sigma\frac{rdA}{r+\eta} = \Sigma dA - \frac{1}{r}\Sigma\eta dA + \frac{1}{r}\Sigma\frac{\eta^2 dA}{r+\eta} = A + \frac{\Omega}{r}.$$

Making these substitutions, we have

$$\frac{T_x}{E} = -\left(\frac{d\Delta\phi}{ds} + \frac{1}{r}\frac{d\Delta s}{ds}\right)\frac{\Omega}{r} - \left(\frac{d\Delta s}{ds} - \epsilon\tau\right)A,$$

$$\frac{M_x}{E} = \left(r\frac{d\Delta\phi}{ds} + \frac{d\Delta s}{ds}\right)\frac{\Omega}{r}.$$

Hence, solving for $\dfrac{d\Delta s}{ds}$ and $\dfrac{d\Delta\phi}{ds}$, we have

$$\frac{d\Delta s}{ds} = -\left(T_x + \frac{M_x}{r}\frac{1}{EA}\right) + \epsilon\tau = Y, \quad (9)$$

$$\frac{d\Delta\phi}{ds} = \left(T_x + \frac{M_x}{r}\right)\frac{1}{EAr} + \frac{M_x}{E\Omega} - \frac{\epsilon\tau}{r} = X. \quad (10)$$

Now, from Fig. 292, we have

$$d(x + \Delta x) = d(s + \Delta s)\cos(\phi + \Delta\phi),$$

$$d(y + \Delta y) = d(s + \Delta s)\sin(\phi + \Delta\phi);$$

but, if we write $\cos\Delta\phi = 1$, and $\sin\Delta\phi = \Delta\phi$,

$$\cos(\phi + \Delta\phi) = \cos\phi - \Delta\phi\sin\phi = \frac{dx}{ds} - \Delta\phi\frac{dy}{ds},$$

$$\sin(\phi + \Delta\phi) = \sin\phi + \Delta\phi\cos\phi = \frac{dy}{ds} + \Delta\phi\frac{dx}{ds}.$$

Hence

$$d\Delta x = -\Delta\phi dy + \frac{d\Delta s}{ds}dx - \left(\frac{d\Delta s}{ds}dy\Delta\phi\right),$$

$$d\Delta y = +\Delta\phi dx + \frac{d\Delta s}{ds}dy + \left(\frac{d\Delta s}{ds}dx\Delta\phi\right):$$

or, omitting the last terms, and integrating,

$$\Delta x = -\int\Delta\phi dy + \int Ydx, \qquad (11)$$

$$\Delta y = \int\Delta\phi dx + \int Ydy; \qquad (12)$$

and, integrating (9) and (10),

$$\Delta s = \int Yds, \qquad (13)$$

$$\Delta\phi = \int Xds. \qquad (14)$$

In these four equations we have

Δx = horizontal deflection due to the loads,
Δy = vertical deflection due to the loads,
Δs = change of length of arc due to the loads,
$\Delta\phi$ = change of slope due to the loads.

If, now, we write

$$M_1 = \frac{M_x}{E\Omega} + \frac{M_x}{EAr^2} + \frac{T_x}{EAr},$$

$$P_1 = \frac{M_x}{EAr} + \frac{T_x}{EA},$$

we shall have

$$X = M_1 - \frac{\epsilon\tau}{r}, \quad Y = -P_1 - \epsilon\tau;$$

and hence (11) to (14) become

$$\Delta x = -\int\int M_1 dsdy - \int P_1 dx + \int\int \frac{\epsilon\tau}{r} dsdy - \int \epsilon\tau dx, \quad (15)$$

$$\Delta y = \int\int M_1 dsdx - \int P_1 dy - \int\int \frac{\epsilon\tau}{r} dsdy - \int \epsilon\tau dy, \quad (16)$$

$$\Delta s = - \int P_1 ds - \int \epsilon\tau ds, \quad (17)$$

$$\Delta \phi = \int M_1 ds - \int \frac{\epsilon\tau}{r} ds. \quad (18)$$

If we neglect the effect of temperature, they become

$$\Delta x = -\int\int M_1 dsdy - \int P_1 dx, \quad (19)$$

$$\Delta y = \int\int M_1 dsdx - \int P_1 dy, \quad (20)$$

$$\Delta s = - \int P_1 ds, \quad (21)$$

$$\Delta \phi = \int M_1 ds. \quad (22)$$

If, on the other hand, we do not neglect the effect of temperature, but omit all terms containing $\frac{1}{r}$ in the values of X and Y, which would be more nearly correct the larger the value of r, i.e., the flatter the arch, we should obtain

$$\Delta x = -\int\int \frac{M_x}{E\Omega} dsdy - \int \frac{T_x}{EA} dx - \int \epsilon\tau dx, \quad (23)$$

$$\Delta y = \int\int \frac{M_x}{E\Omega} dsdx - \int \frac{T_x}{EA} dy - \int \epsilon\tau dy, \quad (24)$$

$$\Delta s = - \int \frac{T_x}{EA} ds - \int \epsilon\tau ds, \quad (25)$$

$$\Delta \phi = \int \frac{M_x}{E\Omega} ds. \quad (26)$$

Moreover, if we put the moment of inertia I for Ω, which will cause but little error, we derive

$$\Delta x = -\int\int\frac{M_x}{EI}dsdy - \int\frac{T_x}{EA}dx - \int\epsilon\tau dx, \qquad (27)$$

$$\Delta y = \int\int\frac{M_x}{EI}dsdx - \int\frac{T_x}{EA}dy - \int\epsilon\tau dy, \qquad (28)$$

$$\Delta s = - \int\frac{T_x}{EA}ds - \int\epsilon\tau ds, \qquad (29)$$

$$\Delta\phi = \int\frac{M_x}{EI}ds. \qquad (30)$$

§ 272. **Manner of using the Fundamental Equations to Determine the Stresses in an Iron Arch.** — In order to be able to determine the stresses in all the members of an iron arch with any given loading, we need to determine the three quantities T_x, S_x, and M_x for each section.

Now, if we let R_x represent the thrust at the section, we shall have

$$R_x = \sqrt{T_x^2 + S_x^2}; \qquad (1)$$

and, if we let H_x and V_x represent the horizontal and vertical components of R_x respectively, we have that we need to determine the three quantities H_x, V_x, and M_x for each section.

If we suppose the arch to be subjected to vertical loads only, we shall have, if we let

H = horizontal component of thrust at all points,
V = vertical component of left-hand support,
V_1 = vertical component of right-hand support,
M = bending-moment at left-hand support,
M' = bending-moment at right-hand support.

Assume origin of co-ordinates at left-hand support, and x + to the right, and y + upwards, and impose the conditions of equilibrium upon the forces acting on the part of the arch

between the section and the left-hand support; then we have, if W is any one load, and a the x of its point of application,

$$H_x = H, \tag{2}$$
$$V_x = V - \Sigma_0^x W, \tag{3}$$
$$M_x = M + Vx - Hy - \Sigma_0^x W(x - a). \tag{4}$$

Hence it is plain that the three quantities which we need to determine are H, V, and M.

Now these are also the three unknown quantities which will, by suitable reductions, become the three unknown constant quantities in equations (27) to (30). The determination of these three quantities requires three conditions; what these conditions are depends upon the manner of building the arch, as will be seen from the following three special cases:—

CASE I.—Let the arch be jointed at three points, viz., the two supports, and one other point whose co-ordinates are $x = x_1$ and $y = y_1$. Then we know, that, for all points where there is a hinge, there can be no bending-moment. Hence

$$M = 0, \qquad M' = 0, \qquad \text{and} \qquad M_{x_1} = 0,$$

which are the three required conditions; and, if these be imposed, it is easy to obtain H_x, V_x, and M_x for every section.

CASE II.—Let the arch be jointed only at the ends. Then $M = M' = 0$ gives us two conditions: and for the third we have $\Delta l = 0$; i.e., if we put l for x in equation (15) or (27), § 271, after having made the integrations, we have the third equation, as this expresses simply the condition that the supports remain at the same horizontal distance apart after the load is put on as before. With these three conditions we can determine H_x, V_x, and M_x for all sections.

CASE III.—Let the arch be fixed in direction at the ends. We must now have three conditions. These will be as follows:—

1°. $\Delta l = 0$; i.e., the supports remain at the same horizontal distance apart after the load is applied as before.

2°. $\Delta h = 0$; (h being the difference of level of the supports); i.e., the supports remain at the same vertical distance apart after as before the load is applied.

3°. $\Delta\phi_1 = 0$; i.e., the tangents at the ends make the same angle with each other after as before the load is applied.

The value of Δl is obtained by integrating (15) or (27), § 271, and then substituting l for x.

The value of Δh is obtained by integrating (16) or (28), § 271, and then substituting l for x, or h for y.

The value of $\Delta\phi_1$ is obtained by integrating (18) or (30), § 271, and then substituting l for x, or h for y.

In this case we often find it convenient to use these equations in a simpler form. Thus, suppose we adopt the case where we neglect the effect of temperature (i.e., equations (19), (20), and (22) of § 271), by making one integration they become

$$\Delta x = -\int M_1 y ds - \int P_1 dx, \tag{5}$$

$$\Delta y = +\int M_1 x ds - \int P_1 dy, \tag{6}$$

$$\Delta\phi = \int M_1 ds; \tag{7}$$

these being more convenient to use.

For very flat arches, P_1 becomes very small: and then the equations become

$$\Delta x = -\int M_1 y ds, \tag{8}$$

$$\Delta y = +\int M_1 x ds, \tag{9}$$

$$\Delta\phi = \int M_1 ds; \tag{10}$$

and these, if all terms containing $\frac{1}{r}$ be omitted, become

$$\Delta x = -\int \frac{M_x}{EI} y ds, \tag{11}$$

$$\Delta y = +\int \frac{M_x}{EI} x ds, \tag{12}$$

$$\Delta\phi = \int \frac{M_x}{EI} ds. \tag{13}$$

EXAMPLES.

1. Given a semicircular arch jointed at each springing-joint and at the crown, radius r. Trace out the effect of a single load W acting upon it at the extremity of a radius making 45° with the horizontal.

Solution.

The presence of three joints gives us the bending-moments at each of these joints equal to zero, the co-ordinates of these joints being respectively (0, 0), (r, r), and $(2r, 0)$.

Hence, using equation (4), we obtain

1°. $M = 0$,
2°. $Vr - Hr - W(0.70711r) = 0$,
3°. $V(2r) - W(1.70711r) = 0$.

Solving, we have, therefore,

$$V = 0.85355\,W = \text{left-hand supporting-force},$$

and

$$H = 0.14645\,W = \text{horizontal component of thrust.}$$

Hence $V_1 = 0.14645\,W =$ right-hand supporting-force.

Hence, for a section whose co-ordinates are (x, y),

$$x < 0.29289r, \qquad V_x = 0.85355\,W;$$
$$x > 0.29289r, \qquad V_x = -0.14645\,W.$$

Hence equation (1) gives, for

$$x < 0.29289r, \qquad R_x = W\sqrt{(0.85355)^2 + (0.14645)^2} = 0.86602\,W,$$

$$x > 0.29289r, \qquad R_x = W\sqrt{(0.14645)^2 + (0.14645)^2} = 0.20711\,W.$$

Now, the angle made by R_x with the horizontal is, for

$$x < 0.29289r, \qquad \alpha_1 = \tan^{-1}\left(\frac{0.85355}{0.14645}\right) = 80^\circ\ 15'\ 51'',$$

$$x > 0.29289r, \qquad \alpha_1 = \tan^{-1}\left(\frac{0.14645}{0.14645}\right) = 45^\circ.$$

Knowing, now, the angle made by R_x with the horizontal, we can find, for the point (x, y), the angle made by a tangent to the circle with the horizon, or $\alpha_2 = \tan^{-1}\left(\frac{r - x}{y}\right)$. Then resolve R_x into two components, respectively tangent to the arch and normal to it at the point x, y, and the tangential component is the direct thrust T_x, while the normal is the shearing-force S_x.

Then, for the bending moment, we have, from (4),

$$x < 0.29289r, \quad M_x = 0.85355\,Wx - 0.14645\,Wy;$$

$$x > 0.29289r, \quad M_x = 0.85355\,Wx - 0.14645\,Wy - W(x - 0.29289r).$$

Hence we determine the direct thrust, the shearing-force, and the bending-moment at any section, and can hence obtain the stresses at all points.

2. Given the same arch with a load W distributed uniformly over the circular arc, find stresses at all points.

3. Given the same arch jointed only at the two springing-points, find stresses at all points.

§ 273. **Position of True Line of Resistance in a Stone Arch.** — The proof will now be given of the proposition already referred to in regard to the position of the true line of resistance; viz., —

"For an arch of constant section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring, as determined by the method of least squares."

Proof. — If we denote by y the ordinate of the axis of the arch for an abscissa x, and by μ that of the line of resistance for the same abscissa, then $\mu - y$ is the vertical distance between the two curves for abscissa x. Now, the condition that the line of resistance should be as near the arch-ring as possible, is, that the sum of the $(\mu - y)^2$ shall be a minimum, or

$$\int (\mu - y)^2 ds = \text{minimum}. \qquad (1)$$

But (x, μ) are the co-ordinates of the point of application of the

actual thrust, and hence $(\mu - y)$ is the distance of the point at which the resultant thrust acts from the centre of gravity of the section. Hence we have

$$\mu - y = \frac{M_x}{H}.$$

Hence (1) becomes

$$\int\left(\frac{M_x}{H}\right)^2 ds = \text{minimum}. \qquad (2)$$

But H is constant for the same line of resistance, though it varies for different lines: hence we can place H outside of the integral sign. Hence we may write

$$u = \frac{1}{H^2}\int M_x^2 ds = \text{minimum}. \qquad (3)$$

Now, from (4), § 272, we have

$$M_x = M + Vx - Hy - \Sigma_0^x W(x - a) = \phi(M, V, H);$$

M, V, and H being constants for the same line of resistance, but varying for different lines. Hence, by differentiating (3), we have

$$\frac{du}{dM} = \frac{du}{dM_x}\frac{dM_x}{dM} = \frac{2}{H^2}\int M_x ds = 0 \qquad \therefore \quad \int M_x ds = 0, \qquad (4)$$

$$\frac{du}{dV} = \frac{du}{dM_x}\frac{dM_x}{dV} = \frac{2}{H^2}\int M_x x ds = 0 \qquad \therefore \quad \int M_x x ds = 0 \qquad (5)$$

$$\frac{du}{dH} = \frac{du}{dM_x}\frac{dM_x}{dH} = -2H^{-3}\int M_x^2 ds + 2H^{-2}\int M_x y ds$$

$$= -H^{-1}\left\{\frac{2}{H^2}\int M_x^2 ds + 2\frac{1}{H}\int M_x y ds\right\} = 0.$$

But the first term must be very small: hence we may write approximately,

$$\int M_x y ds = 0. \qquad (6)$$

Now, the three expressions (4), (5), and (6) are identical with (11), (12), and (13) of § 272; and the conditions that these shall be zero are, as will be seen by referring to § 272, Case III., the

conditions that hold in the case of an arch fixed in direction at the ends. Hence it follows that the condition that the line of resistance shall fall as near the centre of the arch as possible is the condition which, in an elastic arch fixed in direction at the ends, gives us its true position. Hence it would seem that the most probable position for the true line of resistance is the nearest possible to the axis of the arch.

This is the conclusion reached by Winkler; and a more detailed discussion of the matter is to be found in an article by Professor Swain in "Van Nostrand's" for October, 1880.

§ 274. **Domes.** — The method to be used for determining the stability of a dome differs essentially from that used in the case of an arch, for there is no thrust at the crown in a dome. Indeed, the most general case is that of the dome open at the top: we will, therefore, consider this case first in studying the action of the forces required to preserve equilibrium.

Fig. 243 shows a meridional section of an open dome. Suppose that this dome had been entirely built, except the upper ring-course of stones, represented by *LKGH*. Then, suppose that one of the stones only of this course were placed in position without any auxiliary support, its own weight would evidently overturn it, since the line *ab*, along which the weight acts, does not cut the joint; but, if the whole ring-course is put in place, the stones keep each other in position. The way in which this is accomplished is as follows: they press laterally against each other; and the resultant of the

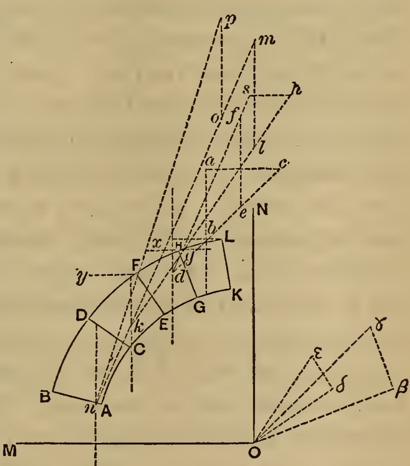


Fig. 293.

pressures exerted upon the two lateral faces of any one stone by the other stones of the course is a horizontal radial force, which, combined with the weight of the stone, gives, as the resultant of the two, a force which cuts the joint between G and H. Moreover, sufficient pressure will be developed to accomplish this result, as a failure to reach the result will only increase the pressure upon the lateral faces.

Moreover, if, when sufficient pressure has been developed to bring the resultant of the weight of the stone and the above-described horizontal radial force within the joint, it should make an angle with the normal to the joint greater than the angle of friction, the tendency of the stone to slide will increase the lateral pressure, and this in turn will increase the outward horizontal force till the angle made by the resultant with the normal to the joint is no greater than the angle of friction of the material of the voussoirs.

This will be made plain by reference to the figure (Fig. 243), where ab represents the weight of the stone $HLKG$, and where $O\beta$ is perpendicular to HG and $O\gamma$ is drawn so that $\gamma O\delta = \phi$, the angle of friction. Now, since ab produced passes outside of HG, horizontal thrust must be developed. And, moreover, were only sufficient horizontal thrust furnished to make the resultant cut HG at G, the angle between this resultant and the normal to the joint would be greater than ϕ; therefore we proceed as follows: assuming the horizontal thrust to act through L, the upper edge of the stone, we lay off from b, the intersection of the horizontal through L with a vertical line drawn through the centre of gravity of the stone, the weight ab to scale, then from b draw bc parallel to $O\gamma$, and draw through a a horizontal line to meet bc. Then will ac be the horizontal force that will be furnished by the other stones of the course to keep this stone in place; and the pressure upon joint HG is bc, and acts at the intersection of bc and HG.

Now prolong bc to meet the vertical drawn through the

centre of gravity of the next stone, $HGFE$, at d. Combine it with the weight of this stone; this is done by laying off $de = bc$, and from e drawing ef vertical, and equal to the weight of $FHGE$. The resultant fd makes an angle with the normal to FE greater than ϕ: hence draw $O\delta$ perpendicular to FE and $O\epsilon$, so that $\epsilon O\delta = \phi$; then from g, the intersection of df with a horizontal line through H, the top of $FHGE$, lay off $gs = df$, through g draw gh parallel to $O\epsilon$, and through s draw sh horizontal. Then is sh the horizontal thrust that will be furnished at H to keep the stone $HGEF$ in place; and this is the pressure upon joint FE, and acts at the intersection of FE with hg.

Next, prolong hg to meet the vertical through the centre of gravity of stone $FEDC$ at k; lay off $kl = gh$, and from l lay off lm = weight of stone $FEDC$; draw km, which cuts the joint within the joint itself, and needs no horizontal thrust to bring it inside; hence mk is the pressure on joint DC.

Then draw mk to meet the vertical through the centre of gravity of $ABCD$ at n, and lay off $no = km$; draw op = weight of $ABCD$, and draw pn, which will be the pressure on the joint BA.

It is necessary, for stability, that all these forces should cut the joint inside of the joint if the stones are reckoned incompressible; or we may adopt the middle third, or other limits, as our criterion of stability.

As long as it is outward thrust that is required to produce stability, it is possible to furnish it; but, if we should reach a joint where inward thrust would be required, this could not be furnished, and the dome would be unstable. Moreover, the resultant pressure on the springing gives us the pressure exerted upon the support of the dome; and it must not cut any joint of the support outside of that joint, as otherwise the support would not stand.

In determining the numerical value and direction of this pressure on the support, we may either construct it graphically,

or we may compute it as follows: (1°) Compound all the vertical forces, i.e., the weights, and find the magnitude and line of action of the resultant of these. (2°) Compound all the horizontal forces, and find the magnitude and line of action of their resultant (in this case the horizontal forces are two; viz., ac applied at L, and sh applied at H); then compound these two resultants. The graphical and analytical method should check if no mistake has been made in the work.

In the above calculation, it has been assumed that the figure represents the portion of a dome included between two meridional planes.

If we desire to ascertain the pressure exerted upon the lateral face of the stone by its neighbors in the same ring-course, we only need to know the angle made by the two meridional planes containing the lateral faces of the stone in question, then resolve the horizontal thrust upon that stone into two equal components, which make with each other an angle equal to the supplement of the angle of the planes; i.e., resolve the outward horizontal thrust into two components normal to the lateral faces.

In regard to the assumption that the outward thrust acts at the top of the stone, it should be said that this is Scheffler's custom, his reason being that less thrust will be required if he assumes it at the top than if he assumes it nearer the middle. The true position of this thrust is probably much nearer the middle of the stone.

An example will next be solved, giving Scheffler's method of working.

Example. — Given the dome shown in the figure, surmounted by a lantern at the top; determine whether it is stable, and what should be the thickness of the support in order that the resultant pressure may not pass outside any joint of the pier.

The dimensions are as follows: —

Diameter of outer vertical circle = 20 feet.

Diameter of inner vertical circle = 18 feet.

Angle made by springing-radius with vertical = $75°$ = angle AOB.

The inner edge of the upper voussior subtends $18°$ on the lower circle; the width of the load of the lantern is 0.6; the voussoirs below that, each subtend $18°$.

Assume 36 stones in a horizontal course. The width of the lowest will, then, be 1.51; the width of the others are determined from their lever arms.

Given height of pier = 8 feet.

Height of the centre of the sphere above base of pier = $8' - 10 \sin 15° = 5.41'$.

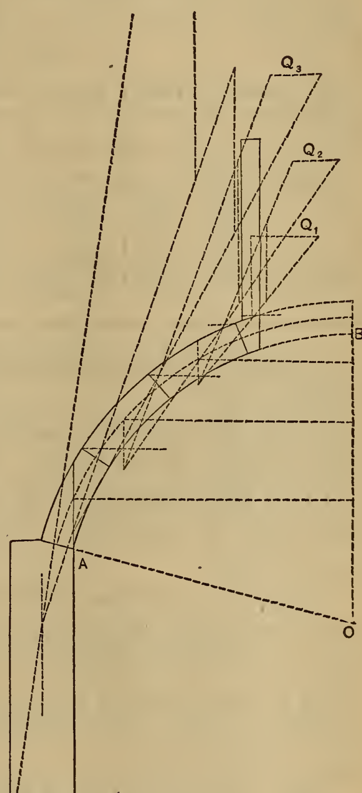


Fig. 294.

The figure may be taken to represent the portion of the dome included between two vertical planes passing through the axis of the dome: hence it shows one vertical series of stones.

We first construct a table giving the weights of the different voussoirs with any superincumbent load, their centres of gravity, and the moments of their weights about an axis passing through O, and perpendicular to the central plane of the portion shown; and we so choose our unit of weight that the volumes of the

voussoirs shall represent their weights. The work is arranged as follows : —

Elementary Forces.						Horizontal Forces.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
No. of Voussoir.	Area of Lateral Face.	Thickness.	Product.	Lever Arms.	Moment.	Horizontal Forces.	Lever Arms.	Moment.
1	0.6×6.680	0.53	2.124	3.07	6.521	1.74	9.60	16.104
2	2.985	0.74	2.209	4.75	10.493	1.26	9.33	11.756
3	2.985	1.23	3.672	7.05	25.888	1.32	7.78	10.273
4	2.985	1.51	4.507	8.68	39.121	–	–	–
–	–	–	12.512	–	82.023	4.32	–	38.730

Column (1) contains the numbers of the voussoirs, counting from the top.

Column (2) contains the areas of the lateral faces of the stones shown in the figure. For the three lower stones, the area of a ring subtending 18° at the centre, and of the dimensions given, is calculated. For the first, the height is 6.68 and the width 0.6.

Column (3) contains the thicknesses of the voussoirs; i.e., the length of arc between their two lateral faces measured on a horizontal circle through the centre of gravity of the voussoir, which is here taken at the middle point of the arc subtended by this voussoir on its middle vertical circle, i.e., one which has a radius 9.5 feet.

Hence, the thickness of the lower stone being 1.51 feet, that of the others will be

$$(1.51)\frac{3.07}{8.68} = 0.53, \qquad (1.51)\frac{4.25}{8.68} = 0.74,$$

$$(1.51)\frac{7.05}{8.68} = 1.23.$$

Column (4) gives the weights of the voussoirs and their loads: it is obtained by multiplying together the numbers in columns (2) and (3).

Column (5) gives the distances of the centres of gravity of the different voussoirs from the axis of the dome: it may be determined graphically or by calculation.

Column (6) gives the moments of the weights about a horizontal axis through O perpendicular to the central plane of this series of voussoirs. The graphical construction for determining the horizontal thrusts required is next made, and the results are recorded in column (7). It will be seen that no thrust is required on voussoir No. 4.

Column (8) contains the lever arms of these forces about the same axis.

Column (9) contains their moments about the same axis.

The construction thus far has shown no case where horizontal tension instead of horizontal thrust is required to cause the thrust on any joint to pass within the joint: hence thus far the dome is stable; and the question comes next as to what should be the width of the pier in order that the line of resistance, if continued down, may remain within it.

For this purpose we proceed as follows: —

Let $t =$ thickness required.

Let breadth be equal to that of the lowest voussoir.

Height $= 8$ feet.

Take moments about the outer edge of the base of the pier.

We shall then have, —

1°. Moment of vertical load on dome, and of weight of dome sector about inner edge of springing, $=$

$$(12.512)(8.68 - 6.56) = 26.52.$$

2°. Moment of same about outer edge of springing of pier $=$

$$26.52 + (12.512)t.$$

3°. Moment of horizontal forces about the same axis =

$$38.730 + (4.32)(5.41) = 62.101.$$

4°. Moment of weight of pier about outer edge =

$$\{8(1.51)t\}\tfrac{1}{2}t = 6.04t^2.$$

Hence we have

$$6.04t^2 + 12.51t + 26.52 = 62.101$$

$$\therefore \quad t^2 + 2.07t = 5.89 \qquad \therefore \quad t = 1.60 \text{ feet.}$$

This is the thickness required in order that the line of resistance may remain within the lower joint.

If, on the other hand, while pursuing the same method with the dome itself, we require that the line of resistance shall remain within the middle third of the pier, we take moments about a point in the springing of the pier at a distance $\frac{2}{3}t$ from its inner edge, we should then have

$$\tfrac{2}{3}t^2 + \tfrac{2}{3}(2.07)t = 5.89$$

$$\therefore \quad t^2 + 2.07t = 8.84 \qquad \therefore \quad t = 2.10 \text{ feet.}$$

On the other hand, we could proceed in a similar way to the above, if we desired to keep the line of resistance in the dome within the middle third, by merely assuming the horizontal thrusts to act at two-thirds the thickness of a joint from the lower edge, and using a point two-thirds the thickness from the top, instead of the lower edge, as the lower limiting-point for the pressure to pass through.

This will not be done here, however.

Example. — As an example, St. Peter's dome will be given, with the dimensions as given by Scheffler reduced to English measures. The dome consists in its upper part, as will be evident from the figure, of two domes; the lantern resting on

the two is assumed to have one-third of its weight resting on the upper, and two-thirds on the lower dome.

Diameter of dome = diameter at the base = 144 feet.

Up to a point 28.48 feet above the point C it is formed of a single dome 11.84 feet thick. In its upper part, on the other hand, it is composed of two domes whose normal distance apart is 5.15 feet; the exterior having a thickness of 2.56 feet, and the inner of 4.13 feet at the top and 5.15 feet at the springing. At the top of these two domes is an opening 12.24 feet radius, surmounted by a cylindrical lantern. The magnitude of the load of the lantern on the dome is represented on the figure by 1.82 feet width and 56.66 feet height.

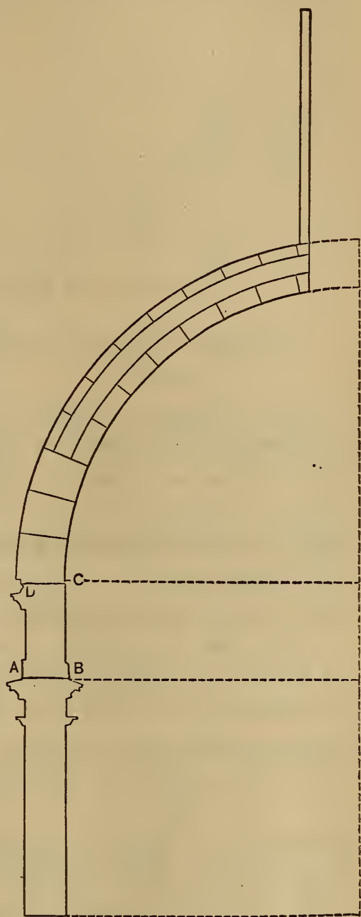


FIG. 295.

Height of the entablature $ABCD = 23.69$.

Width of $ABCD$ normal to plane of paper = 1.02 feet.

Thickness of $ABCD = 10.30$ feet.

Divide the exterior dome into nine parts, the interior into eight of a uniform circumferential width of 10.08 feet, except the first, which has a width of only 1.82 feet.

Determine whether this thickness of $ABCD$ is sufficient to keep the line of resistance within joint AB.

CHAPTER X.

THEORY OF ELASTICITY, AND APPLICATIONS.

§ 275. **Strains.** — When a body is subjected to the action of external forces, and in consequence of this undergoes a change of form, it will be found that lines drawn within the body are changed, by the action of these external forces, in length, in direction, or in both; and the entire change of form of the body may be correctly described by describing a sufficient number of these changes.

If we join two points, A and B, of a body before the external forces are applied, and find, that, after the application of the external forces, the line joining the same two points of the body has undergone a change of length $\Delta(AB)$, then is the limit of the ratio $\frac{\Delta(AB)}{AB}$, as AB approaches zero, called the *strain* of the body at the point A in the direction AB.

If $AB + \Delta(AB) > AB$, the strain is one of tension; whereas, if $AB + \Delta(AB) < AB$, the strain is one of compression.

In order to study the changes of form of the body, let us assume a point O within the body when there are no external forces acting, and let us draw through this point three rectangular axes, OX, OY, and OZ, and assume a small rectangular parallelopipedical particle whose three edges are OA, OB, and

OC, and let us examine the form of this particle after the loads are applied; it will be found that the edges *OA*, *OB*, and *OC* will be of different lengths from what they were before, and that the angles *AOB*, *AOC*, and *BOC* will no longer be right angles, but will differ slightly from 90°. Let the parallelopiped *oabc–gdef* represent the form and dimensions of the particles after the external forces are applied. Then we shall have,

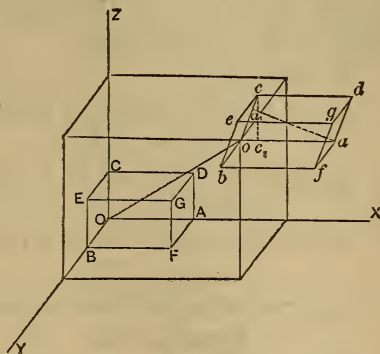


FIG. 296.

if ϵ_x, ϵ_y, and ϵ_z represent the strains in the directions *OX*, *OY*, and *OZ* respectively, that

$$\epsilon_x = \text{limit of } \frac{oa - OA}{OA} \text{ as } OA \text{ approaches zero,}$$

$$\epsilon_y = \text{limit of } \frac{ob - OB}{OB} \text{ as } OB \text{ approaches zero,}$$

$$\epsilon_z = \text{limit of } \frac{oc - OC}{OC} \text{ as } OC \text{ approaches zero.}$$

In the figure, ϵ_x and ϵ_z are tensile strains, and ϵ_y is a compressive strain.

But these strains do not represent completely the distortion of the particle; for the plane *CEGD* has slid by the plane *OABF* through the distance oc_1, the distance apart of these planes being *OC*, and the plane halfway between the two has slid just half as far, so that the amount of shearing, or the *shearing-strain* of planes parallel to *XOY* in the direction *OX*, may be represented by $\dfrac{oc_1}{OC} = \dfrac{oc_1}{cc_1}$ nearly, or the distortion divided by the

distance apart of these planes. This, moreover, is the tangent of the angle occ_1, or the tangent of the angle by which aoc differs from a right angle.

If, now, we let

γ_{zx} = shearing-strain in a plane perpendicular to OZ in the direction OX,

γ_{zy} = shearing-strain in a plane perpendicular to OZ in the direction OY,

γ_{yx} = shearing-strain in a plane perpendicular to OY in the direction OX,

γ_{yz} = shearing-strain in a plane perpendicular to OY in the direction OZ,

γ_{xz} = shearing-strain in a plane perpendicular to OX in the direction OZ,

γ_{xy} = shearing-strain in a plane perpendicular to OX in the direction OY,

and let $boc = \frac{\pi}{2} - \phi$, $aoc = \frac{\pi}{2} - \psi$, $aob = \frac{\pi}{2} - \chi$, then we shall have

$$\gamma_{zx} = \frac{oc_1}{cc_1} = \tan\psi, \qquad \gamma_{yz} = \tan\phi,$$

$$\gamma_{zy} = \tan\phi, \qquad \gamma_{xz} = \tan\psi,$$

$$\gamma_{yx} = \tan\chi, \qquad \gamma_{xy} = \tan\chi.$$

We thus have

$$\gamma_{zy} = \gamma_{yz} = \tan\phi,$$

$$\gamma_{xz} = \gamma_{zx} = \tan\psi,$$

$$\gamma_{xy} = \gamma_{yx} = \tan\chi,$$

three very important equations.

We thus have to determine six strains, in order to define completely the state of strain in a body at a given point; viz., if we assume three rectangular axes, we must know ϵ_x, ϵ_y, ϵ_z, $\gamma_{zy} = \gamma_{yz}$, $\gamma_{zx} = \gamma_{xz}$, $\gamma_{xy} = \gamma_{yx}$, three normal and three tangential strains.

§ 276. **Strains in Terms of Distortions.** — Let us assume a rectangular parallelopipedical particle, the co-ordinates of one corner of which are x, y, z, and of the other, $x + dx$, $y + dy$, $z + dz$; this being the case before the load is applied.

Let the effect of the load be to change x, y, z, respectively, into $x + \xi$, $y + \eta$, $z + \zeta$, and to change $x + dx$, $y + dy$, $z + dz$, into $(x + \xi) + (dx + d\xi)$, $(y + \eta) + (dy + d\eta)$, $(z + \zeta) + (dz + d\zeta)$. Then are dx, dy, dz, the edges of the particle before the load is applied.

Then, from what has preceded, we shall have

$$\epsilon_x = \frac{d\xi}{dx}, \qquad \epsilon_y = \frac{d\eta}{dy}, \qquad \epsilon_z = \frac{d\zeta}{dz};$$

$$\gamma_{xy} = \gamma_{yx} = \frac{d\xi}{dy} + \frac{d\eta}{dx}, \quad \gamma_{xz} = \gamma_{zx} = \frac{d\xi}{dz} + \frac{d\zeta}{dx}, \quad \gamma_{yz} = \gamma_{zy} = \frac{d\eta}{dz} + \frac{d\zeta}{dy}.$$

The first three will be evident at once. As to the last three, if the student will construct the figure indicated, he will see that

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} = \tan\chi, \quad \frac{d\xi}{dz} + \frac{d\zeta}{dx} = \tan\psi, \quad \frac{d\eta}{dz} + \frac{d\zeta}{dy} = \tan\phi.$$

§ 277. **Determination of the Strain in any Given Direction.** — Suppose we are required, knowing the strains ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{xz}, γ_{yz}, to determine the strain in a direction making angles α, β, γ, with OX, OY, OZ respectively. Assume our rectangular parallelopipedical particle in such a way that the diagonal from (x, y, z) to $(x + dx, y + dy, z + dz)$ shall be in the required direction, and call the length of this diagonal ds; then we shall have

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2, \tag{1}$$

$$\cos\alpha = \frac{dx}{ds}, \tag{2}$$

$$\cos\beta = \frac{dy}{ds}, \tag{3}$$

$$\cos\gamma = \frac{dz}{ds}. \tag{4}$$

Let ϵ be the strain in the required direction; then length of diagonal after load is applied will be

$$ds(1 + \epsilon),$$

and we shall have

$$(ds)^2(1 + \epsilon)^2 = (dx + d\xi)^2 + (dy + d\eta)^2 + (dz + d\zeta)^2,$$

or

$$(ds)^2 + 2\epsilon(ds)^2 + \epsilon^2(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 + 2(dxd\xi + dyd\eta + dzd\zeta) + (d\xi)^2 + (d\eta)^2 + (d\zeta)^2. \tag{5}$$

Now, subtracting (1) from (5), and neglecting $\epsilon^2(ds)^2$, $(d\xi)^2$, $(d\eta)^2$, and $(d\zeta)^2$ as being very small compared with the rest, we have

$$2\epsilon(ds)^2 = 2dxd\xi + 2dyd\eta + 2dzd\zeta$$

$$\therefore \quad \epsilon ds = \frac{dx}{ds}d\xi + \frac{dy}{ds}d\eta + \frac{dz}{ds}d\zeta, \tag{6}$$

or

$$\epsilon ds = d\xi\cos\alpha + d\eta\cos\beta + d\zeta\cos\gamma. \tag{7}$$

But

$$d\xi = \frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy + \frac{d\xi}{dz}dz, \tag{8}$$

$$d\eta = \frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy + \frac{d\eta}{dz}dz, \tag{9}$$

$$d\zeta = \frac{d\zeta}{dx}dx + \frac{d\zeta}{dy}dy + \frac{d\zeta}{dz}dz. \tag{10}$$

Hence, substituting these, we have, after dividing by ds, and observing (2), (3), and (4),

$$\epsilon = \frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\cos\beta\cos\gamma$$
$$+ \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\cos\alpha\cos\gamma + \left(\frac{d\eta}{dx} + \frac{d\xi}{dy}\right)\cos\alpha\cos\beta; \qquad (11)$$

or, making use of § 276, we have

$$\epsilon = \epsilon_x\cos^2\alpha + \epsilon_y\cos^2\beta + \epsilon_z\cos^2\gamma + \gamma_{yz}\cos\beta\cos\gamma$$
$$+ \gamma_{xz}\cos\alpha\cos\gamma + \gamma_{xy}\cos\alpha\cos\beta, \qquad (12)$$

which gives us the strain in any direction.

It can be shown that there are three directions, at right angles to each other, that give the maximum strains or minimum strains: and we might deduce the ellipsoid of strains, in which semi-diameters of the ellipsoid represent the strains; but we will pass on to the consideration of the stresses.

§ 278. **Stresses.** — When a body is subjected to the action of external forces, if we imagine a plane section dividing the body into two parts, the force with which one part of the body acts upon the other at this plane is called the *stress* on the plane; and, in order to know it completely, we must know its distribution and its direction at each point of the plane. If we consider a small area in this plane, including the point O, and represent the stress on this area by p, whereas the area itself is represented by a, then will the limit of $\frac{p}{a}$, as a approaches zero, be the intensity of the stress on the plane under consideration at the point O. Observe that we cannot speak of the stress at a certain point of a body unless we refer it to a certain plane of action: thus, if a body be in a state of strain, we do not attempt to analyze all the molecular forces with which any one

particle is acted on by its neighbors: but, when we assume a certain plane of section through the point, the stress on this plane at the point becomes recognizable in magnitude and direction; and what the magnitude and direction of the stress at the given point is, depends upon the direction of the plane section chosen, the magnitude and direction differing for different plane sections through the point.

§ 279. **Simple Stress.** — A simple stress is merely a pull or a thrust. Assume a prismatic body, with sides parallel to OX, subjected to a pull in the direction of its length; the magnitude of the pull being P. Assume first a plane section AA normal to the direction of P, and let area of AA be A. Then, if p_x represent the intensity of stress at any point of this plane,

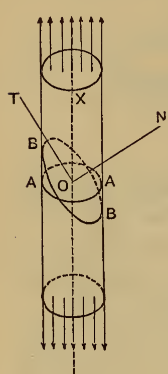


FIG. 297.

$$p_x = \frac{P}{A}.$$

This, which is the intensity of the stress as distributed over a plane normal to its direction, may be called its normal intensity.

On the other hand, if we desire to ascertain the intensity of the stress on the oblique plane BB, making an angle θ with AA, we shall have

$$\text{Area } BB = \frac{A}{\cos\theta}.$$

Hence, if p_r represent the intensity of the stress on this plane in the direction OX, we shall have

$$p_r = \frac{P}{\left(\frac{A}{\cos\theta}\right)} = \frac{P}{A}\cos\theta = p_x \cos\theta. \qquad (1)$$

If we resolve this into two components, acting respectively nor-

mal and tangential to BB, and if we denote the normal intensity by p_n, and the tangential by p_t, we shall have

$$p_n = p_r \cos\theta = p_x \cos^2\theta, \qquad (2)$$

$$p_t = p_r \sin\theta = p_x \cos\theta \sin\theta. \qquad (3)$$

If, now, we assume another oblique plane section, perpendicular to the first, we shall obtain the normal p_n' and the tangential p_t' stress on this plane by substituting for θ, $\frac{\pi}{2} - \theta$; hence we obtain

$$p_n' = p_x \sin^2\theta, \qquad (4)$$

$$p_t' = p_x \cos\theta \sin\theta. \qquad (5)$$

Hence follows

$$p_t' = p_t;$$

or, the tangential components of a simple stress on a pair of planes at right angles to each other are equal.

§ 280. **Compound Stress.** — A compound stress may be accounted to be the resultant of a set of simple stresses, and may be analyzed into different groups of simple stresses.

PROPOSITION. — *Whatever be the external forces applied to a body, if through any point we pass three planes of section at right angles to each other, the tangential components of the stress on any two of these planes in directions parallel to the third must be of equal intensity.*

To prove this proposition, assume three rectangular axes, origin at O, and assume a rectangular parallelopipedical particle, as shown in the figure, so small that we may without appreciable error assume the stress on any one of the faces to be the same as that on the opposite face; resolve these stresses, i.e., the forces exerted upon the faces of the particle by the other parts of the body, into components parallel to the axes.

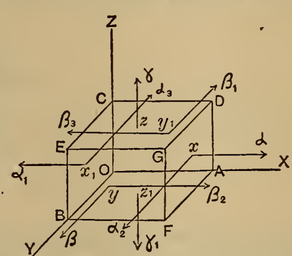


FIG. 298.

Let σ_x = intensity of normal stress on the x plane,
σ_y = intensity of normal stress on the y plane,
σ_z = intensity of normal stress on the z plane,
τ_{xy} = intensity of shearing-stress on x plane in direction OY,
τ_{xz} = intensity of shearing-stress on x plane in direction OZ,
τ_{yx} = intensity of shearing-stress on y plane in direction OX,
τ_{yz} = intensity of shearing-stress on y plane in direction OZ,
τ_{zx} = intensity of shearing-stress on z plane in direction OX,
τ_{zy} = intensity of shearing-stress on z plane in direction OY.

We have thus apparently nine stresses, which must be given, in order to define the stress at the point O completely; but we will now proceed to prove that

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{zy} = \tau_{yz}.$$

In the figure, the only ones of these stresses that are represented are the following: —

$$\begin{aligned} x\alpha &= x_1\alpha_1 = \sigma_x, \\ y\beta &= y_1\beta_1 = \sigma_y, \\ x\alpha_2 &= x_1\alpha_3 = \tau_{xy}, \\ y\beta_2 &= y_1\beta_3 = \tau_{yx}, \\ z\gamma &= z_1\gamma_1 = \sigma_z. \end{aligned}$$

The other four are omitted, in order not to complicate the figure.

Now, it is evident that the total normal force on the face $AFGD$ and the normal force on the face $OBEC$ balance each other independently, and likewise with the other normal forces.

The only forces tending to cause rotation around OZ are the equal and opposite parallel forces τ_{xy} (area $AFGD$), one acting on the face $AFGD$, and the other on the face $OBEC$; and the equal and opposite forces τ_{yx} (area $FBEG$), one acting on the face $FBEG$, and the other on the face $COAD$.

The first pair forms a couple whose moment is τ_{xy} (area $AFGD$) (xx_1), and the second has the moment τ_{yx} (area $FBEG$) (yy_1).

But

$$\text{Area } AFGD = (FA)(zz_1), \quad \text{area } FBEG = (FB)(zz_1)$$

$$\therefore \quad \tau_{xy}(FA)(zz_1)(xx_1) = \tau_{yx}(FB)(zz_1)(yy_1).$$

Cancelling zz_1, we have

$$\tau_{xy}(FA)(xx_1) = \tau_{yx}(FB)(yy_1).$$

But

$$FA = yy_1 \quad \text{and} \quad FB = xx_1$$

$$\therefore \quad \tau_{xy}(xx_1)(yy_1) = \tau_{yx}(xx_1)(yy_1)$$

$$\therefore \quad \tau_{xy} = \tau_{yx},$$

Q. E. D.

In a similar manner we can prove

$$\tau_{xz} = \tau_{zx},$$

$$\tau_{yz} = \tau_{zy}.$$

GENERAL REMARKS.

From what precedes, it follows, that, when we have the six stresses

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz},$$

or, in other words, the normal and tangential components of the stresses on three planes at right angles to each other, given, the state of stress at that point is entirely determined; and, when these are given, it is possible to determine the direction and intensity of the stress on any given plane.

Moreover, if three rectangular axes, OX, OY, and OZ, be assumed, and the direct strains along these axes be given, and also the shearing-strain about these axes, then the direct strain in any given direction can be determined, and also the shearing-strain around this direction as an axis.

The two above-stated propositions furnish two of the fundamental propositions of the theory of elasticity, the third being the determination of the relation between the stresses and the strains.

§ 281. **Relations Governing the Variation of the Stresses at Different Points of a Body.** — If we assume a point whose co-ordinates are (x, y, z), and a small parallelopipedical particle having this point and the point $(x + dx, y + dy, z + dz)$ for the extremities of its diagonal, we shall have, for the edges of this particle, dx, dy, dz, respectively.

Now let the stresses at (x, y, z) be

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz};$$

i.e, σ_x denotes the normal stress on any plane perpendicular to OX, and passing through the point (x, y, z), etc. Then, for the planes passing through $(x + dx, y + dy, z + dz)$, we shall have the stresses

$$\sigma_x + d\sigma_x, \quad \sigma_y + d\sigma_y, \quad \sigma_z + d\sigma_z, \quad \tau_{xy} + d\tau_{xy}, \quad \tau_{xz} + d\tau_{xz}, \quad \tau_{yz} + d\tau_{yz}.$$

We may also have outside forces acting upon the particle in question: if such is the case, let the components of the resultant external force along the axes be respectively

$$Xdxdydz, \qquad Ydxdydz, \qquad Zdxdydz.$$

Now impose the conditions of equilibrium between all the forces acting on the particle. To do this, place equal to zero the algebraic sum of all the forces parallel to each of the axes

respectively, the moment equations having already been incorporated in our demonstration that

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy}.$$

Hence we have three conditions of equilibrium, as follows:—

$$(\sigma_x + d\sigma_x - \sigma_x)dydz + (\tau_{xy} + d\tau_{xy} - \tau_{xy})dxdz + (\tau_{xz} + d\tau_{xz} - \tau_{xz})dxdy + Xdxdydz = 0,$$

$$(\sigma_y + d\sigma_y - \sigma_y)dxdz + (\tau_{xy} + d\tau_{xy} - \tau_{xy})dydz + (\tau_{yz} + d\tau_{yz} - \tau_{yz})dydx + Ydxdydz = 0,$$

$$(\sigma_z + d\sigma_z - \sigma_z)dxdy + (\tau_{yz} + d\tau_{yz} - \tau_{yz})dxdz + (\tau_{xz} + d\tau_{xz} - \tau_{xz})dzdy + Zdxdydz = 0.$$

Hence, reducing, and dividing by $dxdydz$, we have

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xy}}{dy} + \frac{d\tau_{xz}}{dz} + X = 0, \qquad (1)$$

$$\frac{d\tau_{xy}}{dx} + \frac{d\sigma_y}{dy} + \frac{d\tau_{yz}}{dz} + Y = 0, \qquad (2)$$

$$\frac{d\tau_{xz}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + Z = 0. \qquad (3)$$

If the particle is in the interior of the body, so that no external forces act upon it, then $X = Y = Z = 0$.

Equations (1), (2), and (3) give the necessary relations which the variations of stress from point to point must satisfy in order that the conditions of equilibrium may be fulfilled.

§ 282. **Relations between the Stresses and Strains.**— Before proceeding to the general problems of composition of stresses, i.e., of determining from a sufficient number of data the stress upon any plane, we will first discuss the relations between the stresses and the strains; and we will confine ourselves to those bodies that are homogeneous, and of the same elasticity throughout.

From what we have already seen, if to a straight rod whose cross-section is A there be applied a pull P in the direction of

its length, the intensity of the stress on the cross-section wi be

$$\sigma = \frac{P}{A};$$

and, if E be the tensile modulus of elasticity of the material of the rod, the strain in a direction at right angles to the cross-section, or, in other words, in the direction of the pull, will be

$$\epsilon = \frac{\sigma}{E}.$$

Now, another fact, which we have thus far taken no account of, is, that although there is no stress in a direction at right angles to the pull, or, in other words, although a section at right angles to the above-stated cross-section will have no stress upon it, yet there will be a strain in all directions at right angles to the direction of the pull: and this strain will be, for any direction at right angles to the pull,

$$\epsilon_1 = -\frac{\epsilon}{m},$$

being of the opposite kind from ϵ; thus, if ϵ is extension, ϵ_1 is compression, and *vice versa.*

Hence, if, at any point O of such a rod, we assume three rectangular axes, of which OX is in the direction of the pull, and we use the notation already adopted, we shall have

$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0,$$

$$\epsilon_x = \frac{\sigma_x}{E}, \quad \epsilon_y = \epsilon_z = -\frac{1}{m}\frac{\sigma_x}{E} = -\frac{\epsilon_x}{m},$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0.$$

MODULUS OF SHEARING ELASTICITY.

In the case of direct tension or compression, when only a simple stress is applied, we have defined the modulus of elasticity as the ratio of the stress to the strain in its own direction.

Adopting a similar definition in the case of shearing, we shall have

$$\frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xz}}{\gamma_{xz}} = \frac{\tau_{yz}}{\gamma_{yz}} = G,$$

where G is the modulus of shearing elasticity.

GENERAL RELATIONS BETWEEN STRESSES AND STRAINS.

Whenever a compound stress acts on a body at a given point, let the stresses be

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz};$$

then we shall have, for the strain in the direction OX,

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_y}{E} - \frac{1}{m}\frac{\sigma_z}{E}, \qquad \gamma_{xy} = \frac{\tau_{xy}}{G},$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{1}{m}\frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_z}{E}, \qquad \gamma_{xz} = \frac{\tau_{xz}}{G},$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{1}{m}\frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_y}{E}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}.$$

This enables us to determine the strains in terms of the stresses, as soon as the values of E, G, and m are known from experiment, for the material under consideration.

If, on the other hand, the stresses be required in terms of the strains, we can consider ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{xz}, γ_{yz}, as known, and determine σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz}, from the above equations.

We thus obtain

$$E\epsilon_x = \sigma_x - \frac{\sigma_y + \sigma_z}{m}, \qquad (1)$$

$$E\epsilon_y = \sigma_y - \frac{\sigma_x + \sigma_z}{m}, \qquad (2)$$

$$E\epsilon_z = \sigma_z - \frac{\sigma_x + \sigma_y}{m}; \qquad (3)$$

and, by solving these equations for the stresses, we have

$$\sigma_x = \frac{m}{m+1}E\left(\epsilon_x + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (4)$$

$$\sigma_y = \frac{m}{m+1}E\left(\epsilon_y + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (5)$$

$$\sigma_z = \frac{m}{m+1}E\left(\epsilon_z + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (6)$$

and also

$$\tau_{xy} = G\gamma_{xy}, \quad (7) \qquad \tau_{xz} = G\gamma_{xz}, \quad (8) \qquad \tau_{yz} = G\gamma_{yz}. \quad (9)$$

These equations express the stresses in terms of the strains.

The three last might be written as follows (see § 276) : —

$$\tau_{xy} = G\left(\frac{d\xi}{dy} + \frac{d_\eta}{dx}\right), \qquad (10)$$

$$\tau_{xz} = G\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right), \qquad (11)$$

$$\tau_{yz} = G\left(\frac{d_\eta}{dz} + \frac{d\zeta}{dy}\right), \qquad (12)$$

as these forms are often convenient.

§ 283. Case when $\sigma_z = 0$. — Inasmuch as there are many cases in practice where the stress is all parallel to one plane, and where, consequently, the stress on any plane parallel to this plane has no normal component, it will be convenient to have the reduced forms of equations (4), (5), and (6) which apply in this case.

Let the plane to which the stresses are parallel be the Z plane; then $\sigma_z = 0$. Then equation (6) becomes

$$\epsilon_z + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m - 2} = 0$$

$$\therefore \quad \epsilon_z = - \frac{\epsilon_x + \epsilon_y}{(m - 2)(m - 1)};$$

and, substituting this value of ϵ_z in (4) and (5), and reducing, we obtain

$$\sigma_x = \frac{m}{m + 1} E\left(\epsilon_x + \frac{\epsilon_x + \epsilon_y}{m - 1}\right), \qquad (1)$$

$$\sigma_y = \frac{m}{m + 1} E\left(\epsilon_y + \frac{\epsilon_x + \epsilon_y}{m - 1}\right), \qquad (2)$$

which are the required forms.

The other three equations, viz., —

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{xz} = G\gamma_{xz}, \quad \tau_{yz} = G\gamma_{yz},$$

remain the same as before.

§ 284. **Values of** E, G, **and** m. — These three constants need to be known, to use the relations developed above.

1°. As to E, this is the modulus of elasticity for tension, and has been determined experimentally for the various materials, as has been already explained. Moreover, it has also been shown experimentally, that, with moderate loads, the modulus of elasticity for compression is nearly identical with that for tension in cast-iron, wrought-iron, and steel.

2°. As to m, in those few applications that Professor Rankine gives of his theory of internal stress, such as the case of combined twisting and bending, he determines the greatest intensity of the stress acting; and his criterion is, that this shall be kept within the working-strength of the material. This is equivalent to assuming $m = \infty$. The more modern writers,

such as Grashof and others, take account of the fact that m has a finite value, and make their criterion that the greatest strain shall be kept within the quotient obtained by dividing the working-strength by the modulus of elasticity of the material.

Thus, if f is the working-strength, and σ_1 the greatest stress, and ϵ_1 the greatest strain, Rankine's criterion of safety is

$$\sigma_1 \leqq f;$$

whereas the more modern criterion is

$$E\epsilon_1 \leqq f.$$

The resulting formulæ differ in each case; and, as has been stated, those of Rankine could be derived from the more general ones by making

$$\frac{1}{m} = 0 \quad \text{or} \quad m = \infty,$$

which is never the case.

As to the value of m, but few experiments have been made. Those of Wertheim give, for brass, 2.94; for wrought-iron, 3.64.

The values $m = 3$ and $m = 4$ are those most commonly adopted, so that

$$\frac{1}{m} = \frac{1}{3} \quad \text{or} \quad \frac{1}{m} = \frac{1}{4}.$$

3°. The value of G, the shearing-modulus of elasticity, i.e., the ratio of the stress to the strain for shearing, has been determined experimentally, and has generally been found to be about two-fifths that for tension.

According to the theory of elasticity, we must have

$$G = \frac{1}{2}\frac{m}{m+1}E,$$

as may be proved as follows:—

Assume a square particle whose side is a, and let a simple normal stress σ be applied at the face AB; then we shall have, on the planes BD and AC, a shearing-stress (§ 279)

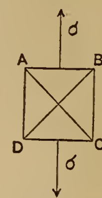


FIG. 299.

$$\tau = \sigma \sin 45^\circ \cos 45^\circ = \tfrac{1}{2}\sigma.$$

On the other hand, if we let

$$\epsilon = \frac{\sigma}{E},$$

the strain of the particle in the direction AD will be ϵ, while that in the direction AB will be $-\frac{\epsilon}{m}$; hence the particle will become a rectangle, the side AD changing its length from a to $a + a\epsilon$, and side AB changing from a to $a - \frac{a\epsilon}{m}$.

The diagonals will no longer be at right angles to each other; and, if we denote by α the angle by which their angle differs from a right angle, we shall have, for the shearing-strain on the planes AC and BD,

$$\gamma = \tan \alpha.$$

But, after the distortion, the angle ADB will become

$$\frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)$$

$$\therefore \quad \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{1 - \tan\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}} = \frac{a - \frac{a\epsilon}{m}}{a + a\epsilon} = \frac{1 - \frac{\epsilon}{m}}{1 + \epsilon};$$

therefore, dividing, and carrying the division only to terms of the first degree, we have

$$1 - 2\tan\frac{\alpha}{2} = 1 - \left(1 + \frac{1}{m}\right)\epsilon$$

$$\therefore \quad 2\tan\left(\frac{\alpha}{2}\right) = \frac{m+1}{m}\epsilon.$$

But

$$\gamma \doteq \tan\alpha = 2\tan\frac{\alpha}{2} \text{ nearly}$$

$$\therefore \quad \gamma = \frac{m+1}{m}\epsilon$$

$$\therefore \quad \frac{\tau}{\gamma} = \frac{\frac{1}{2}\sigma}{\frac{m+1}{m}\epsilon} = \frac{1}{2}\frac{m}{m+1}\left(\frac{\sigma}{\epsilon}\right);$$

but

$$\frac{\tau}{\gamma} = G \quad \text{and} \quad \frac{\sigma}{\epsilon} = E$$

$$\therefore \quad G = \frac{1}{2}\frac{m}{m+1}E.$$

§ 285. **Conjugate Stresses.** — If the stress on a given plane at a given point of a body be in a given direction, the stress at the same point on a plane parallel to that direction will be parallel to the given plane. Let *YOY* represent, in section, a given plane, and let the stress on that plane be in the direction *XOX*.

Consider a small prism *ABCD* within a body, the sides of whose base are parallel respectively to *XOX* and *YOY*. The forces on the plane *AB* are counterbalanced by the forces on the plane *DC*; the resultants of each of these sets being equal and opposite, and acting along a line passing through *O*. Hence the forces acting on the planes *AD* and *BC* must be balanced entirely independently of any of the forces on *AB* or *DC*: and this can be the case only when their direction is parallel to *YOY*; for otherwise their resultants, though equal in magnitude and opposite in direction, would not be directly opposite, but would form a couple, and, as there is no equal and opposite couple furnished by the forces on the other faces, equilibrium could not exist under this supposition.

§ 286. **Composition of Stresses.** — The general problem of the composition of stresses may be stated as follows: —

Knowing the stresses at a given point of a strained body on three planes passing through that point, to find the stress at the same point on any other plane, also passing through the same point. The stresses on the three given planes are not entirely independent; in other words, we could not give the stresses on these three planes, in magnitude and direction, at random, and expect to find the problem a possible one. Thus, suppose that the planes are at right angles to each other, we have already seen that we have the right to give their three normal components, σ_x, σ_y, and σ_z, and the three tangential, τ_{xy}, τ_{yz}, and τ_{xz}, and that $\tau_{yx} = \tau_{xy}$, etc. We will now proceed to special cases.

§ 287. **Problem.** — Given the three planes of action of the stress as the x, y, and z plane respectively, and given the normal and tangential components of the stresses on these planes, viz., σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz}, to find the intensity and direction of the stress on a plane whose normal makes with OX, OY, and OZ the angles α, β, and γ respectively, where, of course, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Draw the line ON, making angles α, β, and γ with OX, OY, and OZ respectively; then draw near O the plane ABC perpendicular to ON. It has the direction of the required plane, and cuts off intercepts OA, OB, and OC on the axes; and, moreover, we shall have, from trigonometry, the relations,

Area $BOC = (ABC) \cos\alpha$,
Area $AOC = (ABC) \cos\beta$,
Area $AOB = (ABC) \cos\gamma$.

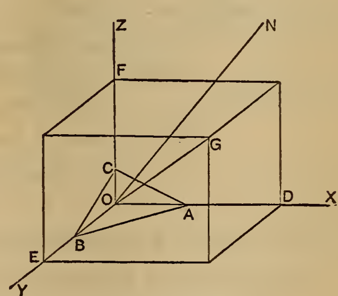


FIG. 300.

Now consider the conditions of equilibrium of the tetrahedron $OABC$. The stress on ABC must be equal and directly

opposed to the resultant of the stresses on the three faces *AOC*, *BOC*, and *AOB*. Now let us proceed to find this resultant.

In the direction *OX* we have the force

$$\sigma_x(BOC) + \tau_{xz}(AOB) + \tau_{xy}(AOC)$$
$$= (ABC)(\sigma_x \cos\alpha + \tau_{xy}\cos\beta + \tau_{xz}\cos\gamma).$$

Lay off *OD* to represent this quantity. In the same way represent the force in the direction *OY* by

$$OE = \sigma_y(AOC) + \tau_{yz}(BOA) + \tau_{xy}(BOC)$$
$$= (ABC)(\sigma_y \cos\beta + \tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma),$$

and that in the direction *OZ* by

$$OF = \sigma_z(AOB) + \tau_{yz}(BOC) + \tau_{xz}(AOC)$$
$$= ABC(\sigma_z\cos\gamma + \tau_{yz}\cos\alpha + \tau_{xz}\cos\beta).$$

Now compound these three forces, and we have, as resultant force,

$$R = OG = \sqrt{OD^2 + OE^2 + OF^2},$$

and as resultant intensity

$$\sigma = \frac{R}{ABC} = \frac{\sqrt{OD^2 + OE^2 + OF^2}}{ABC}$$
$$= \sqrt{\{(\sigma_x\cos\alpha + \tau_{xy}\cos\beta + \tau_{xz}\cos\gamma)^2 + (\sigma_y\cos\beta + \tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma)^2 + (\sigma_z\cos\gamma + \tau_{yz}\cos\alpha + \tau_{xz}\cos\beta)^2\}}$$
$$= \sqrt{\{\sigma_x^2\cos^2\alpha + \sigma_y^2\cos^2\beta + \sigma_z^2\cos^2\gamma + \tau_{xy}^2(\cos^2\alpha + \cos^2\beta) + \tau_{xz}^2(\cos^2\alpha + \cos^2\gamma) + \tau_{yz}^2(\cos^2\beta + \cos^2\gamma) + 2\sigma_x(\tau_{xy}\cos\beta + \tau_{xz}\cos\gamma)\cos\alpha + 2\sigma_y(\tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma)\cos\beta + 2\sigma_z(\tau_{yz}\cos\alpha + \tau_{xz}\cos\beta) + 2\tau_{xy}\tau_{xz}\cos\beta\cos\gamma + 2\tau_{xy}\tau_{yz}\cos\alpha\cos\gamma + 2\tau_{yz}\tau_{xz}\cos\alpha\cos\beta\}};$$

the direction being given by the angles, α_r, β_r, and γ_r, where

$$\cos\alpha_r = \frac{OD}{R}, \quad \cos\beta_r = \frac{OE}{R}, \quad \cos\gamma_r = \frac{OF}{R}.$$

§ 288. **Stresses Parallel to a Plane.** — To solve the same problem when there is no stress in the direction OZ, and when the new plane is perpendicular to XOY, or, in other words, in the case when the planes of action are all perpendicular to one plane, to which the stresses are all parallel: we then have

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad \text{and} \quad \beta = 90^\circ - \alpha,$$

and hence

$$\sigma = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha + \tau_{xy}^2 + 2(\sigma_x + \sigma_y)\tau_{xy}\cos\alpha\sin\alpha}.$$

Or we may proceed as follows: —

Let the normal intensity of the stress on the x plane (i.e., that perpendicular to OX) be σ_x, that on the y plane σ_y, and the tangential intensity τ_{xy}. Let ON be the direction of the normal to the plane on which the stress is to be determined, and let the angle $XON = \alpha$. Then let the plane AB be drawn perpendicular to ON, and let us consider the equilibrium of the forces exerted by the other parts of the body upon the triangular prism whose base is ABO and altitude unity.

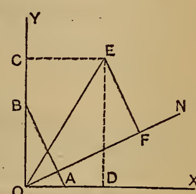


FIG. 301.

If we compound the forces acting on the faces AO and OB, we shall have, in their resultant, the total force on the face AB in magnitude and direction. Moreover, we have the relations,

Area OB = area $AB\cos\alpha$ and Area OA = area $AB\sin\alpha$.

Force acting on OB in direction $OX = \sigma_x(OB)$,

Force acting on OB in direction $OY = \tau_{xy}(OB)$,

Force acting on OA in direction $OX = \tau_{xy}(OA)$,

Force acting on OA in direction $OY = \sigma_y(OA)$.

Hence, if we lay off

$$OD = \sigma_x(OB) + \tau_{xy}(OA) \quad \text{and} \quad OC = \sigma_y(OA) + \tau_{xy}(OB),$$

then will OD represent the total force acting in the direction OX, and OC will represent the total force acting in the direction OY.

Compounding these, we shall have OE as the resultant total force on the face AB, and $\frac{OE}{AB}$ will represent its intensity.

To deduce the analytical values, we have

$$OD = \sigma_x(OB) + \tau_{xy}(OA) = (AB)(\sigma_x \cos\alpha + \tau_{xy}\sin\alpha),$$
$$OC = \sigma_y(OA) + \tau_{xy}(OB) = (AB)(\sigma_y \sin\alpha + \tau_{xy}\cos\alpha)$$

$$\therefore \quad OE = \sqrt{OD^2 + OC^2}$$
$$= AB\sqrt{(\sigma_x \cos\alpha + \tau_{xy}\sin\alpha)^2 + (\sigma_y \sin\alpha + \tau_{xy}\cos\alpha)^2}$$
$$= AB\sqrt{\{\sigma_x^2\cos^2\alpha + \sigma_y^2\sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha(\sigma_x + \sigma_y) + \tau_{xy}^2(\cos^2\alpha + \sin^2\alpha)\}}.$$

Or, if σ_r represent the resultant intensity on the plane AB, and α_r the angle this resultant makes with OX, we shall have

$$\sigma_r = \sqrt{\{\sigma_x^2\cos^2\alpha + \sigma_y^2\sin^2\alpha + 2\tau_{xy}(\sigma_x + \sigma_y)\cos\alpha\sin\alpha + \tau_{xy}^2\}}, \quad (1)$$

and

$$\cos\alpha_r = \frac{OD}{OE} \quad \text{and} \quad \sin\alpha_r = \frac{OC}{OE}.$$

Moreover, it is sometimes desirable to resolve the stress into normal and tangential components. If this be done, and if σ and τ represent respectively the normal and tangential components, we shall have

$$\sigma = \frac{OF}{AB} \quad \text{and} \quad \tau = \frac{EF}{AB};$$

but

$$OF = OD \cos\alpha + ED \sin\alpha \quad \text{and} \quad EF = ED \cos\alpha - OD \sin\alpha$$

$$\therefore \quad \sigma = \frac{OD}{AB}\cos\alpha + \frac{OC}{AB}\sin\alpha$$

$$= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha \qquad (2)$$

and

$$\tau = \frac{OC}{AB}\cos\alpha - \frac{OD}{AB}\sin\alpha$$

$$= (\sigma_y - \sigma_x)\cos\alpha\sin\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

$$= \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha. \qquad (3)$$

§ 289. **Principal Stresses.** — It will next be shown, that, whatever be the state of stress in a body, provided the stresses are all parallel to one plane, the planes of action being all taken perpendicular to this plane, there are always two planes, at right angles to each other, on which there is no tangential stress; these two planes being called the planes of principal stress, the stress on one of these planes being greater, and the other less, than that on any other plane through the same point.

To prove the above, it will be necessary only in the last case, which is a perfectly general one, to determine for what values of α the value of τ is zero, and whether these values of α are always possible. We have

$$\tau = \frac{\sigma_y - \sigma_x}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha:$$

and, if we put this equal to zero, we have

$$\frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y};$$

and this gives us, for all values of σ_x, σ_y, and τ_{xy}, two possible values for 2α, differing from each other by 180°, hence two values for α differing by 90°. Hence follows the first part of the proposition.

The latter part — that these are the planes of the greatest and least stresses — will be shown by differentiating the value of σ_r^2, and putting the first differential co-efficient equal to zero; and, as this gives us

$$\begin{aligned} &2\sigma_x^2 \cos\alpha \sin\alpha + 2\sigma_y^2 \cos\alpha \sin\alpha \\ &\quad + 2\tau_{xy}(\sigma_x + \sigma_y)(\cos^2\alpha - \sin^2\alpha) \\ &\qquad = 2(\sigma_x + \sigma_y)\{(\sigma_y - \sigma_x)\cos\alpha \sin\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)\} \\ &\qquad = 0, \end{aligned}$$

therefore we have the same condition for the maximum and minimum stresses as we have for the planes of no tangential stress.

It follows that the determination of the greatest and least stresses at any one point of a body is identical with the determination of the principal stresses; and it will be necessary, whenever the stresses on any two planes are given, to be able to determine the principal stresses, as one of these is the greatest stress at that point of the body, and the other the least.

§ 290. **Determination of Principal Stresses.** — When the stress is all parallel to one plane, viz., the z plane, and when the stresses on two planes at right angles to each other are given, i.e., their normal and tangential components, we may be required to determine the principal stresses. Proceed as follows: Given normal stresses on X and Y planes respectively, σ_x and σ_y, and tangential stress on each plane τ_{xy}, to find principal stresses.

From § 288 we have, for a plane whose normal makes an angle α with OX,

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha + 2\tau_{xy}(\sigma_x + \sigma_y)\cos\alpha \sin\alpha + \tau_{xy}^2}, \quad (1)$$

$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \cos\alpha \sin\alpha, \quad (2)$$

$$\tau = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) - (\sigma_x - \sigma_y)\cos\alpha \sin\alpha, \quad (3)$$

or

$$\tau = \tau_{xy}\cos 2\alpha - \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha. \qquad (4)$$

Now, the condition that the plane shall be a plane of principal stress is, that $\tau = 0$. Hence write

$$\tau_{xy}(\cos^2\alpha - \sin^2\alpha) - (\sigma_x - \sigma_y)\cos\alpha\sin\alpha = 0,$$

find α, and substitute its value in (2), and we shall have the principal stresses. The operation may be performed as follows; viz., —

From (3) we have

$$(a) \quad 2\cos^2\alpha - 1 = \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha$$

$$\therefore \quad \cos^2\alpha = \frac{1}{2}\left\{1 + \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha\right\}.$$

$$(b) \quad 1 - 2\sin^2\alpha = \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha$$

$$\therefore \quad \sin^2\alpha = \frac{1}{2}\left\{1 - \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha\right\}.$$

Hence

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\cos\alpha\sin\alpha}{2}\left\{\frac{\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2}{\tau_{xy}} + 4\tau_{xy}\right\}$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\cos\alpha\sin\alpha}{2\tau_{xy}}\left\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\right\}.$$

But we have, since (4) equals zero,

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\therefore \quad \sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{\pm 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}.$$

Hence substitute for $\cos\alpha\sin\alpha$ its value, and

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \qquad (5)$$

which gives us the magnitudes of the principal stresses; the plus sign corresponding to the greater, and the minus sign to the less.

EXAMPLES.

1. Let, in the last section, $\sigma_y = 0$, and find the principal stresses. Here we have

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x}$$

and

$$\sigma_n = \frac{\sigma_x}{2} \pm \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}.$$

2. Given two principal stresses, to find the stress on a plane whose normal makes an angle α with OX.

In this case $\tau_{xy} = 0$.

Hence we have the case of § 288, with the reduction of making $\tau_{xy} = 0$. We may therefore obtain the result by substitution in the results of § 288, or we may proceed as follows: —

(*a*) Find stress on new plane in direction OX; this will be, § 279,

$$\sigma_x \cos\alpha.$$

(*b*) Find stress on new plane in direction OY; this will be, § 279,

$$\sigma_y \sin\alpha.$$

(*c*) Compound the two, and the resultant is

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha}. \qquad (1)$$

(*d*) Normal component of $\sigma_x \cos\alpha$ is

$$\sigma_x \cos^2\alpha.$$

(*e*) Normal component of $\sigma_y \sin\alpha$ is

$$\sigma_y \sin^2\alpha.$$

(*f*) Add, and we have, for normal stress,

$$\sigma_n = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha. \qquad (2)$$

(*g*) Tangential component of $\sigma_x \cos \alpha$ is

$$-\sigma_x \cos \alpha \sin \alpha.$$

(*h*) Tangential component of $\sigma_y \sin \alpha$ is

$$+\sigma_y \cos \alpha \sin \alpha.$$

(*k*) Add, and we have, for tangential stress,

$$\tau = (\sigma_y - \sigma_x) \cos \alpha \sin \alpha. \qquad (3)$$

§ 291. **Ellipse of Stress.** — In the case above, i.e., when the two principal stresses are σ_x and σ_y respectively, if we represent them graphically by $OA = \sigma_x$ and $OB = \sigma_y$, and let CD be the plane on which the stress is required, its normal making with OX the angle $XON = \alpha$, then, from what has been shown, if OR represent the intensity of the resultant stress on this plane, we shall have

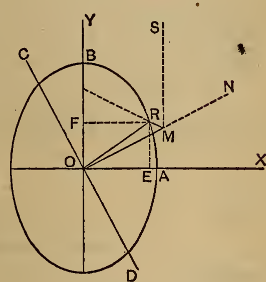


Fig. 302.

$$OR = \sigma_r = \sqrt{\sigma_x^2 \cos^2 \alpha + \sigma_y^2 \sin^2 \alpha};$$

and, moreover,

$$OE = \sigma_x \cos \alpha, \qquad OF = \sigma_y \sin \alpha.$$

If we denote these by x and y respectively, letting (x, y) be the point R, i.e., the extremity of the line representing the stress on AB, then

$$x = \sigma_x \cos \alpha, \qquad y = \sigma_y \sin \alpha,$$

$$\therefore \left(\frac{x}{\sigma_x}\right)^2 = \cos^2 \alpha \quad \text{and} \quad \left(\frac{y}{\sigma_y}\right)^2 = \sin^2 \alpha$$

$$\therefore \frac{x^2}{\sigma_x^2} = \frac{y^2}{\sigma_y^2} = 1,$$

which is the equation of an ellipse whose semi-axes are σ_x and σ_y respectively; hence the stress on any plane will be represented by some semi-diameter of the ellipse.

SPECIAL CASES.

I. When the two given stresses are equal, or $\sigma_x = \sigma_y$, then

$$\sigma_r = \sqrt{a_x^2 \cos^2 \alpha + \sigma_y^2 \sin^2 \alpha} = \sigma_x,$$

and

$$\cos \alpha_r = \frac{\sigma_x \cos \alpha}{\sigma_x} = \cos \alpha \qquad \text{and} \qquad \sin \alpha_r = \sin \alpha;$$

therefore the stress is of the same intensity on all planes, and always normal to the plane.

II. When the two given stresses are equal in magnitude but opposite in sign, or $\sigma_y = -\sigma_x$, then

$$\sigma_r = \sigma_x.$$

But

$$\cos \alpha_r = \cos \alpha \qquad \text{and} \qquad \sin \alpha_r = -\sin \alpha,$$

hence

$$\alpha_r = -\alpha;$$

therefore the stress on any plane whose normal makes an angle α with OX is of the same intensity σ_x, but makes an angle equal to α with OX on the side opposite to that of the normal to the plane.

Problem. — A pair of principal stresses being given, to find the positions of the planes on which the shear is greatest.

Solution. — Let $\tau = (\sigma_y - \sigma_x) \sin \alpha \cos \alpha = \max.$

Therefore differentiate, and

$$\cos^2 \alpha - \sin^2 \alpha = 0$$

$$\therefore \quad \cos \alpha = \pm \sin \alpha \qquad\qquad \therefore \quad \alpha = 45^\circ \text{ or } 135^\circ.$$

§ 292. **Some Special Modes of Solution of some Problems.** — The case where two principal stresses, σ_x and σ_y, are given, to find the stress on any plane whose normal makes an angle α with OX, may be solved as follows, graphically: —

Let, Fig. 302, $\sigma_x = OA$, and $\sigma_y = OB$. Let $XON = \alpha$.

Now,

$$\sigma_y = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2},$$

$$\sigma_x = \frac{\sigma_y + \sigma_x}{2} - \frac{\sigma_y - \sigma_x}{2}.$$

Hence, instead of proceeding at once to find the resultant stress on CD due to the action of σ_x and σ_y, we may first find that due to the action of the two equal principal stresses of the same kind,

$$\frac{\sigma_y + \sigma_x}{2},$$

then that due to the pair

$$\frac{\sigma_y - \sigma_x}{2} \quad \text{and} \quad -\frac{\sigma_y - \sigma_x}{2},$$

and then the resultant of these two resultants.

The first resultant will be evidently laid off on ON, and equal in magnitude to $\dfrac{\sigma_y + \sigma_x}{2}$; hence let $OM = \dfrac{\sigma_y + \sigma_x}{2}$, and OM will be the first resultant.

The second resultant will be of magnitude $\dfrac{\sigma_y - \sigma_x}{2}$, and will have a direction MR such that the angle $NMS = SMR$.

Hence, laying off this angle, and making $MR = \dfrac{\sigma_y - \sigma_x}{2}$, we shall have for the final resultant, OR, as before.

This construction will be useful in the following case: —

To find the most oblique stress, we must find for what value of α the angle MOR is greatest. This will be made

evident if we observe, that, for all positions of the plane, the triangle OMR has always $OM = \frac{\sigma_y + \sigma_x}{2}$, and $MR = \frac{\sigma_y - \sigma_x}{2}$; both of constant length. Hence, if, with M as a centre and MR as a radius, a circle were described, and a tangent were drawn from O to this circle, the point of tangency being taken for R, then will OR be the most oblique stress; i.e., the stress is most oblique when $ORM = 90°$. Therefore greatest obliquity =

$$\sin^{-1}\frac{\sigma_y - \sigma_x}{\sigma_y + \sigma_x}.$$

§ 293. **Converse of the Ellipse of Stress.** — The converse of the ellipse of stress would be the following problem: Given any two planes passing through the point in question; given the intensities and directions of the stresses on these planes, — to find the principal stresses in magnitude and in direction.

The first step to be taken is, to assure ourselves that the conditions are not incompatible, as they are liable to be if the planes and stresses are taken at random. The test of this question is, to resolve each stress into two components, respectively parallel to the two planes; and, if the conditions are not inconsistent, the component of each stress along the plane on which it acts must be equal. The proof of this statement can be made in a similar way to that used in proving that the intensities of the shearing-stresses on two planes at right angles to each other are equal. If, upon applying this test, we find that the conditions are not inconsistent, we may proceed as follows: —

Suppose CD (Fig. 302) were the given plane, and OR the stress upon it, and suppose the position of the principal axes, OX and OY, and, indeed, all the rest of the figure, were absent, i.e., not known. Now, we can easily draw the normal ON; and, if we could determine upon it the point M such that OM

should be one-half the sum of the principal stresses, we should be able to reproduce the whole figure. Hence we will devote ourselves to the determination of the position of the point M.

Let $OB = p =$ stress on plane CD.

Let stress on the other given plane be p_1.

Let $NOR = \theta =$ obliquity of p.

Let $\theta_1 =$ obliquity of p_1.

Then we have

$$MR^2 = OR^2 + OM^2 - 2OM.OR\cos\theta;$$

or, if σ_x and σ_y denote the (unknown) magnitudes of the principal stresses,

$$\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 = p^2 + \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - 2\left(\frac{\sigma_x + \sigma_y}{2}\right)p\cos\theta. \quad (1)$$

From the triangle constructed in the same way, with the stress on the other plane, we should have

$$\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 = p_1^2 + \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - 2\left(\frac{\sigma_x + \sigma_y}{2}\right)p_1\cos\theta_1. \quad (2)$$

Hence, by subtraction,

$$p^2 - p_1^2 = 2\left(\frac{\sigma_x + \sigma_y}{2}\right)(p\cos\theta - p_1\cos\theta_1) \quad (3)$$

$$\therefore \quad \frac{\sigma_x + \sigma_y}{2} = \frac{p^2 - p_1^2}{2(p\cos\theta - p_1\cos\theta_1)}. \quad (4)$$

Having thus found $\frac{\sigma_x + \sigma_y}{2}$, we can next find, from either (1) or (2), the value of $\frac{\sigma_y - \sigma_x}{2}$.

Now, therefore, we know OM and MR, and hence we can lay off this value of OM, and complete the triangle OMR; then

bisect the angle NMR, and the line MS is parallel to the axis of greater principal stress. Hence draw OY parallel to MS, and OX perpendicular to OY, and lay off on OY

$$OB = \sigma_y = \frac{OM + MR}{2},$$

and on OX

$$OA = \sigma_x = \frac{OM - MR}{2},$$

and the problem is solved.

§ 294. **Case of any Stresses in Space.** — In the case of stress which is not all parallel to one plane, we should find that it is always possible, no matter how complicated the state of stress in a body, to find three planes at right angles to each other on which the stress is wholly normal, these being the principal stresses; and a number of propositions follow analogous to those for stresses all parallel to one plane. The discussions of these cases become very complex, and will not be treated here.

§ 295. **Some Applications.** — The following are some of the practical cases which require the theory of elasticity for their solution.

§ 296. **Combined Twisting and Bending.** — This is the case very generally in shafting, as the twist is necessary for the transmission of power, and the bending is due to the weight of the pulleys and shafting, and the pull of the belts, this being especially so when there are pulleys elsewhere than close to the hangers; also in overhanging shafts, in crank-shafts, etc.

Thus far we have no tests of shafting under combined twisting and bending, and therefore the methods used for calculating such shafts vary. With many it is the practice to compute their proper size from the twisting-moment only, but to make up for the bending by using a large factor of safety, the magnitude of this factor depending upon how much

the computer imagines the shaft will be weakened by the particular bending to which it is subjected.

With others it is customary to compute the deflections, under the greatest belt-pulls that can come upon it, by the principles of transverse stress, without any reference to the torsion, and to so determine it that the deflection computed in this way should not exceed $\frac{1}{1200}$ or $\frac{1}{1600}$ of the span.

On the other hand, Unwin and some others give the formulæ, which will be developed here for combined twisting and bending, as deduced by the theory of elasticity. This formula has not, as yet, been very extensively used; and its constants are taken from experiments on tension or torsion alone, and not on a combination of the two. It is to be hoped that we may some time have some experiments on such a combination. We will now proceed to deduce a formula for the greatest intensity of the stress at any point of the shaft.

For this purpose

Let M_1 = bending-moment at any section.

M_2 = twisting-moment at the same section.

I_1 = moment of inertia about neutral axis for bending.

I_2 = moment of inertia about axis of shaft.

r = distance from axis to outside fibre.

Then, if we denote by σ the greatest intensity of the stress due to bending, and by τ the greatest intensity of the stress due to twisting, we have,

$$\sigma = \frac{M_1 r}{I_1}. \qquad (1) \qquad\qquad \tau = \frac{M_2 r}{I_2}. \qquad (2)$$

For a circular or hollow circular shaft,

$$I_2 = 2I_1;$$

hence

$$\sigma = \frac{M_1 r}{I_1}. \qquad (3) \qquad\qquad \tau = \frac{M_1 r}{2I_1}. \qquad (4)$$

Then, at a point at the outside of the shaft in the section under consideration, we shall have, —

1°. On a plane normal to the axis,

(*a*) a normal stress σ,

(*b*) a shearing-stress τ.

2°. On a plane in the direction of the axis,

(*a*) a normal stress σ.

(*b*) a shearing-stress τ.

We thus have the case solved in Example I., § 290.

If, therefore, the greatest and least principal stresses be denoted by σ_1 and σ_2 respectively, we shall have

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2}, \tag{5}$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} + \tau^2}. \tag{6}$$

But, if ϵ_1 and ϵ_2 denote the strains in the directions of the principal stresses, we have

$$E\epsilon_1 = \sigma_1 - \frac{\sigma_2}{m}, \qquad E\epsilon_2 = \sigma_2 - \frac{\sigma_1}{m};$$

Hence, substituting for σ_1 and σ_2 their values, we have

$$E\epsilon_1 = \frac{m-1}{2m}\sigma + \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2}, \tag{7}$$

$$E\epsilon_2 = \frac{m-1}{2m}\sigma - \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2}. \tag{8}$$

We then have, for the greatest stress on any fibre, the greater of the two quantities (7) and (8); and this should not at any section of the shaft exceed the working-strength of the material for tension.

The greater of the two is $E\epsilon_1$: hence we should have, if $f =$ greatest stress,

$$\frac{m-1}{2m}\sigma + \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2} = f. \tag{9}$$

If, now, we let $m = 4$, as is commonly done, we have

$$\tfrac{3}{8}\sigma + \tfrac{5}{8}\sqrt{\sigma^2 + 4\tau^2} = f, \tag{10}$$

this being the formula given by Grashof and others for combined twisting and bending.

On the other hand, Rankine puts the value of σ_1 in (5) equal to f, and hence Rankine's formula is

$$\frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = f. \tag{11}$$

This might be derived from (9) by making $m = \infty$ instead of $m = 4$.

The formulæ developed above are applicable to any section.

APPLICATION TO CIRCULAR AND HOLLOW CIRCULAR SHAFTS.

Substituting for σ and τ in (10) the values from (3) and (4), we should obtain

$$\frac{r}{I_1}\left\{\frac{3}{8}M_1 + \frac{5}{8}\sqrt{M_1^2 + M_2^2}\right\} = f, \tag{12}$$

which is Grashof's formula, and is given by Unwin and others; and, substituting in (11) instead, we should have

$$\frac{r}{2I_1}\left(M_1 + \sqrt{M_1^2 + M_2^2}\right) = f. \tag{13}$$

Equation (12) is equivalent to the following rule:—

Calculate the shaft as though it were subjected to a bending-moment

$$M_0 = \tfrac{3}{8}M_1 + \tfrac{5}{8}\sqrt{M_1^2 + M_2^2};$$

and equation (13) is equivalent to the following rule:—

Calculate the shaft as though it were subjected to a bending-moment

$$M_0 = \frac{M_1}{2} + \frac{1}{2}\sqrt{M_1^2 + M_2^2}.$$

Now, if, as is usually the case, the section where the greatest bending-moment acts is also subjected to the greatest twisting-moment, it will only be necessary to put for M_1 the greatest bending-moment, and for M_2 the greatest twisting-moment.

§ 297. **Thick Hollow Cylinders subjected to a Uniform Normal Pressure.**—Let inside radius $= r$, outside radius $= r_1$, length of portion under consideration $=$ unity, intensity of internal normal pressure $= P$, of external normal pressure $= P_1$.

1°. Divide the cylinder into a series of concentric rings; let radius of any ring be ρ, and thickness $d\rho$, these being the dimensions before the pressure is applied.

Let ρ become $\rho + \xi$, and $d\rho$, $d\rho + d\xi$, after the pressure is applied.

Then at any point of this ring we shall have, for the strain in the radial direction,

$$\frac{d\xi}{d\rho}; \qquad (1)$$

and, since the length of the ring before the application of the pressure is $2\pi r$, and after is $2\pi(r + \xi)$, hence the strain in a direction at right angles to the radius is

$$\frac{2\pi\xi}{2\pi\rho} = \frac{\xi}{\rho}. \qquad (2)$$

2°. Impose, now, the conditions of equilibrium upon the forces exerted by the rest of the cylinder upon the upper half-ring. For this purpose let

p = intensity of normal pressure on inside; i.e., at distance ρ from the axis.

$p + dp$ = intensity of normal pressure on outside; i.e., at distance $\rho + d\rho$ from the axis.

Then we shall have for these forces, —

(*a*) Upward force due to internal pressure,

$$2p(\rho + \xi).$$

(*b*) Downward force due to external pressure,

$$2(p + dp)(\rho + \xi + d\rho + d\xi).$$

(*c*) Upward force at right angles to radius acting at division line between the two half-rings,

$$2t(d\rho + d\xi),$$

where t = intensity of hoop-tension per square unit; i.e., of tension in a circumferential direction. Then we have

$$2(p + dp)(\rho + \xi + d\rho + d\xi) - 2p(\rho + \xi) - 2t(d\rho + d\xi) = 0;$$

and, if this be reduced, and the terms

$$2pd\xi, \quad 2\xi dp, \quad 2dpd\rho, \quad 2dpd\xi, \quad \text{and} \quad 2td\xi$$

be omitted, all of which are very small compared with the remaining ones, we shall have

$$\frac{dp}{d\rho} + \frac{p - t}{\rho} = 0. \qquad (3)$$

Now, the two stresses p and t are principal stresses, since

there are no shearing-stresses on these planes. Hence we have, from equations (1) and (2), § 282,

$$E\frac{d\xi}{d\rho} = p - \frac{t}{m}, \qquad (4)$$

$$E\frac{\xi}{\rho} = t - \frac{p}{m}. \qquad (5)$$

Now eliminate p and t between (3), (4), and (5), and obtain a differential equation between ρ and ξ.

Proceed as follows: —

From (4) and (5),

$$p = \frac{Em^2}{m^2 - 1}\left(\frac{d\xi}{d\rho} + \frac{\xi}{m\rho}\right)$$

$$\therefore \quad \frac{dp}{d\rho} = \frac{Em^2}{m^2 - 1}\left(\frac{d^2\xi}{d\rho^2} + \frac{1}{m\rho}\frac{d\xi}{d\rho} - \frac{\xi}{m\rho^2}\right).$$

From (4) and (5) also,

$$\frac{p - t}{\rho} = \frac{Em}{m + 1}\left(\frac{d\xi}{d\rho} - \frac{\xi}{\rho}\right).$$

Hence, substituting in (3), and reducing, we obtain

$$\frac{d^2\xi}{d\rho^2} + \frac{1}{\rho}\frac{d\xi}{d\rho} - \frac{\xi}{\rho^2} = 0 \qquad (6)$$

$$\therefore \quad \frac{d^2\xi}{d\rho^2} = -\left(\frac{1}{\rho}\frac{d\xi}{d\rho} - \frac{\xi}{\rho^2}\right) = \frac{-d\left(\frac{\xi}{\rho}\right)}{d\rho}.$$

Hence, by integration,

$$\frac{d\xi}{d\rho} = -\frac{\xi}{\rho} + 2a; \qquad (7)$$

$2a$ being an arbitrary constant, to be determined from the conditions of the problem.

From (7) we obtain

$$\rho\frac{d\xi}{d\rho} + \xi = 2a\rho \quad \text{or} \quad \frac{d(\xi\rho)}{d\rho} = 2a\rho.$$

Hence, integrating, we have

$$\xi\rho = a\rho^2 + b, \tag{8}$$

b being another arbitrary constant.

From (8) we obtain

$$\xi = a\rho + \frac{b}{\rho}, \tag{9}$$

which gives us, for the two strains,

$$\frac{d\xi}{d\rho} = a - \frac{b}{\rho^2}, \tag{10}$$

$$\frac{\xi}{\rho} = a + \frac{b}{\rho^2}, \tag{11}$$

Hence, substituting these values in (4) and (5), and solving for p and t successively, we obtain

$$p = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{\rho^2}, \tag{12}$$

$$t = \frac{Em}{m-1}a + \frac{Em}{m+1}\frac{b}{\rho^2}. \tag{13}$$

Now, to determine a and b, we have the conditions, that, when $\rho = r$, $p = P$, and when $\rho = r_1$, $p = -P_1$.

Hence

$$P = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{r^2}, \quad P_1 = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{r_1^2},$$

$$\therefore \quad a = \frac{m-1}{Em}\frac{P_1r_1^2 - Pr^2}{r_1^2 - r^2}, \quad b = \frac{m+1}{m}\frac{P_1 - P}{r_1^2 - r^2}r^2r_1^2,$$

$$\therefore \quad t = \frac{P_1r_1^2 - Pr^2}{r_1^2 - r^2} + \frac{1}{\rho^2}\frac{(P_1 - P)r^2r_1^2}{r_1^2 - r^2}. \tag{14}$$

The greatest value of t, and hence the greatest intensity of the hoop-tension, occurs when $\rho = r$; and hence we obtain

$$\text{Max}\, t = \frac{2P_1 r_1^2 - P(r_1^2 + r^2)}{r_1^2 - r^2}, \qquad (15)$$

this value of t being negative when there is hoop-tension, because the signs were so chosen as to make t positive when denoting compression.

If $P_1 = 0$, i.e., if there is no external pressure, we have

$$\text{Max}\, t = -P\left(\frac{r_1^2 + r^2}{r_1^2 - r^2}\right); \qquad (16)$$

and, according to Professor Rankine's method, we should determine the proper dimensions by keeping max t within the working-strength of the material.

On the other hand, if we decide that we will keep the value of $E\left(\frac{\xi}{\rho}\right)$ within the working-strength, we shall find for this, when we make $\rho = r$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{[2P_1 r_1^2 - P(r_1^2 + r^2)] - \frac{1}{m}P(r_1^2 - r^2)}{(r_1^2 - r^2)}; \qquad (17)$$

and, if $m = 4$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{2P_1 r_1^2 - P(r_1^2 + r^2) - \frac{1}{4}P(r_1^2 - r^2)}{(r_1^2 - r^2)}. \qquad (18)$$

When $P_1 = 0$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{-P(\frac{5}{4}r_1^2 + \frac{3}{4}r^2)}{r_1^2 - r^2}. \qquad (19)$$

Practical cases of thick, hollow cylinders subjected to a uniform normal pressure occur in hydraulic presses and in ordnance.

§ 298. **Strength of Flat Plates.** — In this regard, the formulæ that will be deduced are those of Professor Grashof, the reasoning followed being substantially that given by him in his "Festigkeitslehre."

ROUND PLATES.

Let the curved line CA be a meridian curve of the middle layer of the plate after it is bent. Take the origin at O; let axis OZ be vertical, and axis OX horizontal, and let the axis at right angles to ZOX be $O\Phi$, so that z, x, and ϕ are the co-ordinates of any point in the middle layer of the plate.

Let y denote the (vertical) distance of any horizontal layer from the middle layer of the plate.

Let R = radius of curvature of meridian line at any point (x, z, ϕ).

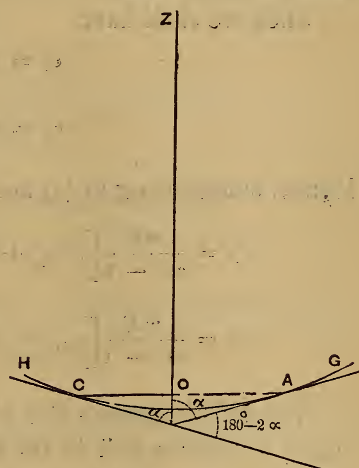


Fig. 303.

Let R_1 = radius of curvature of section of middle layer normal to meridian line.

Then we should have, from the differential calculus,

$$R = -\frac{\frac{d^2z}{dx^2}}{\left(1 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{3}{2}}} = -\frac{d^2z}{dx^2} \text{ nearly,}$$

$$R_1 = -\frac{1}{x}\frac{dz}{dx}.$$

Hence, reasoning in the same way as in the common theory of beams, we should have, for the strains of the layer whose dis-

tance from neutral layer is y at point (x, ϕ), provided there is no stress in the plane of the neutral layer,

$$\epsilon_x = \pm \frac{y}{R}, \quad \epsilon_\phi = \pm \frac{y}{R_1}.$$

When there is such a stress, let the strains due to that stress be ϵ_{x_0} and ϵ_{ϕ_0}.

Then we shall have

$$\epsilon_x = \epsilon_{x_0} - y\frac{d^2z}{dx^2}, \tag{1}$$

$$\epsilon_\phi = \epsilon_{\phi_0} - \frac{y}{x}\frac{dz}{dx}. \tag{2}$$

Hence, substituting in (1) and (2) of § 283, we have

$$\sigma_x = \frac{mE}{m^2 - 1}\left[m\epsilon_{x_0} + \epsilon_{\phi_0} - y\left(m\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right)\right], \tag{3}$$

$$\sigma_\phi = \frac{mE}{m^2 - 1}\left[\epsilon_{x_0} + m\epsilon_{\phi_0} - y\left(\frac{d^2z}{dx^2} + \frac{m}{x}\frac{dz}{dx}\right)\right]. \tag{4}$$

Now let us suppose the plate to be subjected, before loading, to a uniform pull in its own plane, and normal to its circumference; and let the intensity of this pull be p_1. Then

$$\sigma_{x_0} = \sigma_{\phi_0} = p_1;$$

and hence, from (1) and (2), we have

$$\epsilon_{x_0} = \epsilon_{\phi_0} = \frac{p_1(m - 1)}{mE}. \tag{5}$$

Therefore, substituting in (3) and (4), and reducing,

$$\sigma_x = p_1 - \frac{mEy}{m^2 - 1}\left(m\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right), \tag{6}$$

$$\sigma_\phi = p_1 - \frac{mEy}{m^2 - 1}\left(\frac{d^2z}{dx^2} + \frac{m}{x}\frac{dz}{dx}\right). \tag{7}$$

These equations express the stresses in terms of the co-ordinates of the points.

Now impose the conditions of equilibrium upon the forces acting on any half-ring of thickness $dx = d\phi$. These forces are —

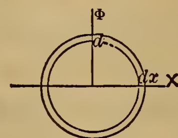


FIG. 304.

1°. Force exerted upon it by the outer part of the plate,

$$\{2x\sigma_x + 2d(x\sigma_x)\}dz.$$

2°. Force exerted by the inner part of the plate,

$$-2x\sigma_x dz.$$

3°. Force exerted upon it by the other half-ring,

$$-2\sigma_\phi dxdz.$$

4°. Force exerted by resistance to shear on top and bottom,

$$\{(\tau + d\tau) - \tau\}2xdx.$$

Hence, equating to zero the algebraic sum of these, and reducing, we obtain

$$\frac{d\tau}{dz} = \frac{d\tau}{dy} = \frac{\sigma_\phi}{x} - \frac{1}{x}\frac{d(x\sigma_x)}{dx}. \qquad (8)$$

Now substitute for σ_x and σ_ϕ their values, and reduce, and we have

$$\frac{d\tau}{dy} = \frac{m^2Ey}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right). \qquad (9)$$

Integrate with regard to y, and we have, since the quantity in brackets is not a function of y,

$$\tau = \frac{m^2Ey^2}{2(m^2 - 1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} + \frac{1}{x^2}\frac{dz}{dx}\right) + c.$$

But, when $y = \frac{h}{2}$ (h being the thickness of the plate), $\tau = 0$, since there is no shearing-force at top or bottom;

$$\therefore \quad c = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right)$$

$$\therefore \quad \tau = \frac{m^2E(h^2-4y^2)}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right). \qquad (10)$$

This gives us the intensity of the shearing-stress at any point (x, z) at distance y from middle layer; and this is the intensity of the shear at that point between two horizontal layers, and hence also along a vertical plane through the point (x, z).

Now let us take the case of a centre load P combined with a distributed load p per unit of area. Then shearing-force at distance x from centre =

$$\pi x^2 p + P,$$

this tending to shear out a circular piece of radius x. Hence we must have this balanced by the whole shearing resistance on the surface subjected to shear;

$$\therefore \quad 2\pi x \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau dy = \pi x^2 p + P$$

$$\therefore \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau dy = \frac{1}{2}\left(px + \frac{P}{\pi x}\right). \qquad (11)$$

Now substitute the value of τ from equation (9), integrate, and reduce, and we obtain

$$\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx} = -\frac{6(m^2-1)}{m^2Eh^3}\left(px + \frac{P}{\pi x}\right). \qquad (12)$$

Hence, for the intensity of the shearing-force, we have

$$\tau = \frac{3}{4}\frac{h^2 - 4y^2}{h^3}\left(px + \frac{P}{\pi x}\right). \qquad (13)$$

This gives the intensity of the shearing-force at any point of the plate.

Next, to find its deflection, or the equation of the meridian line, we have, from (12),

$$\frac{d}{dx}\left(\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right) = -\frac{6(m^2 - 1)}{m^2Eh^3}\left(px + \frac{P}{\pi x}\right)$$

$$\therefore \quad \frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^2}{2} - \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\log_e x + c$$

$$\therefore \quad x\frac{d^2z}{dx^2} + \frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^3}{2} - \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}x\log_e x + cx.$$

But

$$x\frac{d^2z}{dx^2} + \frac{dz}{dx} = \frac{d}{dx}\left(x\frac{dz}{dx}\right);$$

hence, integrating, we have

$$x\frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{8}$$
$$- \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{2}\log_e x + \frac{6(m^2 - 1)}{m^2Eh^3}\frac{x^2}{4} + \frac{cx^2}{2} + d. \qquad (14)$$

Hence, dividing through by x, and integrating,

$$z = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{4}$$
$$- \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{4}(\log_e x - 1) + \frac{cx^2}{4} + d\log_e x + e; \qquad (15)$$

and this is the meridian line of the surface, the constants c, d, and e being as yet undetermined.

This is as far as we can proceed before taking up special cases.

(*a*) *Full Plate.*—When the plate is full, the slope becomes zero, for $x = 0$; therefore (14) gives us

$$d = 0,$$

and in this case (15) becomes

$$z = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{4} - \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{4}(\log_e x - 1) + \frac{cx^2}{4} + e. \quad (16)$$

And, substituting for z, $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$, their values in (1) and (2), we obtain

$$E\epsilon_x = \frac{m-1}{m}\sigma_{x_0} + \left(\frac{3}{8}\frac{6(m^2 - 1)}{m^2Eh^3}p - \frac{c}{2}\right)y, \quad (17)$$

$$E\epsilon_\phi = \frac{m-1}{m}\sigma_{x_0} + \left(\frac{1}{8}\frac{6(m^2 - 1)}{m^2Eh^3}p - \frac{c}{2}\right)y; \quad (18)$$

and (13) gives

$$\tau = \frac{3}{4}\frac{h^2 - 4y^2}{h^3}px. \quad (19)$$

(α) *Uniformly Loaded, no Centre Load.*—$P = 0$;

$$\therefore \quad z = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{4} + \frac{cx^2}{4} + e. \quad (20)$$

But when $x = r$, $z = 0$;

$$\therefore \quad e = \frac{6(m^2 - 1)}{m^2Eh^3}\frac{r^4}{32} - \frac{cr^2}{4}$$

$$\therefore \quad z = \left(\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^2 + r^2}{8} - c\right)\left(\frac{r^2 - x^2}{4}\right). \quad (21)$$

(β) *Supported all around.* — When $x = r$, $\sigma_x = \sigma_{x_0} = p_1$ for all values of y: therefore, from (6),

$$m\left(\frac{d^2z}{dx^2}\right)_r + \frac{1}{r}\left(\frac{dz}{dx}\right)_r = 0;$$

and, substituting the values of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ as determined by differentiating (21), we have, after reducing,

$$c = \frac{3}{2}\frac{(m-1)(3m+1)}{m^2}\frac{pr^2}{Eh^3}.$$

Hence equation of meridian line is

$$z = \frac{3}{16}\frac{m^2-1}{m^2}\frac{p}{Eh^3}\left\{\frac{5m+1}{m+1}r^2 - x^2\right\}(r^2 - x^2). \quad (22)$$

Hence we have maximum deflection by making $x = 0$;

$$\therefore \quad z_0 = \frac{3}{16}\frac{(m-1)(5m+1)}{m^2}\frac{pr^4}{Eh^3}. \quad (23)$$

And, substituting in (17) and (18), we obtain, after reduction,

$$E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}\left\{\frac{3m+1}{m+1}r^2 - 3x^2\right\}y, \quad (24)$$

$$E\epsilon_\phi = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}\left\{\frac{3m+1}{m+1}r^2 - x^2\right\}y. \quad (25)$$

But, in a plate supported all around, $p_1 = 0$; and then the maximum value of either one occurs when $y = \frac{h}{2}$, and hence

$$E\epsilon_0 = \frac{3}{8}\frac{(m-1)(3m+1)}{m^2}\frac{pr^2}{h^2}. \quad (26)$$

On the other hand, τ becomes greatest when $x = r$ and $y = 0$. Hence

$$\text{Max } E\tau = \frac{3}{4}\frac{r}{h}p;$$

and, if ϵ_1 represent the maximum strain due to this shearing-force, we have

$$\text{Max } (E\epsilon_1) = \left(\frac{m+1}{m}\right)(\text{max } E\tau) = \frac{3}{4}\frac{m+1}{m}\frac{r}{h}p. \quad (27)$$

RESULTING FORMULÆ FOR PLATE SUPPORTED ALL ROUND.

$$\text{Max } E\epsilon_0 = \frac{3}{8}\frac{(m-1)(3m+1)}{m^2}\frac{r^2}{h^2}p \quad \text{or} \quad \frac{3}{4}\frac{m+1}{m}\frac{r}{h}p,$$

whichever is greatest.

$$z_0 = \frac{3}{16}\frac{(m-1)(5m+1)}{m^2}\frac{pr^4}{Eh^3}.$$

PLATE FIXED AT ENDS.

Equation (20) applies to this case also.

Now, when $x = r$, $\dfrac{dz}{dx} = 0$;

$$\therefore \quad c = \frac{3}{2}\frac{m^2-1}{m^2Eh^3}pr^2$$

$$\therefore \quad z = \frac{3}{16}\frac{m^2-1}{m^2}\frac{p}{Eh^3}(r^2-x^2)^2. \qquad (28)$$

Hence greatest deflection is

$$z_0 = \frac{3}{16}\frac{m^2-1}{m^2}\frac{pr^4}{Eh^3},$$

and

$$E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}(r^2 - 3x^2)y, \qquad (29)$$

$$E\epsilon_\phi = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}(r^2 - x^2)y. \qquad (30)$$

When p_1 is positive or zero, then $E\epsilon_x$ is maximum for $x = 0$ $y = \frac{h}{2}$, and for $x = r$, $y = -\frac{h}{2}$; and $E\epsilon_\phi$ is maximum for $x = 0$, $y = \frac{h}{2}$: and the maximum value of $E\epsilon_\phi$ is equal to first maximum of $E\epsilon_x$. We have

$$\text{First max } E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{8}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p, \qquad (31)$$

$$\text{Second max } E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p. \qquad (32)$$

Hence the second is the real maximum.

RESULTING FORMULÆ FOR PLATES FIXED AT THE ENDS.

$$\text{Max } E\epsilon_0 = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p,$$

$$z_0 = \frac{3}{16}\frac{m^2-1}{m^2}\frac{pr^4}{Eh^3}.$$

For $p_1 = 0$,

$$\text{Max } E\epsilon_0 = \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p.$$

§ 299. **Thickness of Plates.** — Grashof advises the use of 3 as value of m. If this be adopted, we should have, for the proper thickness of round plates,

Supported.	Fixed.
$h = r\sqrt{\frac{5p}{6f}},$	$h = r\sqrt{\frac{2p}{3f}}.$

where h = thickness, r = radius, p = pressure per square inch, and f = working-strength per square inch. If, now, we use a factor of safety 8, and use as tensile strength of cast-iron 20000, of wrought-iron, 48000, and of steel 80000, we should have:—

	Supported.	Fixed.
Cast-iron . . .	$h = 0.0182570r\sqrt{p}$	$h = 0.0163300r\sqrt{p}$
Wrought-iron .	$h = 0.0117850r\sqrt{p}$	$h = 0.0105410r\sqrt{p}$
Steel	$h = 0.0091287r\sqrt{p}$	$h = 0.0081649r\sqrt{p}$

§ 300. **Rectangular Plates.** — Refer the plate to rectangular axes, as before, OZ, OX, $O\Phi$; the origin being at the middle of its middle layer.

Let y = distance of any point in the plate from the middle layer.

Let ρ_x be the radius of curvature of a normal section parallel to OX at the point (x, z, ϕ).

Let ρ_ϕ be the radius of curvature of a normal section parallel to $O\Phi$ at the point (x, z, ϕ).

Then we shall have, by the principles of the common theory of beams,

$$\epsilon_x = \epsilon_{x_0} \pm \frac{y}{\rho_x}, \qquad \epsilon_\phi = \epsilon_{\phi_0} \pm \frac{y}{\rho_\phi},$$

where ϵ_{x_0} and ϵ_{ϕ_0} are the strains of the middle layer in the directions OX and $O\Phi$ respectively.

Moreover, from the Differential Calculus, we have

$$\rho_x = \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{d\phi}\right)^2}}{\frac{d^2z}{dx^2}\cos^2\lambda + 2\frac{d^2z}{dxd\phi}\cos\lambda\cos\mu + \frac{d^2z}{d\phi^2}\cos^2\mu},$$

where λ = angle between normal and z axis, and μ = angle between normal and x axis. But $\frac{dz}{dx}$ and $\frac{dz}{d\phi}$ being the slopes, and hence small, we shall have nearly

$$\cos\lambda = 1, \qquad \cos\mu = 0,$$

$$\frac{1}{\rho_x} = \mp\frac{d^2z}{dx^2}, \qquad \frac{1}{\rho_\phi} = \mp\frac{d^2z}{d\phi^2}$$

$$\therefore \quad \epsilon_x = \epsilon_{x_0} - y\frac{d^2z}{dx^2}, \tag{1}$$

$$\epsilon_\phi = \epsilon_{\phi_0} - y\frac{d^2z}{d\phi^2}. \tag{2}$$

Hence (1) and (2) of § 283 give us

$$\sigma_x = \frac{mE}{m^2 - 1}(m\epsilon_{x_0} + \epsilon_{\phi_0}) - y\frac{mE}{m^2 - 1}\left\{m\frac{d^2z}{dx^2} + \frac{d^2z}{d\phi^2}\right\},$$

$$\sigma_\phi = \frac{mE}{m^2 - 1}(\epsilon_{x_0} + m\epsilon_{\phi_0}) - y\frac{mE}{m^2 - 1}\left\{\frac{d^2z}{dx^2} + m\frac{d^2z}{d\phi^2}\right\}.$$

And, if σ_{x_0}, σ_{ϕ_0}, denote the stresses in the middle layer, we shall have, since

$$\sigma_{x_0} = \frac{mE}{m^2 - 1}(m\epsilon_{x_0} + \epsilon_{\phi_0}), \qquad \sigma_{\phi_0} = \frac{mE}{m^2 - 1}(\epsilon_{x_0} + m\epsilon_{y_0}),$$

$$\sigma_x = \sigma_{x_0} - \frac{m^2E}{m^2 - 1}y\left\{\frac{d^2z}{dx^2} + \frac{1}{m}\frac{d^2z}{d\phi^2}\right\}, \tag{3}$$

$$\sigma_y = \sigma_{y_0} - \frac{m^2E}{m^2 - 1}y\left\{\frac{1}{m}\frac{d^2z}{dx^2} + \frac{d^2z}{d\phi^2}\right\}. \tag{4}$$

Now, if ξ and η denote the increments in x and ϕ respectively due to the load, we shall have

$$\xi = \int_0^x \epsilon_x dx = x\epsilon_{x_0} - y\frac{dz}{dx} \qquad \therefore \quad \frac{d\xi}{d\phi} = -y\frac{d^2z}{dxd\phi},$$

$$\eta = \int_0^\phi \epsilon_\phi d\phi = \phi\epsilon_{\phi_0} - y\frac{dz}{d\phi} \qquad \therefore \quad \frac{d\eta}{dx} = -y\frac{d^2z}{dxd\phi}.$$

But

$$\tau_{x\phi} = G\gamma_{x\phi} = G\left(\frac{d\xi}{d\phi} + \frac{d\eta}{dx}\right);$$

hence

$$\tau_{x\phi} = -2Gy\frac{d^2z}{dxd\phi}. \qquad (5)$$

Equations (3), (4), and (5) are the expressions giving the stresses on two planes at right angles to each other, parallel to OX and $O\Phi$ respectively. Hence we have a case of stress on two planes at right angles to each other, and we are to find the principal stresses: we thus have —

1°. Normal stress on x plane, σ_x.
2°. Shearing-stress on x plane, $\tau_{x\phi}$.
3°. Normal stress on ϕ plane, σ_ϕ.
4°. Shearing-stress on ϕ plane, τ_ϕ.

Hence, if we denote by σ_1 and σ_2 the maximum and minimum principal stress, we have (§ 290)

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_\phi) + \tfrac{1}{2}\sqrt{(\sigma_x + \sigma_\phi)^2 + 4\tau_{x\phi}^2}, \qquad (6)$$

$$\sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_\phi) - \tfrac{1}{2}\sqrt{(\sigma_x + \sigma_\phi)^2 + 4\tau_{x\phi}^2}; \qquad (7)$$

and hence, if ϵ_1 and ϵ_2 denote the strains in the directions of the principal stresses,

$$E\epsilon_1 = \sigma_1 - \frac{\sigma_2}{m} = \frac{m-1}{2m}(\sigma_x + \sigma_\phi) + \frac{m+1}{2m}\sqrt{(\sigma_x + \sigma_\phi)^2 + 4\tau_{x\phi}^2}, \qquad (8)$$

$$E\epsilon_2 = \sigma_2 - \frac{\sigma_1}{m} = \frac{m-1}{2m}(\sigma_x + \sigma_\phi) - \frac{m+1}{2m}\sqrt{(\sigma_x + \sigma_\phi)^2 + 4\tau_{x\phi}^2}; \qquad (9)$$

and for the strain ϵ_3, parallel to OZ, we have

$$E\epsilon_3 = -\frac{\sigma_x + \sigma_\phi}{m}. \qquad (10)$$

In order to use (8), (9), and (10), however, we must know σ_x, σ_ϕ, and $\tau_{x\phi}$; and for this purpose we must know the equation of the middle layer after bending. For this purpose, apply the equations (1), (2), (3), of § 281 to any particle $dxd\phi dz$ in the interior of the body. We have then, $X = Y = Z = 0$. Therefore

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xz}}{dy} + \frac{d\tau_{x\phi}}{d\phi} = 0 \quad \therefore \quad \frac{d\tau_{xz}}{dy} = -\left(\frac{d\sigma_x}{dx} + \frac{d\tau_{x\phi}}{d\phi}\right),$$

$$\frac{d\sigma_\phi}{d\phi} + \frac{d\tau_{x\phi}}{dx} + \frac{d\tau_{\phi z}}{dy} = 0 \quad \therefore \quad \frac{d\tau_{\phi z}}{dy} = -\left(\frac{d\sigma_\phi}{d\phi} + \frac{d\tau_{x\phi}}{dx}\right),$$

$$\frac{d\sigma_z}{dy} + \frac{d\tau_{\phi z}}{d\phi} + \frac{d\tau_{xz}}{dx} = 0.$$

Therefore, making use of (3), (4), and (5) with the above conditions, we deduce

$$\frac{d\sigma_x}{dx} = -\frac{m^2Ey}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{1}{m}\frac{d^3z}{dxd\phi^2}\right), \qquad (11)$$

$$\frac{d\sigma_\phi}{d\phi} = \frac{m^2Ev}{m^2 - 1}\left(\frac{d^3z}{d\phi^3} + \frac{1}{m}\frac{d^3z}{dx^2d\phi}\right), \qquad (12)$$

$$\frac{d\tau_{xy}}{dx} = -\frac{mEy}{m + 1}\frac{d^3z}{dx^2d\phi}, \qquad (13)$$

$$\frac{d\tau_{x\phi}}{d\phi} = -\frac{mEy}{m + 1}\frac{d^3z}{dxd\phi^2}, \qquad (14)$$

$$\frac{d\tau_{xz}}{dy} = \frac{m^2Ev}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right), \qquad (15)$$

$$\frac{d\tau_{\phi z}}{dy} = \frac{m^2Ev}{m^2 - 1}\left(\frac{dz^3}{dy^3} + \frac{d^3z}{dx^2dy}\right) \qquad (16)$$

Hence, by integrating (15) and (16), we have

$$\tau_{xz} = \frac{m^2Ev^2}{2(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right) + c_1,$$

$$\tau_{\phi z} = \frac{m^2Ev^2}{2(m^2-1)}\left(\frac{d^3z}{d\phi^3} + \frac{d^3z}{dx^2d\phi}\right) + c_2.$$

But when $v = \frac{h}{2}$, $\tau_{\phi z} = \tau_{xz} = 0$;

$$\therefore \quad c_1 = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right)$$

and

$$c_2 = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{d\phi^3} + \frac{d^3z}{dx^2d\phi}\right)$$

$$\therefore \quad \tau_{\phi z} = \frac{m^2E}{m^2-1}\left(\frac{d^3z}{dx^2d\phi} + \frac{d^3z}{d\phi^3}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right)$$

and

$$\tau_{xz} = \frac{m^2E}{m^2-1}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right).$$

Hence

$$\frac{d\tau_{\phi z}}{d\phi} = \frac{m^2E}{m^2-1}\left(\frac{d^4z}{dx^2d\phi^2} + \frac{d^4z}{d\phi^4}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right),$$

$$\frac{d\tau_{xz}}{dx} = \frac{m^2E}{m^2-1}\left(\frac{d^4z}{dx^4} + \frac{d^4z}{dx^2d\phi^2}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right).$$

Now we have $\sigma_z = p$, where p is the intensity of the load; therefore the third equation gives us, on integrating between the limits $\frac{h}{2}$ and $-\frac{h}{2}$,

$$\sigma_z + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{\phi z}}{d\phi}dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{xz}}{dx}dy = 0$$

$$\therefore \quad p + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{\phi z}}{d\phi}dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{xz}}{dx}dy = 0$$

$$\therefore \quad p + \frac{m^2 E}{m^2 - 1}\left\{\left(\frac{v^3}{6} - \frac{h^2 v}{8}\right)\left(\frac{d^4 z}{dx^4} + 2\frac{d^4 z}{dx^2 d\phi^2} + \frac{d^4 z}{d\phi^4}\right)_{-\frac{h}{2}}^{\frac{h}{2}}\right\} = 0$$

$$\therefore \quad p + \frac{m^2 E}{m^2 - 1}\left(\frac{d^4 z}{dx^4} + 2\frac{d^4 z}{dx^2 d\phi^2} + \frac{d^4 z}{d\phi^4}\right)\left(\frac{h^3}{24} - \frac{h^3}{8}\right) = 0$$

$$\therefore \quad \frac{d^4 z}{dx^4} + 2\frac{d^4 z}{dx^2 d\phi^2} + \frac{d^4 z}{d\phi^4} = \frac{12(m^2 - 1)p}{m^2 E h^3}; \qquad (17)$$

and this is the differential equation of the surface, and should be integrated in each special case.

INDEFINITE PLATES WHICH ARE FIRMLY HELD AT A SYSTEM OF POINTS DIVIDING THEM INTO RECTANGULAR PANELS.

Let the sides of the panels be $2a$ and $2b$. Assume the origin at the middle of the panel, the axis of x being parallel to $2a$, and the axis of y parallel to $2b$. We shall in this case have the following conditions; viz., —

(a) $\frac{dz}{dx} = 0$ for $x = \pm a$ and all values of ϕ.

(b) $\frac{dz}{d\phi} = 0$ for $\phi = \pm b$ and all values of x.

(c) $z = 0$ when $x = \pm a$, $\phi = \pm b$.

(d) If we develop the value of z in powers of x and ϕ, there must enter only even powers of x and ϕ, since the value of z remains the same when we put $-x$ for x, or $-\phi$ for ϕ.

Now, if we write

$$z = A + Bx^2 + C\phi^2 + Dx^2\phi^2 + Ex^4 + F\phi^4$$
$$+ Gx^6 + Hx^4\phi^2 + Kx^2\phi^4 + L\phi^6 + Mx^8, \text{ etc.},$$

the above conditions will be fulfilled: —

1°. By making all the co-efficients after the fourth, each zero.

2°. By making $D = 0$, therefore writing

$$z = A + Bx^2 + C\phi^2 + Ex^4 + F\phi^4.$$

Now

$$\frac{dz}{dx} = 2Bx + 4Ex^3, \qquad \frac{dz}{d\phi} = 2C\phi + 4F\phi^3,$$

$$\therefore \quad 2Ba + 4Ea^3 = 0, \qquad 2Cb + 4Fb^3 = 0,$$

and

$$0 = A + Ba^2 + Cb^2 + Ea^4 + Fb^4$$

$$\therefore \quad B = -2Ea^2, \qquad C = -2Fb^2,$$

$$\therefore \quad A = 2Ea^4 + 2Fb^4 - Ea^4 - Fb^4 = Ea^4 + Fb^4.$$

Hence the equation becomes

$$z = Ea^4 + Fb^4 - 2Ea^2x^2 - 2Fb^2\phi^2 + Ex^4 + F\phi^4$$
$$= E(a^2 - x^2)^2 + F(b^2 - \phi^2)^2$$

$$\therefore \quad \frac{dz}{dx} = -4Ex(a^2 - x^2) = 4Ex^3 - 4Ea^2x$$

$$\therefore \quad \frac{d^2z}{dx^2} = 12Ex^2 - 4Ea^2,$$

also

$$\frac{dz}{d\phi} = -4F\phi(b^2 - \phi^2) = 4F\phi^3 - 4Fb^2\phi$$

$$\therefore \quad \frac{d^2z}{d\phi^2} = 12F\phi^2 - 4Fb^2,$$

$$\frac{d^3z}{dx^2d\phi} = 0 \quad \therefore \quad \frac{d^4z}{dx^2d\phi^2} = 0, \qquad \frac{d^3z}{dxd\phi^2} = 0, \qquad \frac{d^4z}{dx^2d\phi^2} = 0,$$

$$\frac{d^3z}{dx^3} = 24Ex, \qquad \frac{d^3z}{d\phi^3} = 24F\phi, \qquad \frac{d^4z}{dx^4} = 24E, \qquad \frac{d^4z}{d\phi^4} = 24F$$

$$\therefore \quad 24(E + F) = \frac{12(m^2 - 1)p}{m^2Eh^3} \qquad \therefore \quad E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}.$$

Hence equation of the middle layer is

$$z = E(a^2 - x^2)^2 + F(b^2 - \phi^2)^2, \text{ where } E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}. \qquad (18)$$

Now, in the case of an ordinary beam fixed at both ends, and loaded uniformly with p lbs. per unit of area, if b is the breadth, we have: —

1°. The points of inflection are at a distance from the middle equal to $\frac{a}{\sqrt{3}}$, where a is the half-span; and

2°. The bending-moment at a section at a distance x from the middle is $\frac{pb}{2}\left(\frac{a^2}{3} - x^2\right)$ when $x < \frac{a}{\sqrt{3}}$, and $\frac{pb}{2}\left(x^2 - \frac{a^2}{3}\right)$ when $x > \frac{a}{\sqrt{3}}$; therefore the value of z is found from the formula

$$z = \frac{1}{EI}\int_0^x\int_0^x Mdx^2 = \frac{6p}{Eh^3}\int_0^x\int_0^x\left(x^2 - \frac{a^2}{3}\right)dx^2$$

or

$$\frac{6p}{Eh^3}\int_0^x\int_0^x\left(\frac{a^2}{3} - x^2\right)dx^2.$$

Either one, when integrated, gives for z the value

$$z = \frac{p}{2Eh^3}(a^2 - x^2)^2.$$

Hence in the flat plate, if $b = 0$, the values of E and F must be such that the formula shall reduce to $z = \frac{p}{2Eh^3}(a^2 - x^2)^2$ when $b = 0$. Now, it does reduce to $z = E(a^2 - x)^2$. Therefore E must be such a function of a and b, that, when $b = 0$, it shall reduce to $\frac{p}{2Eh^3}$. So likewise F must be such a function of a and b, that, when $a = 0$, it shall reduce to $\frac{p}{2Eh^3}$. Suppose, then, we put

$$E = \frac{p}{2Eh^3} + b^n c \qquad \text{and} \qquad F = \frac{p}{2Eh^3} + a^n c,$$

since these functions fulfil the above conditions.

Now we have

$$E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}$$

$$\therefore \qquad \frac{p}{2Eh^3} + c(a^n + b^n) = \frac{(m^2 - 1)p}{2m^2Eh^3}$$

$$\therefore \quad (a^n + b^n)c = -\frac{(m^2 + 1)p}{2m^2Eh^3}$$

$$\therefore \qquad c = -\frac{(m^2 + 1)p}{2m^2Eh^3(a^n + b^n)}$$

$$\therefore \qquad E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} + yE\frac{d^2z}{dx^2},$$

$$E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} - yE\frac{d^2z}{d\phi^2}.$$

Hence, substituting for $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{d\phi^2}$ their values, and observing that

$$\epsilon_x \text{ is greatest for } x = \pm a,\ y = \pm\frac{h}{2},$$

$$\epsilon_\phi \text{ is greatest for } \phi = \pm b,\ y = \pm\frac{h}{2},$$

we obtain

$$\max\,(E\epsilon_x) = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm 2\,\frac{a^n - \frac{1}{m^2}b^n}{a^n + b^n}\,\frac{a^2}{h^2}p, \tag{19}$$

$$\max\,(E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm 2\,\frac{b^n - \frac{1}{m^2}a^n}{a^n + b^n}\,\frac{b^2}{h^2}p. \tag{20}$$

These may be written as follows:

$$\max E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm 2\,\frac{1 - \dfrac{1}{m^2}\left(\dfrac{b}{a}\right)^n}{1 + \left(\dfrac{b}{a}\right)^n}\,\frac{a^2}{h^2}p, \qquad (21)$$

$$\max E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm 2\,\frac{1 - \dfrac{1}{m^2}\left(\dfrac{a}{b}\right)^n}{1 + \left(\dfrac{a}{b}\right)^n}\,\frac{b^2}{h^2}p. \qquad (22)$$

We have also, by substituting for E and F their values in equation (18),

$$z = \frac{p}{2Eh^3}\left\{\frac{a^n - \dfrac{1}{m^2}b^n}{a^n + b^n}(a^2 - x^2)^2 + \frac{b^n - \dfrac{1}{m^2}a^n}{a^n + b^n}(b^2 - \phi^2)^2\right\}. \qquad (23)$$

In these results the exponent n is undetermined, and we have no means of determining it in the general case. We only know, that, since the deflection must increase for a decrease in x and ϕ, therefore we must have, whenever $a > b$,

$$\left(\frac{a}{b}\right)^n < m^2 \qquad \therefore \quad n < \frac{2\log m}{\log\left(\dfrac{a}{b}\right)}.$$

This leaves the general case indeterminate; but a common practical case is not subject to this indetermination, i.e., the case when $a = b$, for then

$$\left(\frac{a}{b}\right)^n = \left(\frac{b}{a}\right)^n = 1,$$

whatever the value of n; and hence equations (21), (22), and (23) give

$$\max (E\epsilon_x) = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm \frac{m^2 - 1}{m^2}\,\frac{a^2}{h^2}p, \qquad (24)$$

$$\max E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm \frac{m^2 - 1}{m^2}\,\frac{a^2}{h^2}p, \qquad (25)$$

$$z = \frac{m^2 - 1}{m^2} \frac{p}{4Eh^3} \{ (a^2 - x^2)^2 + (a^2 - \phi^2)^2 \}, \qquad (26)$$

and

$$\max z = \frac{m^2 - 1}{m^2} \frac{a^4}{2Eh^3} p. \qquad (27)$$

FORMULÆ FOR THE SHEETS OF A LOCOMOTIVE FIRE-BOX.

In this case we have $a = b$; hence (24), (25), and (27) apply: and if we write, with Grashof, $m = 3$, they become

$$\max (E\epsilon_x) = \sigma_{x_0} - \frac{1}{3} \sigma_{\phi_0} + \frac{8}{9} \frac{a^2}{h^2} p, \qquad (28)$$

$$\max (E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{3} \sigma_{x_0} + \frac{8}{9} \frac{a^2}{h^2} p, \qquad (39)$$

$$\max (z) = \frac{4}{9} \frac{pa^4}{Eh^3}. \qquad (30)$$

Now, in the case of the horizontal sheets, $\sigma_{x_0} = \sigma_{\phi_0} = 0$, and we have

$$\max (E\epsilon_x) = \frac{8}{9} \frac{a^2}{h^2} p, \qquad (31)$$

$$\max (z) = \frac{4}{9} \frac{pa^4}{Eh^3}. \qquad (32)$$

In the case of the vertical walls, inasmuch as these have to resist the steam-pressure in a vertical direction, the inner one is called upon to bear compression, and the outer tension, in a vertical direction. If l is the length of the outside of the fire-box, and l_1 its breadth, we shall have for the outer plate, taking axis of x vertical,

$$\sigma_x = \frac{ll_1 p}{2(l + l_1)h}, \qquad \sigma_{\phi_0} = 0;$$

and for the inner plate, if l and l_1' are corresponding dimensions of inside of fire-box,

$$\sigma_{x_0} = \frac{ll_1 p}{2(l' + l_1')h}, \qquad \sigma_{\phi_0} = 0.$$

And, by making these substitutions in (28), (29), and (30), we obtain our formulæ.

RECTANGULAR PLATE FIXED AT THE EDGES.

For this case Grashof deduces the equation of the middle layer as follows: —

1°. This equation must be a function of x and ϕ.

2°. If $2a$ and $2b$ are the sides of the plate, this function must become

(α) When $b = \infty$ for all values of ϕ,

$$z = \frac{p}{2Eh^3}(a^2 - x^2)^2.$$

(β) When $a = \infty$ for all values of x,

$$z = \frac{p}{2Eh^3}(b^2 - \phi^2)^2,$$

because the plate then becomes a beam fixed at the ends.

The function that will satisfy these two conditions is

$$z = \frac{p}{2Eh^3} \frac{(a^2 - x^2)^2(b^2 - \phi^2)^2}{a^4 + b^4}. \tag{1}$$

From this he deduces for max z, when $x = \phi = 0$,

$$\max z = \frac{1}{2} \frac{p}{Eh^3} \frac{a^4 b^4}{a^4 + b^4}. \tag{2}$$

From (1) he deduces

$$\frac{d^2z}{dx^2} = -\frac{2p}{Eh^3}\frac{(a^2 - 3x^2)(b^2 - \phi^2)^2}{a^4 + b^4}, \qquad (3)$$

$$\frac{d^2z}{d\phi^2} = -\frac{2p}{Eh^3}\frac{(a^2 - x^2)^2(b^2 - 3\phi^2)}{a^4 + b^4}, \qquad (4)$$

$$\frac{d^2z}{dxd\phi} = \frac{8p}{Eh^3}\frac{(a^2 - x^2)x(b^2 - \phi^2)\phi}{a^4 + b^4}, \qquad (5)$$

$$\max\frac{d^2z}{dx^2} = \frac{4p}{Eh^3}\frac{a^2b^4}{a^4 + b^4}, \quad \text{for} \quad x = \pm a, \quad \phi = 0, \qquad (6)$$

$$\max\frac{d^2z}{d\phi^2} = \frac{4p}{Eh^3}\frac{a^4b^2}{a^4 + b^4}, \quad \text{for} \quad \phi = \pm b, \quad x = 0, \qquad (7)$$

$$\max\frac{d^2z}{dxd\phi} = \frac{32}{27}\frac{p}{Eh^3}\frac{a^3b^3}{a^4 + b^4}, \quad \text{for} \quad x^2 = \frac{1}{3}a^2, \quad \phi^2 = \frac{1}{3}b^2, \qquad (8)$$

these corresponding to the points of inflection of a loaded beam fixed at the ends.

Hence (1), (2), and (5) of § 300 give

$$\max E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm \frac{2b^4}{a^4 + b^4}\frac{a^2}{h^2}p, \qquad (9)$$

$$\max (E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm \frac{2a^4}{a^4 + b^4}\frac{b^2}{h^2}p, \qquad (10)$$

$$\max (\tau_z) = \frac{8}{27}\frac{m}{m + 1}\frac{2a^2b^2}{a^4 + b^4}\frac{ab}{h^2}p. \qquad (11)$$

At the places where ϵ_x and ϵ_ϕ are greatest,

$$\tau_z = 0.$$

At the place where τ_z is greatest,

$$\sigma_x = \sigma_{x_0}, \qquad \sigma_\eta = \sigma_{\eta_0}.$$

Hence it is either (9) or (10) that gives us the suitable formula to use in any special case.

EXAMPLES OF THEORY OF ELASTICITY.

1. It has been sometimes proposed to use oblique seams in a boiler-shell. Assume the seams at an angle of 45° with the axis of the boiler, a pressure of 100 lbs. per square inch of the steam, and a diameter of 4 feet. Find the tension per inch of length of seam, and its direction.

2. Given a shaft carrying 80 *HP*, and running at 250 revolutions per minute. Suppose the driving-pulley to be at the middle of the length, this being 6 feet, and given that the ratio of the tension on the tight side of the belt to that on the loose side is 3.75. Find the proper size of shaft, assuming 10000 lbs. per square inch as the working-strength of the iron.

3. What should be the thickness of a flat plate to bear 150 lbs. pressure per square inch, and stayed at points forming squares 8 inches on a side, the plate being of wrought-iron, working-strength 10000 lbs. per square inch.

4. Find inner radius of a hydraulic press to bear 1500 lbs. per square inch, given outer radius = 18 inches; material, cast-iron; tensile strength 20000 lbs. per square inch.

INDEX.

65

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